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COVER LEVELS AND RANDOM INTERLACEMENTS[1]

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This note investigates cover levels of finite sets in the random interlacements model introduced in [*Ann. of Math.* (2) **171** (2010) 2039–2087], that is, the least level such that the set is completely contained in the random interlacement at that level. It proves that as the cardinality of a set goes to infinity, the rescaled and recentered cover level tends in distribution to the Gumbel distribution with cumulative distribution function $\exp(-\exp(-z))$.

0. Introduction. The random interlacements model was introduced in [18]. It helps to understand the picture left by a simple random walk in the discrete torus $(\mathbb{Z}/N\mathbb{Z})^d$, $d \geq 3$, or the discrete cylinder $(\mathbb{Z}/N\mathbb{Z})^{d-1} \times \mathbb{Z}$, $d \geq 3$, when the walk is run up to times of a certain scale. The random interlacements are an increasing family of random sets $\mathcal{I}^u \subset \mathbb{Z}^d$, indexed by a parameter $u \geq 0$, and for each u the set $\mathcal{I}^u$ is, intuitively speaking, the trace the paths whose label is at most u from a Poisson cloud of labeled doubly infinite paths in $\mathbb{Z}^d$ modulo time-shift. By analogy with the concept of random walk cover times this note introduces the *cover level* of a set by random interlacements and proves a fine asymptotic limit result for this quantity. Since random interlacements model random walk in the discrete torus and cylinder on certain suitable time scales, our result should eventually lead to a better understanding of the distributional limits of cover times in these graphs.

We now briefly recall how $\mathcal{I}^u$ is constructed. We denote by W the space of doubly infinite nearest neighbor paths in $\mathbb{Z}^d$ that spend finite time in bounded subsets of $\mathbb{Z}^d$. We also introduce the equivalence relation $\sim$ on W by letting $w \sim v$ if w is a time-shift of v, that is, if there exists an $N \in \mathbb{Z}$ such that $w(n) = v(N+n)$ for all $n \in \mathbb{Z}$. The space $W^\star$ of doubly infinite paths modulo time-shift is defined by $W^\star = W/\sim$. The "Poisson cloud" mentioned above is then the interlacement Poisson point process, that is, a Poisson point process $\omega = \sum_i \delta_{(w_i^\star, u_i)}$ on the space $W^\star \times [0,\infty)$ with intensity measure given by the product measure of a certain σ-finite measure ν on $W^\star$ and Lebesgue measure. If $K \subset \mathbb{Z}^d$ is finite, the total mass assigned by ν to the set of trajectories modulo time-shift that enter K is the capacity of K [see (1.5)]. If one normalizes the measure ν on this set and then considers for each trajectory modulo time-shift the representative from W which

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enters K for the first time at time 0, then ν corresponds to picking a position at time 0 distributed according to the normalized equilibrium distribution [see (1.4)], and conditionally on the position at time 0 the forward and backward trajectories are, respectively, distributed as simple random walk and simple random walk conditioned never to reenter K. We refer to Section 1 of [18] for details. We denote the law governing ω by $\mathbb{P}$. The random interlacement $\mathcal{I}^u$ is then defined as

$$\mathcal{I}^u = \bigcup_{i\,:\,u_i \le u} \text{range}(w_i^\star), \; u \ge 0, \; \text{where } \omega = \sum_i \delta_{(w_i^\star, u_i)}, \tag{0.1}$$

and range$(w^\star)$ denotes the set of all vertices of $\mathbb{Z}^d$ visited by the path $w^\star \in W^\star$. The measure ν is constructed so that, intuitively speaking, for a value u related to the time up to which the random walk in the torus or cylinder is run, the trace of the random walk "looks like" $\mathcal{I}^u$ [16, 22]. The law of the indicator function of $\mathcal{I}^u$ on $\{0,1\}^{\mathbb{Z}^d}$ has a simple characterization (see [18], (2.16)): it is the unique law with the property that

$$\mathbb{P}(A \cap \mathcal{I}^u = \varnothing) = \exp(-u \cdot \text{cap}(A)) \qquad \text{for all finite } A \subset \mathbb{Z}^d, \tag{0.2}$$

where cap(A) denotes the capacity of A [see (1.5) for the definition].

The sets $\mathcal{I}^u$ are naturally increasing in u. Taking inspiration from the concept of cover time of a finite graph one may consider the *cover level* of a finite set $A \subset \mathbb{Z}^d$, defined as the least level u such that A is completely contained in $\mathcal{I}^u$,

$$M(A) = \inf\{u \ge 0 : A \subset \mathcal{I}^u\} = \max_{x \in A} U_x, \tag{0.3}$$

where U_x denotes the cover time of the vertex x,

$$U_x = \inf\{u \ge 0 : x \in \mathcal{I}^u\}, \; x \in \mathbb{Z}^d. \tag{0.4}$$

The main result of this note is the following theorem.

THEOREM 0.1 (Rescaled and recentered cover levels have a distribution close to Gumbel). *For any finite nonempty $A \subset \mathbb{Z}^d$ we have*

$$\sup_{z \in \mathbb{R}} \left| \mathbb{P}\left(\frac{M(A)}{g(0)} - \log|A| \le z \right) - \exp(-e^{-z}) \right| \le c|A|^{-c_1}, \tag{0.5}$$

where $c_1 > 0$ is the constant given in (2.5) and $g(\cdot)$ is the $\mathbb{Z}^d$ Green's function [see (1.1)]. In particular, $\frac{M(A)}{g(0)} - \log|A|$ tends in distribution to the Gumbel distribution, as $|A|$ tends to infinity.

We will now give some comments on the scope of the above theorem. If $G_1, G_2, \dots$ denotes a sequence of finite graphs whose cardinality tends to infinity and C_N denotes their cover times (i.e., the first time simple random walk has visited every vertex of the graph) it is sometimes possible (see, e.g., [4, 8]) to show that $\frac{C_N}{c|G_N|} - \log|G_N|$ tends in law to the Gumbel distribution as $N \to \infty$.

Theorem 0.1 should be seen as a result of this flavor. There are, however, simple families of graphs for which one can show that $\frac{C_N}{c|G_N|\log|G_N|} \to 1$ in probability but for which the finer distributional limit result remains out of reach to this day. For example, when $G_N = (\mathbb{Z}/N\mathbb{Z})^d, d \geq 3$, is the discrete torus, it is known that $\frac{C_N}{g(0)|G_N|\log|G_N|} \to 1$ in probability but only conjectured that $\frac{C_N}{g(0)|G_N|} - \log|G_N|$ tends to the Gumbel distribution (see [1], Chapter 7, Section 2.2, pages 22 and 23). In [15, 17] and [20] results are proved that couple random interlacements and the trace of random walk in the discrete cylinder $(\mathbb{Z}/N\mathbb{Z})^d \times \mathbb{Z}, d \geq 2$, and discrete torus $(\mathbb{Z}/N\mathbb{Z})^d, d \geq 3$, in nearly macroscopic boxes of side length $N^{1-\varepsilon}, \varepsilon > 0$. In these works the couplings are used as a "transfer mechanism" to allow one to study the disconnection time of the cylinder and the so-called "fragmentation" of the torus by studying a related problem formulated completely inside the random interlacements model. We believe that in a similar way the results in the present note for cover levels of sets in the random interlacements model will lead to progress in the study of cover times of sets by random walk in the torus and cylinder (for more on this see Remark 2.9).

The second result of this note is a corollary of Theorem 0.1. For any $1 \leq l \leq d$ and $z \in \mathbb{R}$ we let $B_N^l = [0, N-1]^l \times \{0\}^{d-l}$ and define a sequence $(\mathcal{N}_N^{l,z})_{N \geq 1}$ of point measures on $\mathbb{R}^d$:

$$\mathcal{N}_N^{l,z} = \sum_{x \in B_N^l} \delta_{x/N} 1_{\{U_x > g(0)\{\log|B_N^l| + z\}\}}, \quad N \geq 1. \tag{0.6}$$

In other words, $\mathcal{N}_N^{l,z}$ collects the points of $[0,1]^l \times \{0\}^{d-l}$ which under scaling correspond to the sites of B_N^l not yet covered by the random interlacements at level $g(0)\{\log|B_N^l| + z\}$.

COROLLARY 0.2 (Convergence of point process of uncovered points to a homogeneous Poisson point process).

$$\mathcal{N}_N^{l,z} \textit{ converges in law to } \mathcal{N}^{l,z} \textit{ as } N \to \infty, \tag{0.7}$$

where $\mathcal{N}^{l,z}$ *is a Poisson point process with intensity* $\exp(-z)\lambda_l$ *and* λ_l *is Lebesgue measure on* $[0,1]^l \times \{0\}^{d-l}$.

Incidentally, it follows from this corollary that the last few sites of B_N^l to be covered by the random interlacements are "far apart," at typical distance of about N. This fact is proved in Proposition 2.8.

We now comment on the proofs of Theorem 0.1 and Corollary 0.2. For each $x \in \mathbb{Z}^d$ the random variable U_x is known to have exponential distribution with parameter $\frac{1}{g(0)}$ [see (0.4) and (1.10)]. If the U_x were independent, then standard extreme value theory would tell us that the rescaled and recentered maxima $\frac{M(A)}{g(0)} - \log|A|$ tend in distribution to the Gumbel distribution as $|A| \to \infty$ (see [5], Example 3.2.7, page 125). However, in the random interlacements model the U_x are not independent, in fact, there is a long-range correlation; cf. (1.12). There is a theory that gives mixing conditions called D and D' for stationary sequences ([5], Section 4.4, page 209) and also similar conditions for stationary random fields [7, 12] which are sufficient for the rescaled and recentered maxima to converge in distribution to the Gumbel distribution. When we prove (2.9) of Lemma 2.5 we will prove something similar to D'. However, conditions similar to D are difficult to verify in our context because of the slow decay of the correlation; cf. (1.12). We, therefore, take a different approach and prove the convergence in distribution directly.

The key to proving Theorem 0.1 is to exploit Lemma 2.1, which says in a quantitative way that in the random interlacements model spatial separation implies approximate independence. We do this in Proposition 2.2 by considering sets A that are "well separated," that is, that consist of isolated points that are far apart. This spatial separation allows us to use Lemma 2.1 to show that the points of a well-separated set are covered approximately independently.

We then consider arbitrary nonempty finite sets A and condition on the subset left uncovered at level $g(0)(1-\varepsilon)\log|A|$ for a value of ε that satisfies $0 < \varepsilon \leq 12c_1$. In Lemma 2.6 we show that with high probability this "uncovered set" is well separated and has cardinality concentrated around its expected cardinality, which equals $|A|^{\varepsilon}$. When the "uncovered set" is well separated, Proposition 2.2, mentioned above, implies that after level $g(0)(1-\varepsilon)\log|A|$ the points of the uncovered set are covered approximately independently. This allows us to finish the job in the proof of Theorem 0.1 by showing (in a quantitative way) that, when we restrict to a certain "good event" that has probability tending to one as $|A| \to \infty$, the cover level $M(A)$ is the maximum of approximately $|A|^{\varepsilon}$ random variables, which are "essentially" independent and exponentially distributed. It is then not hard to show that if we rescale and recenter $M(A)$ appropriately, it is close in distribution to the Gumbel distribution, just as would be the case if the cover levels of the points of A were truly independent.

As alluded to above, random interlacements model the picture left by random walk in the discrete torus and the discrete cylinder when run up to times of suitably chosen scales. In this light, the uncovered set A_{ε} (when $A = B_N^d$) can, in particular, be thought of as a counterpart of the uncovered set in the discrete torus discussed in [2, 9]. See Remarks 2.7 and 2.9 for more on this.

The proof of Corollary 0.2 uses Theorem 0.1 and Kallenberg's theorem ([13], Proposition 3.22, page 157) which allow us to verify the convergence of point processes by checking some straightforward conditions involving convergence of the intensity measure and the probability that the point measure does not charge a set.

1. Notation and a review of random interlacements.

Constants denoted by c may change from line to line and within formulas. Unless otherwise indicated, all constants depend only on the dimension d of $\mathbb{Z}^d$. Further dependence on, for example, parameters α, β, is denoted by $c(\alpha, \beta)$. The norm $|\cdot|$ on $\mathbb{Z}^d$ is taken to be the Euclidean norm. We denote by $|A|$ the cardinality of the set A. The notation $A \subset\subset B$ means that A is a finite subset of B. For two sets $A, B \subset \mathbb{Z}^d$ we denote their mutual distance $\inf_{x \in A, y \in B} |x - y|$ by $d(A, B)$. We define a path to be a sequence $x_i, i \geq 0$, of elements in $\mathbb{Z}^d$ such that $|x_{i+1} - x_i| = 1$ for $i \geq 0$.

We denote by W_+ the space of paths in $\mathbb{Z}^d, d \geq 3$, going to infinity as time goes to infinity. Furthermore, $(X_n)_{n \geq 0}$ denotes the canonical coordinates, $\mathcal{W}_+$ the σ-algebra on W_+ generated by these coordinates and $\theta_n : W_+ \to W_+$ the canonical shift on W_+. We let P_x be the probability measure on $(W_+, \mathcal{W}_+)$ that turns $(X_n)_{n \geq 0}$ into a simple random walk starting at x (since for $d \geq 3$ the simple random walk is transient, its law is supported on W_+). If $q : \mathbb{Z}^d \to [0, \infty)$ then P_q denotes the measure $\sum_{x \in \mathbb{Z}^d} q(x) P_x$.

Green's function is given by

$$g(x, y) = \sum_{n \geq 0} P_x(X_n = y) \quad \text{and} \quad g(\cdot) = g(\cdot, 0). \tag{1.1}$$

Recall the following standard bounds for Green's function ([6], Theorem 1.5.4, page 31):

$$c|x|^{2-d} \leq g(x) \leq c|x|^{2-d}. \tag{1.2}$$

Furthermore, by the invariance principle, if A is a set of diameter at most L (i.e., $|x - y| \leq L$ for all $x, y \in A$) then

$$\sum_{x \in A} g(x) \leq cL^2, \tag{1.3}$$

where we have used that the left-hand side of (1.3) is bounded by the expected time spent by a random walk in a ball containing A.

For $U \subset\subset \mathbb{Z}^d$, we define the entrance time $H_U = \inf\{n \geq 0 : X_n \in U\}$ and the hitting time $\tilde{H}_U = \inf\{n \geq 1 : X_n \in U\}$. The escape probability (or equilibrium measure) $e_U : \mathbb{Z}^d \to [0, \infty)$ is given by

$$e_U(x) = P_x(\tilde{H}_U = \infty) 1_U(x) \tag{1.4}$$

and the capacity of U by

$$\operatorname{cap}(U) = \sum_{x \in U} e_U(x). \tag{1.5}$$

Moreover, for each $y \in K$, we have the equality

$$P_x(H_K < \infty) = \sum_{y \in K} g(x, y) e_K(y). \tag{1.6}$$

We now recall some further facts about random interlacements. As mentioned in the Introduction, to construct the random interlacements one defines on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ the so-called interlacement Poisson point process ω, which is a Poisson point process on the space $W^\star \times [0,\infty)$ of labeled doubly infinite paths modulo time-shift (see the second paragraph of the Introduction for notation) whose intensity is given by the product measure of a certain σ-finite measure ν and Lebesgue measure. For a detailed construction of the measure ν see [18], Theorem 1.1. In this note we will only need the existence on the space $(\Omega, \mathcal{A}, \mathbb{P})$ of a family of Poisson point processes $\mu_{K,u}$ and $\mu_{K,u,u'}$ on $(W_+, \mathcal{W}_+)$, defined for any $K \subset\subset \mathbb{Z}^d$ and any $0 \le u \le u'$. Loosely speaking, these point measures on W_+ keep track of those doubly infinite paths modulo time-shifts in ω that enter K, and have labels at most u and labels between u and u', respectively (i.e., they assign weight 1 to the paths in W_+ which the double infinite paths modulo time-shifts induce after their entrance in K). The random interlacement $\mathcal{I}^u$, already defined in terms of the interlacement Poisson process ω in (0.1), can also be constructed from the point processes $\mu_{K,u}$ (see [14], (1.16)),

$$\mathcal{I}^u = \bigcup_{K \subset\subset \mathbb{Z}^d} \bigcup_{w \in \mu_{K,u}} \mathrm{range}(w), \qquad u \ge 0. \tag{1.7}$$

[For any point process μ we write $x \in \mu$ as a shorthand for x belonging to $\mathrm{Supp}(\mu)$, the support of μ.] For the definitions and some properties of μ_K, $\mu_{K,u,u'}$ we refer to [14], (1.13)–(1.15). We will need the following facts:

(1.8) $\mu_{K,u}$ and $\mu_{K,u,u'}$ are independent Poisson point processes on $(W_+, \mathcal{W}_+)$ with respective intensities $u \cdot P_{e_K}$ and $(u'-u) \cdot P_{e_K}$,

(1.9)
$$\mu_{K,u'} = \mu_{K,u} + \mu_{K,u,u'} \quad \text{and}$$
$$\mu_{K,u} = \sum_{i=0}^m \delta_{\theta_{H_K}(w_i)} 1_{\{H_K(w_i)<\infty\}} \text{ for } K \subset K' \subset\subset \mathbb{Z}^d \text{ and } \mu_{K',u} = \sum_{i=0}^m \delta_{w_i}.$$

This last compatibility relation also holds with $\mu_{K,u,u'}$ and $\mu_{K',u,u'}$ replacing $\mu_{K,u}$ and $\mu_{K',u}$.

From the characterization equation (0.2) of the law of $\mathcal{I}^u$ we see that

$$\mathbb{P}(x \notin \mathcal{I}^u) = \exp\left\{-\frac{u}{g(0)}\right\} \tag{1.10}$$

and

$$\mathbb{P}(x, y \notin \mathcal{I}^u) = \exp\left\{-u \frac{2}{g(0)+g(x-y)}\right\}, \tag{1.11}$$

since $\mathrm{cap}(\{x\}) = \frac{1}{g(0)}$ and $\mathrm{cap}(\{x,y\}) = \frac{2}{g(0)+g(x-y)}$ (see [18], (1.62) and (1.64)). As noted in [18], (1.68), (1.11) together with the bounds on Green's function from (1.2) implies

$$\mathrm{cov}_{\mathbb{P}}\big(1_{\{x\in\mathcal{I}^u\}}, 1_{\{y\in\mathcal{I}^u\}}\big) \sim \frac{cu}{|y-x|^{d-2}} \exp(-cu) \qquad \text{as } |x-y| \to \infty. \tag{1.12}$$

For brevity we write

$$u_A(z) = g(0)\{\log|A| + z\}, \tag{1.13}$$

so that $\{\frac{M(A)}{g(0)} - \log|A| \le z\} = \{M(A) \le u_A(z)\}$.

2. Proofs of Theorem 0.1 and Corollary 0.2. We begin by discussing the overall structure of the proofs of Theorem 0.1 and Corollary 0.2. The first step is to extract some independence in the random interlacement model. This is done in Lemma 2.1 which says in a quantitative fashion that the picture left by the random interlacements in a set K_1 and the picture left in a set K_2 are approximately independent if K_1 and K_2 are far apart.

Next we will prove that for well-separated sets, that is, sets consisting of isolated points that are far apart, the cover levels of the individual points are approximately independent. This is done in Proposition 2.2, which follows easily from Lemma 2.1. If we were only interested in well-separated sets then Proposition 2.2 would be enough to prove convergence to the Gumbel distribution of the rescaled and recentered cover levels. Intuitively speaking, this is because the cover level of a well-separated set is the maximum of a set of essentially independent random variables (namely, the cover levels of the individual points).

But of course we are not dealing only with well-separated sets, but with arbitrary finite nonempty sets A. Therefore, we introduce the random set A_ε, defined in (2.7), which consists of all the points of A left uncovered at the level $g(0)(1-\varepsilon)\log|A|$ for a parameter ε such that

$$0 < \varepsilon \le 12c_1, \tag{2.1}$$

where c_1 is the constant defined in (2.5). For a fixed $\varepsilon \in (0, 12c_1]$ our methods yield (2.19) (uniformly for all z) of which (0.5) is the special case $\varepsilon = 12c_1$. Whenever ε appears below, it is always understood to satisfy (2.1).

Next we need to show that A_ε is "well behaved." This is done in Lemma 2.6 which states that with probability tending to one, A_ε is well separated and has cardinality close to $\mathbb{E}|A|$ [which equals $|A|^\varepsilon$ by (2.17)], or in other words, with probability tending to one, A_ε belongs to the collection $G_{A,\varepsilon}$ of "good sets" defined in (2.13).

Finally, we turn to the proof of Theorem 0.1 in which the goal essentially is to show that $\mathbb{P}(M(A) \le u_A(z))$ is close to the cumulative distribution function of the Gumbel distribution, that is, close to $\exp(-\exp(-z))$. The major step will be to condition on the set of sites of A not yet covered at level $(1-\varepsilon)u_A(0) \overset{(1.13)}{=} g(0)(1-\varepsilon)\log|A|$, that is, on A_ε. This will be useful because by (1.8) we have $\mathbb{P}(M(A) \le u_A(z)|A_\varepsilon = K) = \mathbb{P}(M(K) \le u_A(z) - (1-\varepsilon)u_A(0))$. Next we will show that this latter probability is close to $\exp(-\exp(-z))$. Then, multiplying by

$\mathbb{P}(A_\varepsilon = K)$, summing over all $K \in G_{A,\varepsilon}$ and using that $A_\varepsilon \in G_{A,\varepsilon}$ with probability tending to one will allow us to show that $\mathbb{P}(M(A) \le u_A(z))$ is close to $\exp(-\exp(-z))$.

The key to proving that $\mathbb{P}(M(K) \le u_A(z) - (1-\varepsilon)u_A(0))$ is close to $\exp(-\exp(-z))$ is to use the fact that all $K \in G_{A,\varepsilon}$ are well separated and that $|K|$ is close to $|A|^\varepsilon$. The well-separatedness of K allows us to use Proposition 2.2 to prove that the points of K are covered approximately independently. Thus $M(K)$ is the maximum of $|K|$ approximately i.i.d. random variables, meaning that with the correct rescaling and recentering $M(K)$ is approximately a Gumbel random variable, or in other words, $\mathbb{P}(M(K) \le u_K(z))$ is close to $\exp(-\exp(-z))$. But since K is close to $|A|^\varepsilon$, it follows that $u_K(z) \overset{(1.13)}{=} g(0)\{\log|K| + z\}$ is close to $u_A(z) - (1-\varepsilon)u_A(0) \overset{(1.13)}{=} g(0)\{\varepsilon \log|A| + z\}$. This will allow us to conclude that $\mathbb{P}(M(K) \le u_A(z) - (1-\varepsilon)u_A(0))$ is close to $\mathbb{P}(M(K) \le u_K(z))$ and thus also close to $\exp(-\exp(-z))$.

Corollary 0.2 then follows easily from Theorem 0.1 using Kallenberg's theorem ([13], Proposition 3.22, page 157).

We begin by stating and proving Lemma 2.1. The proof is just the calculation leading up to [18], (2.15), but it is included here for completeness.

LEMMA 2.1 (Approximate independence of distant sets in random interlacements). *Assume $u \ge 0$. Let $K_1, K_2 \subset\subset \mathbb{Z}^d$ be disjoint sets and let B_1, B_2 be events depending only on $\mathcal{I}^u \cap K_1$ and $\mathcal{I}^u \cap K_2$, respectively. Then*

$$|\mathbb{P}(B_1 \cap B_2) - \mathbb{P}(B_1)\mathbb{P}(B_2)| \le cu \frac{\operatorname{cap}(K_1)\operatorname{cap}(K_2)}{d(K_1, K_2)^{d-2}}. \tag{2.2}$$

PROOF. Decompose $\mu_{K_1 \cup K_2, u} = \sum_{n \ge 0} \delta_{w_n}$ as follows:

$$\mu_{K_1 \cup K_2, u} = \mu_{1,1} + \mu_{1,2} + \mu_{2,1} + \mu_{2,2},$$

where

$$\mu_{1,1} = \sum_{n \ge 0} \delta_{w_n} 1_{\{X_0 \in K_1, H_{K_2} = \infty\}}, \qquad \mu_{1,2} = \sum_{n \ge 0} \delta_{w_n} 1_{\{X_0 \in K_1, H_{K_2} < \infty\}},$$

$$\mu_{2,1} = \sum_{n \ge 0} \delta_{w_n} 1_{\{X_0 \in K_2, H_{K_1} < \infty\}}, \qquad \mu_{2,2} = \sum_{n \ge 0} \delta_{w_n} 1_{\{X_0 \in K_2, H_{K_1} = \infty\}}.$$

The $\mu_{i,j}$ are simply the restriction of the Poisson point process $\mu_{K_1 \cup K_2, u}$ to disjoint sets and are thus independent Poisson point processes with respective intensity measures

$$u 1_{\{X_0 \in K_1, H_{K_2} = \infty\}} P_{e_{K_1 \cup K_2}}, \qquad u 1_{\{X_0 \in K_1, H_{K_2} < \infty\}} P_{e_{K_1 \cup K_2}},$$

$$u 1_{\{X_0 \in K_2, H_{K_1} < \infty\}} P_{e_{K_1 \cup K_2}}, \qquad u 1_{\{X_0 \in K_2, H_{K_1} = \infty\}} P_{e_{K_1 \cup K_2}}.$$

There exist measurable functions of point measures F_1 and F_2 such that

$$F_1(\mu_{K_1,u}) = 1_{B_1} \quad \text{a.s.} \quad \text{and} \quad F_2(\mu_{K_2,u}) = 1_{B_2} \quad \text{a.s.}$$

and thus $\mathbb{P}(B_1 \cap B_2) = \mathbb{E}[F_1(\mu_{K_1,u})F_2(\mu_{K_2,u})]$. Note that $\mu_{K_1,u} - \mu_{1,1} - \mu_{1,2}$ is a point process determined by $\mu_{2,1}$ and thus independent from $\mu_{1,1}, \mu_{1,2}, \mu_{2,2}$ and similarly $\mu_{K_2,u} - \mu_{2,2} - \mu_{2,1}$ is a point process independent from $\mu_{2,2}, \mu_{2,1}, \mu_{1,1}$. So we can define auxiliary point processes $\mu'_{2,1}$ and $\mu'_{1,2}$ such that $\mu'_{2,1}$ has the same distribution as $\mu_{K_1,u} - \mu_{1,1} - \mu_{1,2}$ and $\mu'_{1,2}$ has the same distribution as $\mu_{K_2,u} - \mu_{2,2} - \mu_{2,1}$ and $\mu'_{2,1}, \mu'_{1,2}, \mu_{i,j}, 1 \le i, j \le 2$ are independent, so that $\mu_{1,1} + \mu_{1,2} + \mu'_{2,1} \overset{\text{law}}{=} \mu_{K_1,u}$ and $\mu_{2,2} + \mu_{2,1} + \mu'_{1,2} \overset{\text{law}}{=} \mu_{K_2,u}$. Then $\mathbb{P}(B_1)\mathbb{P}(B_2) = \mathbb{E}[F_1(\mu_{1,1} + \mu_{1,2} + \mu'_{2,1})F_2(\mu_{2,2} + \mu_{2,1} + \mu'_{1,2})]$. So $|\mathbb{P}(B_1 \cap B_2) - \mathbb{P}(B_1)\mathbb{P}(B_2)|$ is bounded above by

$$\begin{aligned} &\mathbb{P}(\mu'_{1,2} \neq 0 \text{ or } \mu'_{2,1} \neq 0 \text{ or } \mu_{1,2} \neq 0 \text{ or } \mu_{2,1} \neq 0) \\ &\quad \le 2\big(\mathbb{P}(\mu_{1,2} \neq 0) + \mathbb{P}(\mu_{2,1} \neq 0)\big). \end{aligned} \tag{2.3}$$

We can bound the probabilities in (2.3) by the total mass of the intensity measures of the point processes $\mu_{1,2}$ and $\mu_{2,1}$ so that

$$\begin{aligned} &|\mathbb{P}(B_1 \cap B_2) - \mathbb{P}(B_1)\mathbb{P}(B_2)| \\ &\quad \le 2u\big(P_{e_{K_1 \cup K_2}}(X_0 \in K_1, H_{K_2} < \infty) + P_{e_{K_1 \cup K_2}}(X_0 \in K_2, H_{K_1} < \infty)\big). \end{aligned}$$

But note,

$$\begin{aligned} P_{e_{K_1 \cup K_2}}(X_0 \in K_1, H_{K_2} < \infty) &\overset{(1.4)}{\le} \sum_{x \in K_1} e_{K_1}(x) P_x(H_{K_2} < \infty) \\ &\overset{(1.6)}{=} \sum_{x \in K_1, y \in K_2} e_{K_1}(x) g(x, y) e_{K_2}(y) \\ &\overset{(1.2), (1.5)}{\le} c d(K_1, K_2)^{2-d} \operatorname{cap}(K_1) \operatorname{cap}(K_2). \end{aligned}$$

Applying a similar calculation for $P_{e_{K_1 \cup K_2}}(X_0 \in K_2, H_{K_1} < \infty)$ we get (2.2). □

We are now ready to prove Proposition 2.2 which says that the points of well-separated sets are covered approximately independently. When we use it in the proof of Theorem 0.1 we will use a value for the parameter λ which depends on ε.

PROPOSITION 2.2 (The points of well-separated sets are covered almost independently). *Let $\lambda > 0$ be a parameter and let $A \subset\subset \mathbb{Z}^d$ be nonempty and such that $|x - y| \ge |A|^{(2+\lambda)/(d-2)}$ for all distinct $x, y \in A$. Then for $u \ge 0$ we have*

$$\big|\mathbb{P}\big(M(A) \le u\big) - [\mathbb{P}(U_0 \le u)]^{|A|}\big| \le cu|A|^{-\lambda}. \tag{2.4}$$

PROOF. Fix an arbitrary $x \in A$ and let $B_1 = \{M(A \setminus \{x\}) \le u\}$ and $B_2 = \{M(\{x\}) \le u\} = \{U_x \le u\}$. Applying Lemma 2.1 we get

$$\begin{aligned} &\left|\mathbb{P}\big(M(A) \le u\big) - \mathbb{P}(U_0 \le u)\mathbb{P}\big(M(A \setminus \{x\}) \le u\big)\right| \\ &\quad \le cu \frac{\operatorname{cap}(A \setminus \{x\}) \operatorname{cap}(\{x\})}{d(\{x\}, A \setminus \{x\})^{d-2}} \overset{(1.5),(1.2)}{\le} cu \frac{|A|}{|A|^{2+\lambda}} \\ &\quad = cu|A|^{-1-\lambda}. \end{aligned}$$

Now applying the same step $|A| - 1$ more times, with the other elements of A substituted for x, and the appropriate subsets of A substituted for $A \setminus \{x\}$, and using the triangle inequality we get

$$\left|\mathbb{P}\big(M(A) \le u\big) - [\mathbb{P}(U_0 \le u)]^{|A|}\right| \le cu|A||A|^{-1-\lambda} = cu|A|^{-\lambda}. \qquad \square$$

REMARK 2.3. Assume $A_1, A_2, \dots$ is a sequence of sets with $|A_i| \to \infty$ as $i \to \infty$ that are well separated in the sense that they satisfy the hypothesis of Proposition 2.2 for some fixed $\lambda > 0$. Then convergence in distribution of the rescaled and recentered cover levels of the A_i to the Gumbel distribution follows, since, in the notation of (1.13), $[\mathbb{P}(U_0 \le u_{A_i}(z))]^{|A_i|} = (1 - \frac{\exp(-z)}{|A_i|})^{|A_i|}$ tends to $\exp(-\exp(-z))$ and the right-hand side of (2.4) tends to zero as $|A_i| \to \infty$. This observation is a first step on the way to arbitrary sets.

We now define the constant c_1 that appears in Theorem 0.1:

$$c_1 = \frac{1}{4} \min\left(\frac{1}{14}\frac{d-2}{d-1}, \frac{c_2}{9-c_2}\right) \tag{2.5}$$

and

$$c_2 = P_0(\tilde{H}_0 = \infty). \tag{2.6}$$

Since the random walk is transient in $\mathbb{Z}^d$ for $d \ge 3$, we have $c_2 > 0$, so that $c_1 > 0$.

REMARK 2.4. Since $P_0(\tilde{H}_0 < \infty) \sim \frac{1}{2d}$ as $d \to \infty$ we see that $\frac{c_2}{9-c_2} \to \frac{1}{8} > \frac{1}{14}$ as $d \to \infty$, so for large d we have $c_1 = \frac{1}{56}\frac{d-2}{d-1}$. Actually it can be shown that $\frac{c_2}{9-c_2} > \frac{1}{14}$ for all $d \ge 3$, and hence, $c_1 = \frac{1}{56}\frac{d-2}{d-1}$ for all $d \ge 3$. We do not include the details, but to do this, one uses the expression for $g(0)$ (when $d = 3$) in terms of an integral given in [10], (4.1), and the explicit computation of this integral (scaled by a factor $\frac{1}{3}$) from [21], together with the trivial bound $K(k) \le \frac{\pi}{2}\frac{1}{\sqrt{1-k^2}}$ on $K(k)$, the complete elliptic integral of the first kind.

The next result, Lemma 2.5, encapsulates a calculation used in Lemma 2.6 to prove that with probability tending to one A_ε, defined in (2.7), is a "good set," that is, belongs to the collection $G_{A,\varepsilon}$ from (2.13). We recall that we tacitly assume $0 < \varepsilon \le 12c_1$.

LEMMA 2.5. *For any $A \subset\subset \mathbb{Z}^d$ let*

$$A_\varepsilon = \{x \in A : U_x > g(0)(1-\varepsilon)\log|A|\} \tag{2.7}$$

denote the set of points of A not yet covered at level $(1-\varepsilon)u_A(0) \overset{(1.13)}{=} g(0)(1-\varepsilon)\log|A|$. Then for all $A \subset\subset \mathbb{Z}^d$ and $b \geq 1$ we have

$$\sum \mathbb{P}(x, y \in A_\varepsilon) \leq |A|^{2\varepsilon} + c(\varepsilon)|A|^{-\varepsilon/3} \tag{2.8}$$

and

$$\sum \mathbb{P}(x, y \in A_\varepsilon) \leq c(\varepsilon) b^d |A|^{-\varepsilon/3}, \tag{2.9}$$

where the first sum is over all distinct $x, y \in A$ and the second sum is over all $x, y \in A$ such that $0 < |x-y| < b|A|^{1/(2(d-1))}$.

PROOF. We will prove that

$$\sum_{x,y\in A, 0<|x-y|<a} \mathbb{P}(x, y \in A_\varepsilon) \leq \min(|A|^{2\varepsilon}, c|A|^{2\varepsilon-1}a^d) + c(\varepsilon)|A|^{-\varepsilon/3}. \tag{2.10}$$

This implies (2.9) by taking $a = b|A|^{1/(2(d-1))}$ and noting that $2\varepsilon - 1 + \frac{d}{2(d-1)} \leq -\frac{1}{3}\varepsilon$ because $1 - \frac{d}{2(d-1)} = \frac{1}{2}\frac{d-2}{d-1} \overset{(2.5)}{\geq} 28c_1 \overset{(2.1)}{\geq} \frac{7}{3}\varepsilon$. Also (2.8) follows from (2.10) by letting $a \to \infty$. We begin by splitting the sum in (2.10) into

$$I_1 = \sum_{x,y\in A, 0<|x-y|\leq(\log|A|)^2} \mathbb{P}(x, y \in A_\varepsilon)$$

and

$$I_2 = \sum_{x,y\in A, (\log|A|)^2<|x-y|<a} \mathbb{P}(x, y \in A_\varepsilon).$$

To bound I_1 we note

$$\begin{aligned} I_1 &\overset{(1.11)}{=} \sum_{x,y\in A, 0<|x-y|\leq(\log|A|)^2} \exp\left(-(1-\varepsilon)\log|A|\frac{2g(0)}{g(0)+g(x-y)}\right) \\ &\leq c(\log|A|)^{2d}|A|^{1-2(1-\varepsilon)/(1+g(e_1)/g(0))}, \end{aligned} \tag{2.11}$$

where in the inequality we have used that $\frac{g(e_1)}{g(0)} = \mathbb{P}_{e_1}(H_0 < \infty) \geq \mathbb{P}_x(H_0 < \infty) = \frac{g(x)}{g(0)}$ for all $x \in \mathbb{Z}^d, x \neq 0$. The exponent of $|A|$ equals $1 - (1-\varepsilon)\frac{2}{2-c_2}$ since $\frac{g(e_1)}{g(0)} \overset{(2.6)}{=} 1 - c_2$. The definition of c_1 and (2.1) immediately imply that $\varepsilon \leq 12c_1 \leq 3\frac{c_2}{9-c_2} < 3\frac{c_2}{8-c_2}$ and rearranging gives $c_2 > \frac{8\varepsilon}{\varepsilon+3}$. Plugging this in we have $1-(1-\varepsilon)\frac{2}{2-c_2} < 1-(1-\varepsilon)\frac{2}{2-8\varepsilon/(\varepsilon+3)} = 1-(1-\varepsilon)\frac{2\varepsilon+6}{6-6\varepsilon} = 1-\frac{1}{3}(\varepsilon+3) = -\frac{\varepsilon}{3}$.

So I_1 is bounded above by $c(\varepsilon)|A|^{-\varepsilon/3}$. To bound I_2 first note that using the elementary inequality $\frac{1}{1+x} \geq 1 - x$, we find

$$
\begin{aligned}
I_2 &\overset{(1.11)}{=} \sum_{x,y\in A,(\log|A|)^2<|x-y|<a} \exp\biggl(-2(1-\varepsilon)\log|A|\frac{1}{1+g(x-y)/g(0)}\biggr) \\
&\leq \sum_{x,y\in A,(\log|A|)^2<|x-y|<a} \exp\biggl(-2(1-\varepsilon)\log|A|\biggl(1-\frac{g(x-y)}{g(0)}\biggr)\biggr) \qquad (2.12)\\
&\leq |A|^{-2(1-\varepsilon)} \sum_{x,y\in A,(\log|A|)^2<|x-y|<a} \exp\bigl(c\log|A|g(x-y)\bigr).
\end{aligned}
$$

Now note that $g(x-y) \overset{(1.2)}{\leq} c|x-y|^{2-d} \leq c(\log|A|)^{-2}$ for $|x-y| \geq (\log|A|)^2$ so the quantity in the exponential in (2.12) is bounded. Thus we can conclude that (2.12) itself is bounded by

$$
\begin{aligned}
&|A|^{-2(1-\varepsilon)} \sum_{x,y\in A,(\log|A|)^2<|x-y|<a} \bigl(1+c\log|A|g(x-y)\bigr) \\
&\quad \leq \min(|A|^{2\varepsilon}, c|A|^{2\varepsilon-1}a^d) + c|A|^{-2(1-\varepsilon)}\log|A| \sum_{y\in A}\sum_{z\in A-y} g(z).
\end{aligned}
$$

We have no control on the diameter of A. But let $A - y = \{a_1, a_2, \dots, a_n\}$ with $|a_1| \leq |a_2| \leq \cdots \leq |a_n|$, $|A| = n$, and let $b_1, b_2, \dots$ be an enumeration of $\mathbb{Z}^d$ with $|b_1| \leq |b_2| \leq \cdots$. Then by Green's function estimates in (1.2)

$$g(a_i) \leq c|a_i|^{2-d} \leq c|b_i|^{2-d} \leq cg(b_i).$$

So $\sum_{z\in A-y} g(z) \leq c\sum_{i=1}^n g(b_i) \leq c|A|^{2/d}$ since the diameter of $\{b_1, \dots, b_n\}$ is bounded by $cn^{1/d}$. Hence,

$$
\begin{aligned}
I_2 &\leq \min(|A|^{2\varepsilon}, c|A|^{2\varepsilon-1}a^d) + c|A|^{-2(1-\varepsilon)}\log|A||A|^{2/d+1} \\
&\leq \min(|A|^{2\varepsilon}, c|A|^{2\varepsilon-1}a^d) + c\log|A||A|^{-10\varepsilon/9},
\end{aligned}
$$

where we have used that $-2(1-\varepsilon) + \frac{2}{d} + 1 = 2\varepsilon - \frac{d-2}{d} \leq 2\varepsilon - \frac{2}{3}\frac{d-2}{d-1} \overset{(2.5)}{\leq} 2\varepsilon - \frac{112}{3}c_1 \overset{(2.1)}{\leq} -\frac{10}{9}\varepsilon$. Combining this with $I_1 \leq c(\varepsilon)|A|^{-\varepsilon/3}$ then gives (2.10). □

Our next task is to use the above lemma to prove that with high probability A_ε is "well behaved."

LEMMA 2.6 (The good event is likely). *For $A \subset\subset \mathbb{Z}^d$ and ε as in* (2.1) *let*

$$
\begin{aligned}
G_{A,\varepsilon} = \bigl\{&K \subset A : \bigl||K| - |A|^\varepsilon\bigr| \leq |A|^{2\varepsilon/3}, K \neq \varnothing \text{ and} \\
&|x-y| \geq (2^{1/\varepsilon}|A|)^{1/(2(d-1))} \text{ for all distinct } x, y \in K\bigr\} \qquad (2.13)
\end{aligned}
$$

denote the collection of subsets of A *that are well separated and close in cardinality to* $|A|^{\varepsilon}$*. Then for all* $A \subset\subset \mathbb{Z}^d$ *one has*

$$\mathbb{P}(A_\varepsilon \notin G_{A,\varepsilon}) \le c(\varepsilon)|A|^{-\varepsilon/3}. \tag{2.14}$$

PROOF. The following two statements together clearly imply (2.14):

$$\mathbb{P}\big(\exists x, y \in A_\varepsilon \text{ s.t. } 0 < |x-y| < (2^{1/\varepsilon}|A|)^{1/(2(d-1))}\big) \le c(\varepsilon)|A|^{-\varepsilon/3}, \tag{2.15}$$

$$\mathbb{P}\big(\big||A_\varepsilon| - |A|^{\varepsilon}\big| > |A|^{2\varepsilon/3}\big) \le c(\varepsilon)|A|^{-\varepsilon/3}. \tag{2.16}$$

By the union bound (2.15) follows directly from (2.9) with $b = 2^{(1/\varepsilon)(1/(2(d-1)))} \ge 1$. To prove (2.16) note that

$$\mathbb{E}|A_\varepsilon| = |A|\mathbb{P}\big(U_0 > (1-\varepsilon)g(0)\log|A|\big) \stackrel{(1.10)}{=} |A|^{\varepsilon}. \tag{2.17}$$

So by Chebyshev's inequality,

$$\mathbb{P}\big(\big||A_\varepsilon| - |A|^{\varepsilon}\big| > |A|^{2\varepsilon/3}\big) \le \frac{\mathbb{E}|A_\varepsilon|^2 - |A|^{2\varepsilon}}{|A|^{4\varepsilon/3}}. \tag{2.18}$$

But

$$\begin{aligned} \mathbb{E}|A_\varepsilon|^2 &= \sum_{x,y\in A} \mathbb{P}(x, y \in A_\varepsilon) \\ &= \sum_{x\in A} \mathbb{P}(x \in A_\varepsilon) + \sum_{x,y\in A, x\neq y} \mathbb{P}(x, y \in A_\varepsilon) \\ &\stackrel{(2.8)}{\le} |A|^{\varepsilon} + |A|^{2\varepsilon} + c(\varepsilon)|A|^{-\varepsilon/3}. \end{aligned}$$

Plugging this bound for $\mathbb{E}|A_\varepsilon|^2$ into (2.18) then gives (2.16). □

REMARK 2.7. In bounding the numerator of the right-hand side of (2.18) we showed that the variance of $|A_\varepsilon|$ is bounded from above by $|A|^{\varepsilon} + c(\varepsilon)|A|^{-\varepsilon/3}$. The inequality $\mathbb{E}|A_\varepsilon|^2 = |A|^{\varepsilon} + \sum_{x\neq y} \mathbb{P}(x, y \in A_\varepsilon) \stackrel{(1.11)}{\ge} |A|^{\varepsilon} + |A|(|A|-1)|A|^{-2(1-\varepsilon)}$ gives a matching lower bound and proves that $\operatorname{Var}|A_\varepsilon| \sim |A|^{\varepsilon}$ as $|A| \to \infty$. In particular, if $A = B_N^d$, then $\operatorname{Var}|A_\varepsilon| \sim N^{d\varepsilon}$. As might be expected given the connection between random interlacements and random walk in the discrete torus (see [20, 22]), this agrees with the value which was found in the theoretical physics paper [2] for the variance of the number of points of the torus $(\mathbb{Z}/N\mathbb{Z})^d, d \ge 3$, not covered by random walk run up to time $(1-\varepsilon)g(0)N^d \log N^d$ (see [2], (3.15), (3.9)). Note that the time $(1-\varepsilon)g(0)N^d \log N^d$ is a fraction $1-\varepsilon$ of the "typical" cover time $g(0)N^d \log N^d$ (see [1], Chapter 7, Section 2.2, page 22, Corollary 24) of the torus, just as $(1-\varepsilon)g(0)\log N^d$ is a fraction $1-\varepsilon$ of the "typical" cover level $g(0)\log N^d$ of B_N^d by random interlacements.

We are now ready to prove the main theorem. Recall the definition of $u_A(z)$ from (1.13).

PROOF OF THEOREM 0.1. If $z \le -\frac{1}{4}\varepsilon \log|A|$ and $|A| > 1$, then

$$\mathbb{P}\big(M(A) \le u_A(z)\big) \overset{(2.7)}{\le} \mathbb{P}(A_{\varepsilon/4} = \varnothing) \overset{(2.13)}{\le} \mathbb{P}(A_{\varepsilon/4} \notin G_{A,\varepsilon/4}) \overset{(2.14)}{\le} c(\varepsilon)|A|^{-\varepsilon/12}.$$

Also, $\exp(-\exp(-z)) \le \exp(-|A|^{\varepsilon/4}) \le |A|^{-\varepsilon/4}$ so that

$$\big|\mathbb{P}\big(M(A) \le u_A(z)\big) - \exp(-\exp(-z))\big| \le c(\varepsilon)|A|^{-\varepsilon/12}. \tag{2.19}$$

Furthermore, if $z \ge \log|A|$ then

$$\begin{aligned}\mathbb{P}\big(M(A) > u_A(z)\big) &\le \mathbb{P}\big(A \not\subset \mathcal{I}^{2g(0)\log|A|}\big)\\ &\overset{(1.10)}{\le} |A|\exp(-2\log|A|) = |A|^{-1}.\end{aligned}$$

Also, for such z we have $\exp(-\exp(-z)) \ge \exp(-|A|^{-1}) \ge 1 - |A|^{-1}$ so (2.19) also holds for $z \ge \log|A|$. It thus remains to show (2.19) for $z \in (-\frac{1}{4}\varepsilon\log|A|, \log|A|)$, and in what follows we assume z to be in this range.

Let $\mu_1 = \mu_{A,(1-\varepsilon)u_A(0)}, \mu_2 = \mu_{A,(1-\varepsilon)u_A(0),u_A(z)}$ and $\mu_3 = \mu_{A,u_A(z)}$. If we fix any $K \in G_{A,\varepsilon}$ then $\{A_\varepsilon = K\}$ is simply the event $E_1 = \{A \setminus K = A \cap \bigcup_{w\in\mu_1} \operatorname{range}(w)\}$ [recall from the remark above (1.7) that $w \in u_1$ means $w \in \operatorname{Supp}(\mu_1)$]. Furthermore, $\{M(A) \le u_A(z)\}$ is simply the event $\{A \subset \bigcup_{w\in\mu_3} \operatorname{range}(w)\}$, and since by (1.9) we have $\mu_1 + \mu_2 = \mu_3$, the intersection $E_1 \cap \{M(A) \le u_A(z)\}$ coincides with $E_1 \cap \{A \subset (A \setminus K) \cup \bigcup_{w\in\mu_2} \operatorname{range}(w)\} = E_1 \cap \{K \subset \bigcup_{w\in\mu_2} \operatorname{range}(w)\}$. Denote the last event in the latter intersection by E_2, and recall that by (1.8) we have that μ_1 and μ_2 are independent, so that $\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_1) \times \mathbb{P}(E_2)$. Also by (1.8) the point process $\mu_{A,u_A(z)-(1-\varepsilon)u_A(0)} \overset{(1.13)}{=} \mu_{A,g(0)\{\varepsilon\log|A|+z\}}$ has the same law as μ_2, so

$$\mathbb{P}(E_2) = \mathbb{P}\Bigg(K \subset \bigcup_{w\in u_{A,g(0)\{\varepsilon\log|A|+z\}}} \operatorname{range}(w)\Bigg) = \mathbb{P}\big(M(K) \le g(0)\{\varepsilon\log|A| + z\}\big).$$

It follows that (2.20) holds for all $K \in G_{A,\varepsilon}$ by noting that both the right- and the left-hand side equal $\mathbb{P}(E_1 \cap E_2)/\mathbb{P}(E_1)$.

$$\mathbb{P}\big(M(A) \le u_A(z) | A_\varepsilon = K\big) = \mathbb{P}\big(M(K) \le g(0)\{\varepsilon\log|A| + z\}\big). \tag{2.20}$$

Then consider some $K \in G_{A,\varepsilon}$. Define $\lambda = \frac{1}{2\varepsilon}\frac{d-2}{d-1} - 2 \overset{(2.1),(2.5)}{\ge} \frac{14}{6} - 2 = \frac{1}{3}$ and note that $\frac{\lambda+2}{d-2}\varepsilon = \frac{1}{2(d-1)}$, so that for distinct x, y in K we have

$$|x - y| \overset{(2.13)}{\ge} (2^{1/\varepsilon}|A|)^{1/(2(d-1))} = (2|A|^\varepsilon)^{(2+\lambda)/(d-2)} \overset{(2.13)}{\ge} |K|^{(2+\lambda)/(d-2)}.$$

We can now use Proposition 2.2 to get that for $|A|>1$,

$$\begin{aligned}&\big|\mathbb{P}\big(M(K)\le g(0)\{\varepsilon\log|A|+z\}\big)-\mathbb{P}\big(U_0\le g(0)\{\varepsilon\log|A|+z\}\big)^{|K|}\big|\\ &\quad\overset{z\le\log|A|}{\le} c\log|A||K|^{-\lambda}\\ &\quad\le c(\varepsilon)\log|A||A|^{-\varepsilon/3},\end{aligned}\tag{2.21}$$

since $|K|^{-\lambda}\overset{(2.13)}{\le}(|A|^{\varepsilon}-|A|^{2\varepsilon/3})^{-\lambda}\le c(\varepsilon)|A|^{-\lambda\varepsilon}\le c(\varepsilon)|A|^{-\varepsilon/3}$. Furthermore, using again that $||A|^{\varepsilon}-|K||\le|A|^{2\varepsilon/3}$ and (1.10) we see that

$$\begin{aligned}\left(1-\frac{e^{-z}}{|A|^{\varepsilon}}\right)^{|A|^{\varepsilon}+|A|^{2\varepsilon/3}}&\le\mathbb{P}\big(U_0\le g(0)\{\varepsilon\log|A|+z\}\big)^{|K|}\\ &\le\left(1-\frac{e^{-z}}{|A|^{\varepsilon}}\right)^{|A|^{\varepsilon}-|A|^{2\varepsilon/3}}.\end{aligned}\tag{2.22}$$

But note that if $|A|\ge c(\varepsilon)$ one has the inequality

$$\left|\exp(-e^{-z})-\left(1-\frac{e^{-z}}{|A|^{\varepsilon}}\right)^{|A|^{\varepsilon}\pm|A|^{2\varepsilon/3}}\right|\overset{z\ge-(\varepsilon/4)\log|A|}{\le}c|A|^{-\varepsilon/12}.$$

Combining (2.20), (2.21) and (2.22) with the above formula yields that if $|A|\ge c(\varepsilon)$,

$$\big|\mathbb{P}\big(M(A)\le u_A(z)|A_\varepsilon=K\big)-\exp(-e^{-z})\big|\le c(\varepsilon)|A|^{-\varepsilon/12}.\tag{2.23}$$

Multiplying by $\mathbb{P}(A_\varepsilon=K)$, and summing over all $K\in G_{A,\varepsilon}$, we see that

$$\big|\mathbb{P}\big(M(A)\le u_A(z),A_\varepsilon\in G_{A,\varepsilon}\big)-\exp(-e^{-z})\mathbb{P}(A_\varepsilon\in G_{A,\varepsilon})\big|\le c(\varepsilon)|A|^{-\varepsilon/12}.$$

Finally, two applications of (2.14) gives us that (2.19) holds for $|A|\ge c(\varepsilon)$ and $z\in(-\frac{1}{4}\varepsilon\log|A|,\log|A|)$. Thus (2.19) holds for all $z\in\mathbb{R}$, and (0.5) follows by taking $\varepsilon=12c_1$. □

We now use Theorem 0.1 to prove Corollary 0.2 which states that the point process of uncovered points of a box converges to a homogeneous Poisson point process.

PROOF OF COROLLARY 0.2. We drop the superscripts on $\mathcal{N}_N^{l,z}$ and $\mathcal{N}^{l,z}$ to lighten the notation. By Kallenberg's theorem (see [13], Proposition 3.22, page 157) it suffices to check that

$$\lim_{N\to\infty}\mathbb{E}\mathcal{N}_N(I)=\mathbb{E}\mathcal{N}(I)\text{ for all }I\in\mathcal{J}\tag{2.24}$$

and

$$\lim_{N\to\infty}\mathbb{P}\big(\mathcal{N}_N(I)=0\big)=\mathbb{P}\big(\mathcal{N}(I)=0\big)\text{ for all }I\in\mathcal{J},\tag{2.25}$$

where $\mathcal{J} = \{$Finite unions of open rectangles $\prod_{i=1}^{d}(a_i, b_i)$ in $\mathbb{R}^d\}$. To verify (2.24) simply note that

$$\mathbb{E}\mathcal{N}_N(I) = \sum_{x \in NI \cap B_N^l} \mathbb{P}\big(U_x > g(0)\{\log|B_N^l| + z\}\big)$$
$$\overset{(1.10)}{=} \frac{|NI \cap B_N^l|}{|B_N^l|} \exp(-z) \to \lambda_l(I) \exp(-z) \qquad \text{as } N \to \infty.$$

Since $\mathbb{E}\mathcal{N}(I) = \lambda_l(I)\exp(-z)$ by the definition of $\mathcal{N}$ we have proved (2.24). To verify (2.25) note

$$\mathbb{P}\big(\mathcal{N}_N(I) = 0\big) = \mathbb{P}\big(M(NI \cap B_N^l) \le g(0)\{\log|B_N^l| + z\}\big)$$
$$= \mathbb{P}\big(M(NI \cap B_N^l) \le g(0)\{\log|NI \cap B_N^l| + z'\}\big),$$

where $z' = \log \frac{|B_N^l|}{|NI \cap B_N^l|} + z$. So applying Theorem 0.1 we get that for all $N \ge 1$,

$$\left|\mathbb{P}\big(\mathcal{N}_N(I) = 0\big) - \exp\left(-\frac{|NI \cap B_N^l|}{|B_N^l|}\exp(-z)\right)\right| \le c|NI \cap B_N^l|^{-c_1}.$$

By taking the limit $N \to \infty$ and using that $\frac{|NI \cap B_N^l|}{|B_N^l|} \to \lambda_l(I)$ we get

$$\mathbb{P}\big(\mathcal{N}_N(I) = 0\big) \to \exp(-\lambda_l(I)\exp(-z)).$$

But $\mathbb{P}(\mathcal{N}(I) = 0) = \exp(-\lambda_l(I)\exp(-z))$, so (2.25) follows. □

Corollary 0.2 has the following interesting implication.

PROPOSITION 2.8. *Let the random vector* $(X_1, X_2, \dots, X_{N^l})$ *be the sites of* B_N^l *ordered by the level at which they are covered* (*where we use*, *e.g.*, *the lexicographic order when several sites are covered at the same level*) *so that*

$$M(B_N^l) = U_{X_1} \ge U_{X_2} \ge \cdots \ge U_{X_{N^l}}.$$

Then for all $k \ge 2$,

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \mathbb{P}(\exists 1 \le i < j \le k \text{ such that } |X_i - X_j| \le \delta N) = 0, \tag{2.26}$$

or in other words, *the last sites of* B_N^l *to be covered by the random interlacements are separated*, *at typical distance of order* N.

PROOF. Fix a $\delta > 0$ and let $f : \mathbb{R}^d \to [0, \delta^{-1}]$ be a continuous function such that $f(x) = \delta^{-1}$ when $|x| \le \delta$ and $f(x) = 0$ when $|x| \ge 2\delta$. Consider the sum $\sum_{x,y \in \mathcal{N}_N, x \ne y} f(x - y) = \mathcal{N}_N \otimes \mathcal{N}_N(f(\cdot - \cdot)) - f(0)\mathcal{N}_N([0,1]^d)$ [recall that $\mathcal{N}_N = \mathcal{N}_N^{l,z}$ and $\mathcal{N} = \mathcal{N}^{l,z}$ depend on l and z and also the remark above (1.7) about the notation $x \in \mathcal{N}_N$]. We have that $\mathcal{N}_N$ tends weakly to $\mathcal{N}$, so the product $\mathcal{N}_N \otimes \mathcal{N}_N$ tends weakly to $\mathcal{N} \otimes \mathcal{N}$, so that for all $z \in \mathbb{R}$,

$$\begin{aligned}\lim_{N\to\infty} \mathbb{E}\Bigg[\sum_{x,y\in\mathcal{N}_N, x\neq y} f(x-y)\Bigg] &= \mathbb{E}\big[\mathcal{N}\otimes\mathcal{N}\big(f(\cdot-\cdot)\big) - f(0)\mathcal{N}([0,1]^d)\big] \\ &= \mathbb{E}\Bigg[\sum_{x,y\in\mathcal{N}, x\neq y} f(x-y)\Bigg].\end{aligned} \tag{2.27}$$

Let $\tilde{\mathcal{N}}$ be a homogeneous Poisson point process on $\mathbb{R}^l \times \{0\}^{d-l}$ (identified with $\mathbb{R}^l$) with intensity measure $\exp(-z)\lambda_l$. Recall that the Palm measure of $\tilde{\mathcal{N}}$ (viewed as a point process on $\mathbb{R}^l$) is simply $\exp(-z)$ times the law of $\tilde{\mathcal{N}} + \delta_{\{0\}}$ ([11], Chapter 2, Exercise 3). So by the definition of the Palm measure ([11], Chapter 2, Theorem II.4) we get

$$\begin{aligned}\mathbb{E}\Bigg[\sum_{x,y\in\tilde{\mathcal{N}}} 1_{\{x\in[0,1]^l, y\neq x\}} f(x-y)\Bigg] &= \exp(-z)\int_{[0,1]^l} \mathbb{E}[\tilde{\mathcal{N}}(f)]\,dx \\ &= \exp(-2z)\int_{\mathbb{R}^l} f(x)\,dx \\ &\le c(z)\delta^{l-1}.\end{aligned}$$

The left-hand side of the above equality is an upper bound for the right-hand side of (2.27). So for any $z \in \mathbb{R}$ we find that

$$\begin{aligned}&\lim_{\delta\to 0}\limsup_{N\to\infty} \mathbb{P}\big(\exists x \neq y \text{ in } \operatorname{Supp}(\mathcal{N}_N) \text{ such that } |x-y| \le \delta\big) \\ &\quad\le \lim_{\delta\to 0}\limsup_{N\to\infty} \mathbb{P}\Bigg(\sum_{x,y\in\mathcal{N}_N, x\neq y} f(x-y) \ge \delta^{-1}\Bigg) \\ &\quad\le \lim_{\delta\to 0} \delta \limsup_{N\to\infty} \mathbb{E}\Bigg(\sum_{x,y\in\mathcal{N}_N, x\neq y} f(x-y)\Bigg) = 0.\end{aligned}$$

Finally, we have the inequality

$$\begin{aligned}&\mathbb{P}(\exists 1 \le i < j \le k \text{ such that } |X_i - X_j| \le \delta N) \\ &\quad\le \mathbb{P}\big(\exists x \neq y \text{ in } \operatorname{Supp}(\mathcal{N}_N) \text{ such that } |x-y| \le \delta\big) + \mathbb{P}\big(\mathcal{N}_N([0,1]^d) < k\big).\end{aligned}$$

So taking, in order, the limits $N \to \infty$, $\delta \to 0$ and $z \to -\infty$ (and noting that $\lim_{z\to-\infty} \mathbb{P}(\mathcal{N}([0,1]^d) < k) = 0$) we get (2.26). □

We finish with a remark about the possible applicability of our results to the study of random walk cover times, a comment about the connection between the uncovered set A_ε and the uncovered set in the discrete torus and an open question about whether our results can be generalized.

REMARK 2.9. (1) Using our results and the known connection between random interlacements and simple random walk in the discrete cylinder (see [15, 17]) it should be possible to determine the finer asymptotic behavior of the cover time of the cylinder's zero level $(\mathbb{Z}/N\mathbb{Z})^{d-1} \times \{0\}, d \geq 3$, by random walk. Present knowledge states that the cover time is asymptotic to $N^{2d(1+o(1))}$ (see [3], Theorem 1).

(2) As already explained in the Introduction in the paragraph after the statement of Theorem 0.1, it is tempting to use the coupling result from [20] together with Theorem 0.1 to devise a proof of the conjecture that $\frac{C_N}{g(0)N^d} - \log N^d$ tends in law to the Gumbel distribution, where C_N denotes the cover time of the discrete torus of side length N and dimension $d \geq 3$.

(3) In this note the uncovered set A_ε is studied as one step in proving fine results about the covering of sets by random interlacements. The corresponding uncovered set in the torus (cf. Remark 2.7) has been studied for its own sake. Further illustrating the connection between random interlacements and random walk in the torus, A_ε and the uncovered set in the torus share some properties. Other than the agreement of the variance of the cardinality of the uncovered sets mentioned in Remark 2.7, [9] also shows that in the torus the uncovered set is (in a certain sense) well separated ([9], Lemma 6.4), a result similar in spirit to our Lemma 2.6.

(4) Random interlacements can be constructed for any infinite graph on which simple random walk is transient (see [19]). It is an open question whether a result like Theorem 0.1 can be proved for random interlacements on more general graphs. It seems plausible that on a transient graph G such that Green's function decays "fast enough" and such that $a = \text{cap}(\{x\})$ is independent of $x \in G$ one can use the same method to prove that for sequences of finite sets $A \subset G$

$$\frac{M(A)}{a^{-1}} - \log|A| \overset{\text{law}}{\to} \text{Gumbel distribution, as } |A| \to \infty,$$

where $M(A)$ is the cover level of A.

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