The Opinion Game: Stock price evolution from microscopic market modelling

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We propose a class of Markovian agent based models for the time evolution of a share price in an interactive market. The models rely on a microscopic description of a market of buyers and sellers who change their opinion about the stock value in a stochastic way. The actual price is determined in realistic way by matching (clearing) offers until no further transactions can be performed. Some analytic results for simple special cases are presented. We also propose basic interaction mechanisms and show in simulations that these already reproduce certain particular features of prices in real stock markets.

Keywords: Stock prices, financial markets, statistical mechanics, stochastic dynamics

1. Introduction

The financial markets constitute an intriguing and complex system that has not failed to attract mathematicians and scientists from other fields for a long time. Only rather recently, however, has mathematical finance, and more specifically the theory of derivatives on the stock markets become a major field of mathematics and one of the major sources of inspiration for probability theory in general and stochastic analysis in particular.

One of the crucial issues in the financial mathematics is the modelling of prices of

commodities (stocks, currencies *etc.*) with help of stochastic processes. This theory originated with the thesis of Bachelier [1], who argued that the price process was a result of many small random actions of market participants and that this should lead to the price performing, effectively, a (geometric) Brownian motion. This basic model was the original basis of the tremendously successful theory of Black, Scholes [4], and Mertens [22] for the pricing of options. Extensions of this theory still make up the main stream of modern financial mathematics (for good textbook exposition see, *e.g.,* [18,15]).

On the other hand, it has become evident from financial data, that real life price processes have statistical features that are incompatible with geometric Brownian motion. This has led to a wealth of work attempting to improve Bachelier's modelling ansatz and to derive more realistic models that should reflect the observed "stylised facts" of real price processes. The basic premise of most models is, in the words of H. Föllmer [14], that "As prices are generated by the demand of agents who are active on the financial market for the given asset, a [...] model [...] should be explained in terms of the interaction of these agents."

We want to distinguish two strategies that are followed today in agent based modelling of financial markets. One consists of introducing a behavioural model of a market consisting of a certain number of agents that typically are given a rather small possible states respective to the market, typically of the type "buy", "sell", or "hold". Agents at each time step may change their state according to deterministic or stochastic rules that are supposed to reflect more or less rational behaviours of real world traders. Popular examples of such models are the "minority game" (MG) and its more or less sophisticated versions; for reviews, see Jeffries and Johnson [17], Bouchaud and Giardina [16], and references therein. In such models the price of the traded commodity is usually obtained through an empirical rule that links the market imbalance at a given time to the increment of the price.

Another approach to agent based modelling is concentrated more on the actual price formation process and replaces the empirical mechanism alluded to above by the actual mechanism of double auctioning that takes place at a stock exchange. This requires to model the behaviour of agents in more detail. An agent does not just want to buy, sell, or hold, but actually must place an order to buy or sell a certain number of units of a given stock at a set price (limit order) or at any price (market order). The set of all these orders placed at a stock exchange is called the "order book". The price of the commodity traded is then obtained by matching executable orders (following some set rules that may depend on the type of exchange), until the lowest price of sell orders is higher than the highest price of buy orders, fixing the "ask" and the "bid" price of the stock. A natural way of modelling the price process is thus to model the time evolution of this order book, which then in principle yields the price process as a derived quantity. This seems even more appealing since actual order book data are available and can be (and have been [26,6]) statistically analysed.

There are several ways of modelling the order book dynamics that have been proposed in the literature. They roughly fall into two classes: One possibility is to consider orders arriving with a given statistics at certain prices, staying there a certain amount of time, before disappearing, or being executed. The advantage of such models is that it is possible to obtain statistics of these phenomena from real data on stock exchanges. Thus the models can reflect the actual dynamics of a market. The disadvantage, from our point of view, is that the modelling level here is rather remote from the level of the decision making process that is entered on a purely statistical basis. This type of models seems to go back to Mendelson [21] and was further developed by Cohen *et al* [10], Domowitz and Wang [12], and Bollerslev *et al* [5]. More recently, Farmer *et al* [11,23] have studied variants of such models using dimensional analysis and have obtained a number of interesting analytical results.

Another type of model has been proposed by Bak *et al* [2] and further investigated by Eliezer *et al* [13,9], and Tang and Tian [25]. Here the price range is fixed, and orders arrive only at the upper or lower end of the range. Once an order is placed, it performs a random walk on the price range, until it leaves the range or encounters a matching order of opposite type, whereupon both orders disappear. The purpose of this model is to provide a microscopic model for the dynamics of the price front in terms of a simple but realistic stochastic model. The authors provide a wide range of numerical and (heuristic) analytical results. The model is not intended as a model for the global market or for long time scales. It is also not meant to model the whole market, but only the fraction of highly active professional traders. We will see that in the context of the models we propose, the model of Bak *et al* emerges in a natural way.

A general feature of all order book dynamics models seems to be that they attempt to model the actual order book, rather then the behaviour of all market participants. In this respect they differ from the other class of agent based models discussed above, there at each time the state of each agent it known. In order book models, one typically sees only the orders placed by agents, while we know nothing about them when they have no order placed. But the total volume of orders placed at any given time represents only a small fraction ($< 1\%$) of the total amount of existing stock (resp. demand). On the other hand, agents that have not placed orders will still have some opinion on the value of the stock, *i.e.,* may be willing to buy or sell at some price, which may happen to be so far from the current price that they do not bother to place an order. However, their opinion will be relevant: as the price changes, they will be prompted to place orders. (Zovko and Farmer [26] and Bouchaud [6] have studied the shape of the order books at the London resp. Paris stock markets far away from the current price and found interesting power low behaviour. In all cases the number of orders placed decreases rapidly (although not as rapidly as one might think) away from the current price, which naturally explains the (reasonable) reluctance of people to place orders that are unlikely to have a chance of being ever executed).

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In this paper we will formulate a class of models that combines the agent based approach with the order book dynamics. From the agent based point of view, we extend the state space of each agent to be the set of all possible opinions on the (logarithm of) value of the stock (*i.e.,* $\mathbb{R}$, resp. $\mathbb{Z}$), and we replace the empirical derivation of the price by a rigorous double auction mechanism of the order book models. From the order book point of view, we extend the real order book to a "virtual order book" that keeps track of the evolution the opinions on the value of all agents, whether they have placed an order or not. In principle, this should be a fully realistic description of the market, provided we could model the dynamics of the opinion changes correctly. On the modelling side, the main challenge is to find a reasonable model for this dynamics that is simple enough to be handled, and realistic enough to allow to recover key features of price processes. Mathematically, the main challenge is, as we will see, the derivation of properties of the price process as a derived quantity within the model.

The remainder of this article is organised as follows. In Section 2 we explain the basic principles of our modelling approach and illustrate them in some special cases. Further, we introduce a number of additional features that should be implemented to obtain more realistic models, and discuss some of the additional effects they produce. In Section 3 we study our model in two very simple cases. We analyse some of the phenomena that can already be observed in this simplest setup and relate them to certain features of real world data. In Section 4 we present our conclusions.

2. The model.

We want to consider the time evolution of a virtual order book or a "trading state" containing the opinions of each participating agent about the "value"[a] of the stock. The evolution is driven by the change in opinion of the agents and the action of the market maker.

A minimal model in which this idea can be implemented can be described as follows. We consider trading in one particular stock, and we assume that there are N "traders" and $M < N$ shares of the stock. We make the simplifying assumption that each trader can own at most one share. The state of *each* trader i is given by its opinion, $p_i \in \mathbb{R}$, resp. $\varepsilon\mathbb{Z}$ (the latter is the case we will mostly consider, ε corresponding to the tick-size), on the logarithm of the value of the stock, and by the number of shares, $n_i \in \{0, 1\}$, he owns (*i.e.,* whether this is a buying or selling opinion). This is to say, the trader i would be willing to sell his share at the price e^{p_i}, if he owns one ($n_i = 1$), respectively to buy a share at this price, if he does not

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[a]We will distinguish the notion of the value from that of the price. The *value* is what agents have an opinion about, while the *price* is determined by the market. The opinion on the value can be driven by fundamental considerations (*e.g.,* earning or dividend expectations, typically coming from outside information), or speculative considerations (*e.g.,* predictions based on partial knowledge on the current state of the opinions of other traders), or both.

own one ($n_i = 0$). We say that a trading state is *stable*, if the M traders having the M highest opinions p_i all own a share[b]. This means, in particular, that, in a stable state, one can infer the set of owners of shares from the knowledge of the state of opinions, $\mathbf{p} = (p_1, \ldots, p_N)$. Thus, a stable trading state is completely determined by the set of N values, p_i, and we will in the sequel identify stable trading states with the vector $\mathbf{p}$. As we will normally only work with stable trading states, we suppress the qualifier stable when no confusion can arise.

Given a stable trading state $\mathbf{p}$ we denote by $\hat{\mathbf{p}} = (\hat{p}_1, \ldots, \hat{p}_N)$ its order statistics, that is, $\hat{p}_i = p_{\pi_i}$, for a permutation, $\pi \equiv \pi(\mathbf{p})$, of the set of N elements, such that $\hat{p}_1 \leq \cdots \leq \hat{p}_N$. Then, the number of shares owned by traders, $\mathbf{n}(\mathbf{p}) = (n_1(\mathbf{p}), \ldots, n_N(\mathbf{p}))$, satisfies

$$n_i(\mathbf{p}) = \begin{cases} 0, & \text{if } \pi_i(\mathbf{p}) \leq N - M, \\ 1, & \text{if } \pi_i(\mathbf{p}) \geq N - M + 1. \end{cases} \tag{2.1}$$

With a trading state, $\mathbf{p}$, we associate the *ask price*

$$p^a \equiv p^a(\mathbf{p}) = \hat{p}_{N-M+1}\,, \tag{2.2}$$

and the *bid price*

$$p^b \equiv p^b(\mathbf{p}) = \hat{p}_{N-M}\,. \tag{2.3}$$

For convenience we will refer to the mid-price, $\frac{1}{2}(p^a + p^b)$, as the *current price*.

Any dynamics $\mathbf{p}(t)$, defined on the trading state, induces a dynamics of $\hat{\mathbf{p}}(t)$, and in particular of the pair $(p^a(t), p^b(t))$.

Our basic assumption is that the trading state, $\mathbf{p}$, evolves in time as a (usually time-inhomogeneous) Markov chain[c] $\mathbf{p}(t)$ with state space $(\varepsilon\mathbb{Z})^N$. We will further assume that time is discrete (this is inessential but more convenient for computer simulations) and that the updating proceeds asynchronously, *i.e.,* typically at a given instant only a single opinion changes. In fact, in general the dynamics can be described as follows:

- at time $t + 1$, a trader $i \in \{1, \ldots, N\}$ is selected according to a probability distribution $g(\,\cdot\,; \mathbf{p}(t), t)$, depending in principle on $\mathbf{p}(t)$ and t;
- trader i decides to change its opinion to $p_i(t) + d$, according to a probability distribution $f(\,\cdot\,; \mathbf{p}(t), i, t)$ on $\varepsilon\mathbb{Z}$;
- if the trading state $\mathbf{p}'(t) \equiv (p_1(t), \ldots, p_i(t) + d, \ldots, p_N(t))$, is stable, then $\mathbf{p}(t + 1) = \mathbf{p}'(t)$;
- otherwise, a *transaction* move takes place, in which the trader i exchanges its ownership state $n_i(t)$ with the highest bidder, resp. lowest asker j. Moreover, the opinions of both i and j are updated according to some probability

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[b]One may prefer not to think of our "traders" as real traders, but just consider the numbers p_i abstractly as opinions attached to single shares, or demands of single shares. This is not unrealistic, as a "real" trader may at a given moment of time have a strategy to sell and buy various amounts of shares at different prices.
[c]We will discuss the Markovian assumption later.

law away from the bid, resp. ask price in such a way that a new stable market state, $\mathbf{p}(t + 1)$, is achieved.

This dynamics can be considered as an interacting particle system with the transaction mechanism, which in the context of particle systems resembles to some extent an interactive phase boundary.

To implement this general scheme one must make reasonable choices for the different probability distributions mentioned above, as these should mimic the behaviour of real agents. Our hope is that already reasonably simple choices, based on elementary principles, allow to recover some of the particularities of real price processes.

Before we discuss specific implementations of this general setup, a few remarks concerning some features that may appear offending are in place.

The first is doubtlessly the assumption that the process is Markovian. This appears unnatural because the most commonly available information on a stock is the history of its price, the "chart", and most serious traders will take this information at least partly into account when evaluating a stock, with some making it the main basis for any decision. Certainly one could retain such information and formulate a non-Markovian model. However, if one starts to think about this, one soon finds that it is very difficult to formulate reasonable transition rules on the basis of the price history. On a more fundamental level, one will also come to the conclusion that the analysis of the history of the share price is in fact performed in order to obtain information of the *current opinion of the traders concerning the value of the stock* with the hope of inferring information on the future development of the price. For instance, if one knew that there are many people willing to sell shares at a price not much higher than p^a, one knows that it will be difficult for the price to break through this level (this is known as a "resistance" by chart-analysts and usually inferred from past failures to break through such a level). Therefore, instead of devising rules based on past price history, we may simply assume that the market participants have some access to the prevailing current opinions, obtained through various sources (chart analysis, rumours, newspaper articles, *etc.*) and take them into account when changing their own estimates.

The second irritating point is that money does not appear in our model except in the form of opinions about values. In particular, we do not keep track of the cash-flow of a given investor (that is to say we do not care whether a given investor wins or looses money). There are various reasons to justify this. First, we consider that the market participants do not invest a substantial fraction of their assets in this one stock, so that shortage of cash will not prevent anyone to buy, if she deems opportune to do so (in the worst case money can be obtained through credits). Then, money is not conserved, but the total value of the stock can inflate as long as there is enough confidence. Also, we do not keep track of the objective success of a trader, because we do not know how this will eventually influence her decisions. While a given trader may follow her personal strategy with the hope of making

profits, we cannot be sure that these strategies will succeed. What is important and what is built into our model, however, is the fact that any trader[d] will have the subjective impression to make a profit at any transaction.[e] Thus we feel that opinions about values are the correct variables to describe such a market rather than the actual flow of capital, at least at the level of a simple model.

The above setting suggests a rather general and flexible class of models of a stock market. Its main feature is that it describes the time evolution of a share price as the result of an interacting random process that reflects the change of the opinions of individual traders concerning the value of the stock. Even when this last process is modelled as a Markov process, the resulting price process $\big(p^a(t), p^b(t)\big)$ will in general not be a Markov process.

2.1. *Implementation of the model*

We will now discuss an implementation of the model introduced above that is rather simple but contains already enough elements to make its behaviour reasonably complex and its mathematical analysis rather hard.

Basically, one expects that the behaviour of a trader is influenced by the information received from the market as well as external influences. Moreover, the opinion held by a trader with respect to the current price should somehow reflect some his intrinsic psychological characteristics. Moreover, when a transaction is performed, both traders should probably feel that they have made a good deal; in other words, the same person at the same time would buy only at a lower price than she would sell, and vice versa.

This leads to the following set up:

- The derived process of the price (ask and bid) is the most easily accessible piece of information about the trading state of the market for any trader. It is natural that the updating rules should take the current value of this process into account. The simplest and natural modification is to introduce a bias towards the actual price $p(t) = \big(p^a(t) + p^b(t)\big)/2$ into the distribution of opinion change.
- Traders whose opinion is far from the current price are likely not to pay much attention to what is happening on the market. It is reasonable to assume that they update their opinion less frequently. This feature can be included by reducing the overall transition rates as a function of $p_i(t) - p(t)$.
- Finally, it is natural to assume that the traders performing a transaction, that is exchange of a share, will update their opinions according to some

[d]We could enlarge the model to incorporate a small fraction of traders which do not act according to common sense, or against their own convictions (*e.g.,* traders that have bought their stock on credit and the latter are executed by their creditors on falling prices. It may be interesting to consider the effect of that in the context of market crashes).
[e]Which also implies that this subjective opinion must be wrong at least for one of the traders involved, but this seems to reflect reality.

special rules reflecting the fact that someone buying or selling a share at a given price believes that she has struck a favourable deal, *i.e.,* they attribute a higher value to the share then what they paid, respectively a lower one then what they got.

In the following we suggest some concrete framework in which these features are implemented. We describe the construction of the process algorithmically.

Change of opinion: At any time step we first select a trader at random. We will allow this probability to depend on p_i, and we choose trader i with a probability proportional to $h(p_i(t) - p(t))$, with some function $h(x) \geq 0$. We recall that we have defined the "current price" via $p(t) = \big(p^a(t) + p^b(t)\big)/2$. That means that we choose the trader i with probability

$$g(i; \boldsymbol{p}(t), t) = h(p_i(t) - p(t))\big/Z_g(\boldsymbol{p}(t)), \qquad (2.4)$$

where the normalisation $Z_g(\boldsymbol{p}(t))$ is defined by $\sum_{i=1}^{N} g(i; \boldsymbol{p}(t), t) = 1$. To reflect the slower updating of opinions of traders not in the vicinity of the price, the function $h(x)$ must decrease with $|x|$. A possible choice for this functions is

$$h(x) = 1/(1 + |x|)^{\alpha}, \qquad \alpha > 0. \qquad (2.5)$$

We have found in simulations that $\alpha \sim 1.5$ seems to be reasonable, however we do not have any particular reason why this value should be preferred.

Once a trader has been selected, she changes her opinion from p_i to $p_i' = p_i + d$ with probability equal to $f(d; \boldsymbol{p}(t), i, t)$. In typical cases, f depends on $\boldsymbol{p}(t)$ through $p^a(t)$ and $p^b(t)$ only. In simulations we used

$$f(d; \boldsymbol{p}(t), i, t) = \frac{1}{2l+1}\big[(\delta_{p_i, p(t)})^d e^{V(p_i) - V(p_i + d)} \wedge 1\big] \qquad \text{for } d \in ([-\varepsilon l, \varepsilon l] \cap \varepsilon \mathbb{Z}) \setminus \{0\}, \qquad (2.6)$$

and $f(0; \boldsymbol{p}(t), i, t) = 1 - \sum_{0 < |k| \leq l} f(\varepsilon k; \boldsymbol{p}(t), i, t)$. That means that d is first picked uniformly in the set $[-\varepsilon l, \varepsilon l] \cap \varepsilon \mathbb{Z}$, and then accepted (resp. refused) as a new opinion according to a bias $\delta_{p_i, p(t)}$ and some external potential V. As we want that opinion changes are biased towards the price, we typically set $\delta_{p_i, p(t)} = \delta_s < 1$ for sellers (*i.e.,* $p_i > p(t)$), and $\delta_{p_i, p(t)} = \delta_b > 1$ for buyers ($p_i < p(t)$). The role of the external potential V will be discussed later, typically we use $V \equiv 0$.

Once p_i' is chosen, we check whether $p_i' < p^a$, if $n_i = 0$, resp. whether $p_i' > p^b$, if $n_i = 1$. If this is the case, we set $p_i(t+1) = p_i'$, and $p_j(t+1) = p_j(t)$ for all $j \neq i$, and continue to the next time step. Otherwise, we perform

Transaction: Assume first that $n_i(t) = 0$ and $p_i' \geq p^a(t)$. This means that the buyer i has decided to buy at the current asked price. Since by definition there is at least one seller who asks only the price $p^a(t)$, we select from all these one at random with equal probabilities. Call this trader j. Then we set

$$p_i(t+1) = p^a(t) + g, \qquad p_j(t+1) = p^a(t) - g, \qquad (2.7)$$

where $g \in \varepsilon \mathbb{Z}$ is a fixed or possibly random positive number. Similarly, if $n_i(t) = 1$ and $p_i' \leq p^b(t)$, the seller i sells to one of the buyers that offer the price $p^b(t)$, and

we set

$$p_i(t+1) = p^b(t) - g, \qquad p_j(t+1) = p^b(t) + g. \tag{2.8}$$

The final state in all cases represents a new stable trading state and the process continues. Note that g should be at least as large as to cover the transaction cost. In simulations we used g being uniformly distributed on the set $\{\varepsilon r_1, \varepsilon r_2\} \cap \varepsilon \mathbb{Z}$ for some integers $0 \leq r_1 \leq r_2 < \infty$.

All features introduced in the previous paragraphs make the mathematical analysis of the model much more difficult, but they introduce some interesting effects that are somewhat similar to phenomena observed in real markets. Let us briefly comment on these.

Without any drift (*i.e.,* setting $\delta_{p_i, p(t)} \equiv 1$ in the above algorithm) and in the absence of any confining potential all opinions would in the long run spread over all real numbers and the individual opinions would get arbitrarily far from each other. This is avoided by the mechanism of the *attraction to the current price.*

Slowing down the jump rate of the particles far away from the current price naturally introduces long time memory effects that lead to special features in the distribution of the price process. One of them is possible existence of resistances: if there is a large population of traders in a vicinity of some price p far from (say above) the current price $p(t)$, then this population tends to persist for a long time, unless the current price approaches this value. If that happens, *e.g.,* due to the presence of an upward drift, one observes a slow-down of the upward movement of the price when it approaches this value. The market has a resistance against increase of the price through this value, reflected in multiple returns to essentially the same extremal values for a long period.

A further effect of the slowing down is the tendency of the creation of "bubbles" in the presence of strong drifts. In this case one observes a fast motion of the price accompanied by a depletion of the population below this price. Effectively, a few (buying) traders move with the drift, while most are left behind. Such a situation can lead to a crash, if at some moment the drift is removed (due to external effects). Both effects of slowing down were played out in simulations shown in Figure 1. Note that after the crash on the second figure there was a strong increase of volatility.

The effect of pushing the opinions from the current price after the *transaction* depletes the vicinity of the price and therefore increases the volatility. This goes in the opposite direction to the attraction towards the price and the interplay of both effects can lead to a non-trivial quasi-equilibrium state; the resulting price fluctuations will be studied in the forthcoming paper [8].

2.2. *Coupling with external influence*

Price processes obtained from models of the above type tend to show rather boring long-time behaviour. The main reason for this is that the attraction towards the price effectively produces a potential valley around the price and all traders prefer to
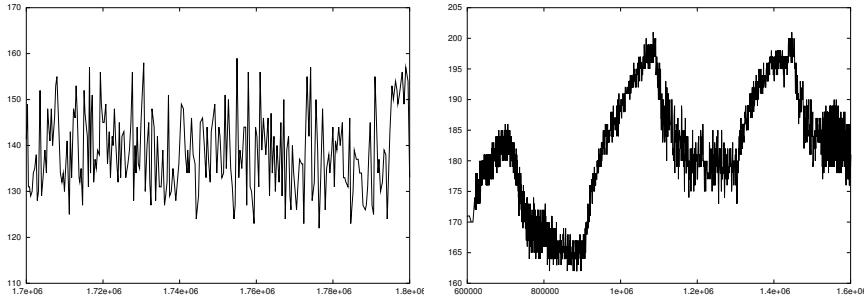
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Fig. 1. Trading in between resistances(left), a sequence of "bubbles"(right)

stay in this valley. The density of traders around the price becomes quite large and, consequently, the price has huge difficulties to move considerably, also the volatility stays constant in time. Without the presence of the drift towards the price, the situation is less dramatic, however it was already observed that models of this type manifest some features that are not observed in real markets. In particular, the value of the Hurst exponent obtained (~ 0.25) is too low compared to values found in real data.

Since, in a usual situation, time-homogeneous Markov chains have a tendency to converge to equilibrium, we believe that some inhomogeneity should be introduced into the model to resolve this problem. In the real market such inhomogeneity is present due to the arrival of the external information which makes the traders more optimistic, or pessimistic about the value of the stock.

We propose the following simple model for the arrival of the external information. Let $\eta_0 = 0$, and let $(\eta_i, i \in \mathbb{N})$ be a homogeneous Poisson point process on $\mathbb{N}$ with intensity ω. Further let s_i be a sequence of i.i.d. random variables whose common distribution is centred around 0. For time step t, $\eta_i \le t < \eta_{i+1}$ we define the external drift by

$$\delta_{\text{ext}}(t) = \exp(s_i). \tag{2.9}$$

We then use δ_{ext} to modify the constant values δ_b, δ_s of $\delta_{p(i),p(t)}$ (which were introduced in the paragraph after (2.6)) and set

$$\delta_{p(i),p(t)} = \begin{cases} \delta_b \cdot \delta_{\text{ext}}(t) & \text{for buyers,} \\ \delta_s \cdot \delta_{\text{ext}}(t) & \text{for sellers.} \end{cases} \tag{2.10}$$

The introduction of the external process brings some interesting features into the behaviour of the model. As an illustration we decided to observe the volatility clustering. We have run two simulations of the model, once with external influence and once without it. All other parameters stay fixed.[f] On Figure 2 we show the

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[f] $N = 2000$, $M = 1000$, $\varepsilon = 1$, $h(x) = (1+|x|)^{1.5}$, $\delta_b = 1/\delta_s = \exp(0.1)$, $l = 4$, and g being uniform

graphs of the empirical volatility correlation function, $\langle(\sigma^2(t+\tau) - \sigma^2(t))^2\rangle$. The volatility $\sigma^2(t)$ was computed as the mean of last 100 square returns $r(t)^2, \ldots, r(t-99)^2$, where the returns were computed as the change of the price in each 100 steps of the algorithm, $r(t) = p(100t) - p(100(t-1))$. We have found that the
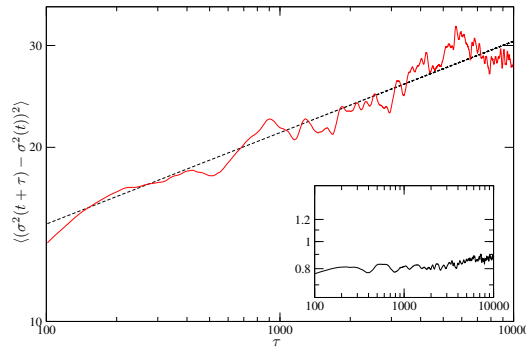

Fig. 2. Volatility clustering in model with external influence. In the inset the same quantity for the original model

correlation function for the process with the external influence increases like the power, τ^ν, with $\nu \sim 0.15$. Remark that the same function computed from the real data ([7], Section 7.2.2) shows the power-law scaling with $\nu \sim 0.22$. Observe also that the power law behaviour remains valid up to $\tau \sim 10^4$ which corresponds to 10^6 time steps of the algorithm. This time is much longer than the mean number of steps between η_i, η_{i+1}, which was 2000. For the comparison, the graph of the same function for the model without the external influence is presented in the inset. In this case only a very slow increase is observed that is in the range of measurement error.

3. Examples

To get any rigorous result for the complete model presented in the previous section is out of reach at this stage. However, there are simple special cases that allow mathematical analysis. In this section we present two of them together with some rigorous results.

3.1. *Ideal gas approximation*

Obviously the simplest model for the dynamics of the trading state $\mathbf{p}(t)$ is to choose it as a collection of independent identically distributed one-dimensional Markov

in [5, 20]. Parameters of external process: $\omega = 1/2000$, and $s_i = \varepsilon_i s_i'$, where ε_i are Bernoulli with $\mathbb{P}[\varepsilon_i = \pm 1] = .5$ and s_i' are Exponential with mean 0.12.

processes ("random walks") $p_i(t)$. This corresponds to the ideal gas approximation in statistical mechanics. In this case, the price process is simply obtained from the order statistics of independent processes and asymptotic results for M and/or N large can be obtained rather easily (recall that M denotes the number of traded shares and N the total number of traders). While this model is somewhat simplistic, some rather interesting phenomena can already be modelled in this context, as we will explain now.

We may be interested in a situation where a macro-economic model may predict several stable (respectively metastable) values of the stock price, realised as the minima of some utility function[g] V. A trader knowing this will tend to adjust her opinion towards one of the stable prices, her current state reflecting to what degree he believes in one or the other scenario.

In such a situation it seems not unreasonable to model the process of a single trader as a one-dimensional simple random walk in the price space $\varepsilon\mathbb{Z}$ with drift obtained from a potential function V. That means that in our algorithmic description we first choose uniformly a trader i, this trader then makes a step of size ε right, resp. left with probabilities

$$f(\pm\varepsilon; \boldsymbol{p}(t), i, t) = \frac{1}{2}\big(\exp\{V(p_i) - V(p_i \pm \varepsilon)\} \wedge 1\big), \tag{3.1}$$

otherwise he does not move.

Let us consider the situation when there are two (meta)stable values of the price, q_1 and q_2, $q_1 < q_2$, *i.e.,* the situation where the potential V has two minima (wells) at q_1 and q_2. If the potential is strong, resp. ε is small, an individual trader would typically spent long periods of time near one of the favoured values q_1 or q_2.

Let w_1 be the escape rate from the well q_1 to the well q_2 and let w_2 be the escape rate in the opposite direction. It is well known that these escape rates are exponentially small, $w_i \asymp \exp\big\{-2(V^* - V_{q_i})/\varepsilon\big\}$, if ε is small. Here V^* denotes the value of the maximum of V on the interval $[q_1, q_2]$. Denote by A_t the number of traders in the right well q_2 at time $t \geq 0$. Suppose that A_0 is much larger than M implying that the actual price at the initial moment is situated near q_2. We are interested in describing the moment of the "crash", *i.e.,* when the price moves from the right well q_2 to the left well q_1.

Then we can approximate the individual processes $p_i(t)$ by a two-state Markov chains with state space $\{q_1, q_2\}$ and transition rates w_1 and w_2. In this approximation we can compute the normalised expected number $a_t \equiv \mathbf{E}A_t/N$ of traders in state q_2 at time t as the solution to the ordinary differential equation

$$\frac{d}{dt}a_t = -w_2 a_t + w_1(1 - a_t). \tag{3.2}$$

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[g]This will occur naturally in non-linear models where the effect of the price of a stock on the global economy is taken into account; e.g. high stock prices make sock owners feel wealthy and more given to spending, which boosts the overall economy, of which the company and in turn its stock price benefits, while a low stock price may trigger the opposite effect.

We get

$$a_t = \frac{w_1}{w_2 + w_1} + \Big(a_0 - \frac{w_1}{w_2 + w_1}\Big) \exp\{-(w_2 + w_1)t\} \tag{3.3}$$

and the crash time T_c can be defined as t such that $a_t = M/N$,

$$T_c = \frac{1}{w_2 + w_1} \log \frac{a_0(w_2 + w_1) - w_1}{\frac{M}{N}(w_2 + w_1) - w_1} \tag{3.4}$$

If the energy barrier $\Delta V \equiv V^* - V_{q_2}$ is large enough, the time for each single buyer to escape from the initial well is much larger than the relaxation time for the system of A_t particles in the right well, and thus it is natural to expect that the system will pass through the sequence of local equilibrium states corresponding to A_t independent random walkers. Using this observation, the evolution of the price can be described in terms of the $B_t \equiv A_t - M$'s order statistic in a system of A_t random variables whose distribution is approximately Gaussian with parameters q_2 and $\big(V''(q_2)\big)^{-1}$.

To do this, let $F(\cdot)$ denote the distribution function of an individual walker conditioned to stay near q_2 and define $q_B = q(B_t)$ as the solution to the equation

$$\frac{M}{A_t} = 1 - F(q_B) \approx 1 - \Phi\big((q_B - q_2)\sqrt{V''(q_2)}\big), \tag{3.5}$$

where $\Phi(\cdot)$ is the distribution function of the standard Gaussian variable. Using the well-known asymptotics for the tail distribution of $\Phi(\cdot)$,

$$\frac{u}{u^2 + 1}\frac{e^{-u^2/2}}{\sqrt{2\pi}} \leq \frac{1}{\sqrt{2\pi}}\int_u^\infty e^{-x^2/2}\,dx \equiv 1 - \Phi(u) \leq \frac{1}{u}\frac{e^{-u^2/2}}{\sqrt{2\pi}}, \qquad u \to \infty, \tag{3.6}$$

we immediately get

$$q_B \approx q_2 - \Big(\frac{2\log(A_t/B_t)}{V''(q_2)}\Big)^{1/2} + \frac{\log(4\pi\log(A_t/B_t))}{\sqrt{8V''(q_2)\log(A_t/B_t)}}, \qquad B_t/M \to 0. \tag{3.7}$$

Consequently, in the limit of large A_t, M such that $\rho = 1 - M/A_t$ is fixed we have

$$q_B \approx q_2 - \Big(\frac{2\log(\rho^{-1})}{V''(q_2)}\Big)^{1/2}. \tag{3.8}$$

In this regime of "increasing ranks", the fluctuations of the B_t's order statistic have Gaussian behaviour and their scaling can be derived from [20, Theorem 2.5.2]. To do this, consider a small enough y such that

$$A_t\, F(q_B + y)\big(1 - F(q_B + y)\big) \tag{3.9}$$

is large for large B_t and such that

$$\frac{M - A_t\big(1 - F(q_B + y)\big)}{(B_t\, M/A_t)^{1/2}} \to \tau \tag{3.10}$$

as A_t, $M = (1 - \rho)A_t$, and $B_t = \rho A_t$ are getting large. In view of the definition of q_B the left-hand side of the expression above equals

$$\sqrt{\frac{B_t A_t}{M}} \left(\frac{F(q_B + y)}{F(q_B)} - 1 \right), \tag{3.11}$$

where the last ratio can be approximated by

$$\frac{q_2 - q_B}{q_2 - q_B - y} \frac{\exp\{-(q_B + y - q_2)^2 V''(q_2)/2\}}{\exp\{-(q_B - q_2)^2 V''(q_2)/2\}} \approx 1 + (q_2 - q_B)V''(q_2)\, y, \tag{3.12}$$

assuming that $(q_2 - q_B)^2 V''(q_2)$ is large enough. As a result, in the limit of large M and B_t, we have

$$\tau \approx \frac{\sqrt{\rho M}(q_2 - q_B)V''(q_2)}{1 - \rho}\, y \approx \frac{\sqrt{2\rho M V''(q_2) \log(\rho^{-1})}}{1 - \rho}\, y. \tag{3.13}$$

It remains to observe that Theorem 2.5.2 from [20] implies then that the ask price $p^a(t)$ satisfies

$$\mathbf{Pr}\left(p^a(t) \leq q_B + y\right) \to \Phi(\tau). \tag{3.14}$$

In other words, the price corresponding to a system of A_t such traders has mean q_B, shifted away from the well q_2 on a distance of order $\left\{ \log(\rho^{-1})/V''(q_2) \right\}^{1/2}$ and the variance of the price (*i.e.*, the volatility) diverges as $\left\{ M V''(q_2)\rho \log(\rho^{-1}) \right\}^{-1}$ in the limit of small $\rho = B_t/(M + B_t)$.

Finally, recalling the hydrodynamic description of B_t and the definition of T_c, we can also describe the time dependence of the mean $\mathbf{E}\rho$,

$$\mathbf{E}\rho = \mathbf{E}\rho_t \approx \frac{1 - \exp\{-(w_2 + w_1)(T_c - t)\}}{1 + w_1 \exp\{(w_2 + w_1)t\}/\big((w_2 + w_1)a_0 - w_1\big)}, \qquad t \leq T_c. \tag{3.15}$$

Figure 3(a) shows the simulated crash in double well potential in the ideal gas approximation. To illustrate the influence of different features discussed in Section 2, we present the typical pictures of a crash, if (b) attraction to the current price, and (c) slowdown of traders far from price are introduced. In all simulations the same potential function V having two local minima (one metastable and one stable) was used. All simulations start with the same initial condition where all traders are located near the metastable minimum. Notice the increased volatility in the slowdown case. The volatility increase before the crash is particularly marked.

3.2. $B + S \to \varnothing$ *model*

Another simple model that could be studied is so called $B + S \to \varnothing$ model. This model has been proposed as a model of the order book dynamics already in [2]. A lot of interest in this model comes also from chemistry, where it is used to describe the behaviour of the boundary between two diffusing, irreversibly reacting chemicals [3]. In this section we announce some rigorous results about this model. An interested reader can find their proofs in the forthcoming paper [8].
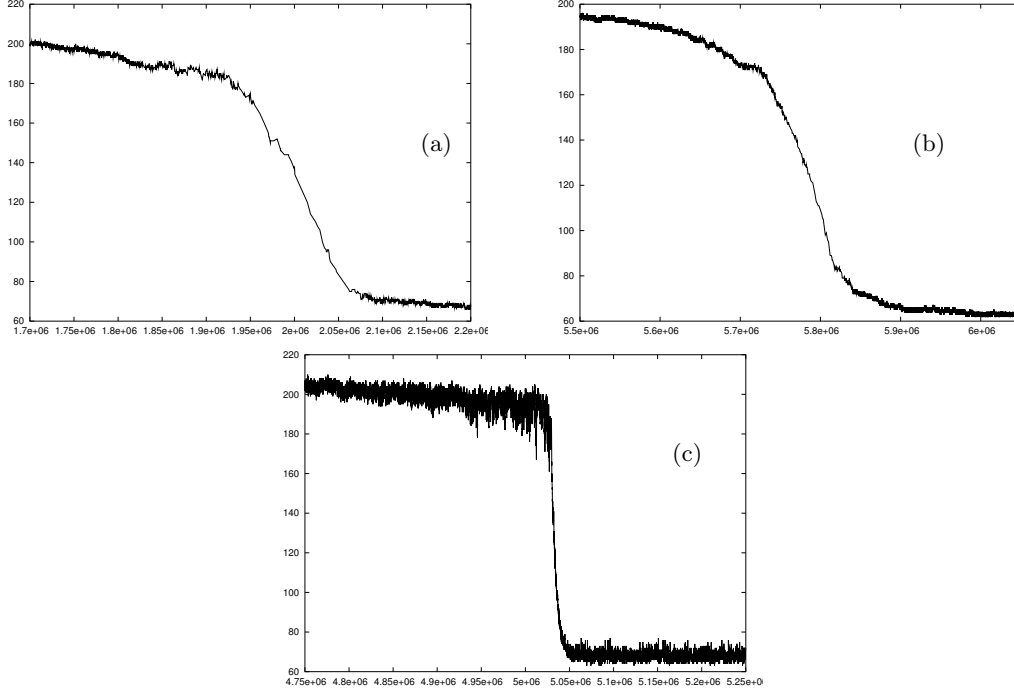
Fig. 3. Crash scenario

Let us first describe the model. The traders take opinions that belong to the set $\Lambda_L = \{-L, -L+1, \ldots, L\}$. The opinion of any trader perform a time-continuous simple random walk (possibly with a drift) in this set. If the opinion leaves Λ_L, then it disappears from the game. On the other hand, the sellers and buyers are injected at L, resp. $-L$ with rates ρ_+, and ρ_-. If a buyer meets a seller they interchange the stock and then both change their opinion in such a way that they leave Λ_L and are no longer considered.

Formally, we use η to denote a state of the model, $\eta = \{\eta(x)\}_{x \in \Lambda_L} \in \mathbb{Z}^{\Lambda_L}$. If $\eta(x) > 0$, then $\eta(x)$ sellers and no buyer have opinion x. Similarly, if $\eta(x) < 0$ then $|\eta(x)|$ buyers and no seller have opinion x. Since the opinions of all sellers must be higher than the opinions of the buyers, all possible configurations η are contained in a subset Ω_L of $\mathbb{Z}^{\Lambda_L}$ defined by

$$\Omega_L = \{\eta \in \mathbb{Z}^{\Lambda_L} : \eta(x) < 0 < \eta(y) \implies x < y\}. \qquad (3.16)$$

As we have already discussed it seems reasonable to introduce a drift toward the price into the dynamics. To this end we choose a function $\delta_L(x, \eta)$ with values in $(0, 1)$; given the state η an trader with opinion x then increases (decreases) her opinion with probability $(1 + \delta_L(x, \eta))/2$, resp. $(1 - \delta_L(x, \eta))/2$. In the simplest case treated later $\delta_L(x, \eta)$ will be function only of $\text{sign}(\eta(x))$, *i.e.*, the drift will be the

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same for all the buyers, resp. sellers. If $\delta_L(x, \eta) \equiv 0$ then there is no drift. Given L, $\rho_\pm$, and δ_L, the $B + S \to \varnothing$ dynamics is defined as a continuous-time Markov $\eta_t(x)$ chain on Ω_L with generator

$$(\mathcal{L}_L f)(\eta) = \sum_{x \in \Lambda_L} \frac{|\eta(x)|}{2} \big[\big(1 + \delta_L(x, \eta)\big) f(\eta^{x, x+1}) + \big(1 - \delta_L(x, \eta)\big) f(\eta^{x, x-1}) - 2f(\eta) \big]$$
$$+ \rho_+ \big(f(\eta^{L, +}) - f(\eta)\big) + \rho_- \big(f(\eta^{-L, -}) - f(\eta)\big) \tag{3.17}$$

for any bounded continuous function $f : \Omega_L \to \mathbb{R}$. Here, the configurations $\eta^{x, x \pm 1}$ and $\eta^{x, \pm}$ are defined by

$$\eta^{x, x \pm 1}(y) = \begin{cases} \eta(y) & \text{if } y \notin \{x, x \pm 1\}, \\ \eta(x \pm 1) + \text{sign}(\eta(x)) & \text{if } y = x \pm 1, \\ \eta(x) - \text{sign}(\eta(x)) & \text{if } y = x, \end{cases} \tag{3.18}$$

and

$$\eta^{x, \pm}(y) = \begin{cases} \eta(y) & \text{if } y \neq x, \\ \eta(x) \pm 1 & \text{if } y = x. \end{cases} \tag{3.19}$$

Note that due to the suitable definition of $\eta^{x, x \pm 1}$, the transaction is not represented visibly in the generator.

The reader has surely remarked that this model does not have all features that we have incorporated into our modelling framework. The two most offending differences are: the presence of the boundary connected with the necessity of the injection of the traders, and a very non-realistic definition of the behaviour of traders after the transaction. Both of these differences are so significant that it is difficult to see what is the relevance of the $B + S \to \varnothing$ model to the complete framework. We argue that the model can be regarded, to some extent, as an approximation of the dynamics in a small (moving) window around the price in the complete model. The injection at boundaries is then easy to explain as the arrival of traders from the outside of the window into B_L. Similarly, the disappearance of the traders after the transaction can be related to the change of opinion after the transaction in the original model. Indeed, as we have already argued, after the transaction the traders should take opinion that is not in the closest vicinity of the price. If the size of the window L is smaller than the typical distance between the price and the newly taken opinions of the trading traders after a transaction, then both traders disappear from the window. The quality of such approximation can be disputed and we do not want to enter in such discussion. In any case, the $B + S \to \varnothing$ is an interesting model and we find that it deserves a rigorous study.

Let us first present our results for $\delta \equiv 0$. In this case both buyers and sellers perform independent simple random walks. This property allows us to construct a coupling between two systems of *non-interacting* simple random walks. These two systems are defined by their generators that are the same as $\mathcal{L}_L$ with ρ_+, resp.

ρ_- being set to zero. The existence of such coupling allows to prove the following theorem.

Theorem 3.1.   *Assume that $\delta_L(x,\eta) \equiv 0$.*

   (i) The Markov chain η_t has an unique invariant measure, $\mu^L_{\rho_+,\rho_-}$, which satisfies

$$\mu^L_{\rho_+,\rho_-}(\eta(x)) = \frac{\rho_+ + \rho_-}{L+1}\, x + \rho_+ - \rho_-. \tag{3.20}$$

   (ii) There exists a measure, μ_ρ, on $\mathbb{Z}^{\mathbb{Z}}$ such that for any bounded, continuous, cylinder function $f : \mathbb{Z}^{\mathbb{Z}} \to \mathbb{R}$

$$\lim_{L\to\infty} \mu^L_{L\rho,L\rho}(f) = \mu_\rho(f). \tag{3.21}$$

This theorem implies that, if $\rho_+ = \rho_-$, then the properties of the equilibrium measure around the interface depend only on the *slope* of the mean profile $\rho = \rho_+/L$. It seems from simulations (see Figure 4) that in equilibrium, the fluctuation of the interface between two types of the traders are Gaussian, however we were not able to show it rigorously.
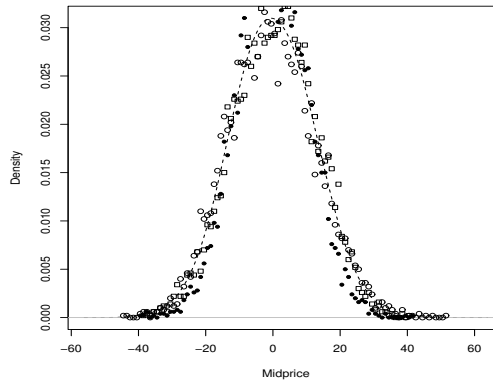

Fig. 4. Histograms of mid-price in drift free $B + S \to \varnothing$ model for different L with constant slope $\rho = 1/200$, ($\bullet\, L = 62$, $\circ\, L = 200$, $\square\, L = 620$)

   For financial application the study of the equilibrium properties of the dynamics is only partially important. The market is constantly put out of the equilibrium by various external impulses. There is therefore a need to study the out-of-equilibrium dynamical properties of the model. It is possible to get description of such properties using the formalism of *hydrodynamic limit* [24,19].

   This formalism allows to treat also the model with drift. As an example we study here the system with small but temporarily and spatially constant drift towards the price, $\delta_L(x,\eta) = \delta \operatorname{sign}(\eta(x))/L$.

To present the result we need some definitions. For any $\eta^L \in \Omega_L$ we define a measure on $[-1, 1]$

$$\pi_L(\eta^L)(dx) = \sum_{x=-L}^{L} \frac{\eta^L(x)}{(2L+1)^2} \, \delta_{x/L}(dx). \qquad (3.22)$$

For a sequence of (possibly random) measures π^L on $[-1, 1]$ we write $\pi^L \to \pi$, if for every continuous function $f : [-1, 1] \to \mathbb{R}$, $\pi^L(f) \to \pi(f)$ in probability. We use η_t^L to denote the Markov chain with generator $\mathcal{L}_L$, and with initial configuration η^L.

The following theorem can be proved using large deviations methods for the hydrodynamic limit of interacting particle systems.

Theorem 3.2. *Assume that $\pi^L(\eta^L) \to \rho_0(x) \, dx$ for some continuous function $\rho_0 : [-1, 1] \to \mathbb{R}$. Define $\rho_\pm = 2Lc_\pm$ with $c_\pm > 0$, and set $\delta_L(x, \eta) = -\delta \operatorname{sign}(\eta(x))/2L$. Then for any finite time t*

$$\pi^L(\eta_{L^2 t}^L) \to \rho(t, x) dx, \qquad (3.23)$$

where $\rho(t, x)$ is the unique solution of the non-linear Cauchy problem

$$\begin{aligned}
\frac{\partial}{\partial t}\rho(t, x) &= \frac{\partial^2}{\partial x^2}\rho(t, x) + \delta \operatorname{sign}(\rho(t, x))\frac{\partial}{\partial x}\rho(t, x) &\qquad & x \in (-1, 1), t > 0, \\
\rho(0, x) &= \rho_0(x) &\qquad & x \in [-1, 1], \\
\rho(t, -1) &= -c_- &\qquad & t \geq 0, \qquad (3.24) \\
\rho(t, +1) &= c_+ &\qquad & t \geq 0.
\end{aligned}$$

We use this theorem to stress how important is the presence of the drift towards the price in the model. To simplify the reasoning we will consider only the equilibrium profile as a function of c_+, c_-, and δ. The equilibrium profile is defined as the unique solution of the system of equations (3.24) with the left hand side of the first equation set to 0. Let $x(\delta) = x(c_+, c_-, \delta)$ be the position of the unique zero of such solution. It is not difficult to find that $x(\delta)$ satisfies $x(\delta) = \delta^{-1} \log(y(\delta))$, where $y(\delta)$ is the larger root of the quadratic equation

$$c_- y^2 + e^\delta(c_+ - c_-)y - c_+ = 0. \qquad (3.25)$$

Figure 5 shows the graph of x as function of δ and c_+/c_-. Remark that as δ increases, the equilibrium position becomes more sensitive to the change of boundary conditions.

4. Conclusions

We have presented a class of Markov models that allow a realistic description of the price evolution of a commodity under trading. The model is formulated in the language of interacting particle systems and describes the time evolution of a virtual order-book, *i.e.,* the collection of all opinions of all traders on the current value of the traded commodity. The price process is inferred from this state according to
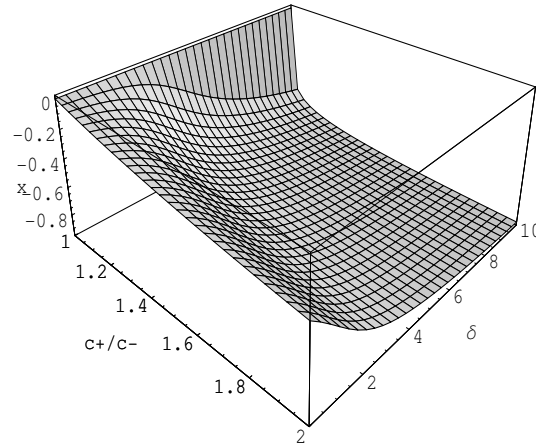
Fig. 5. The position of the equilibrium price

rules analogous to those used in real markets. In the simplest case of independent traders, explicit computations are possible, and we have analysed a crash scenario in a bistable market in this context.

We discussed several basic mechanisms that we think should be taken into account when modelling financial markets. These include attraction to the current price, slowdown of activity for the traders with opinions far away from the current price, and repulsion from the price of traders having performed a transaction. These effects lead to interesting properties of the price process which are analogous to those observed in reality.

We hope to have motivated that the interacting particle systems have place in modelling of financial and economic phenomena. For such modelling to be fruitful, one has to consider interactions taking into account special features of these phenomena and to ask questions that are natural in financial and economic context. In particular, the analysis of the price process raises rather "non-traditional" but very interesting problems regarding order statistics in the interacting particle model. As we will discuss in the forthcoming article [8], these problems are closely related to the study of interfaces and phase boundaries in particle systems.

Acknowledgements

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