Random walks in dynamic random environments and ancestry under local population regulation

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Abstract

We consider random walks in dynamic random environments, with an environment generated by the time-reversal of a Markov process from the 'oriented percolation universality class'. If the influence of the random medium on the walk is small in space-time regions where the medium is 'typical', we obtain a law of large numbers and an averaged central limit theorem for the walk via a regeneration construction under suitable coarse-graining.

Such random walks occur naturally as the spatial embedding of an ancestral lineage in spatial population models with local regulation. We verify that our assumptions hold for logistic branching random walks when the population density is sufficiently high, thus partly settling a question from Depperschmidt (2008) on the behaviour of their ancestral lines.

1 Introduction

Let $\eta_n(x)$ be a random number of particles located at position $x \in \mathbb{Z}^d$ at time n. The joint dynamics of $(\eta_n)_{n\in\mathbb{Z}} = (\eta_n(x) : x \in \mathbb{Z}^d)_{n\in\mathbb{Z}}$ is a stationary (discrete time) particle system with 'local rules' and we assume that η is in its unique non-trivial ergodic equilibrium. Prototypical examples are the super-critical discrete-time contact process, see (2.3) below, or systems of logistic branching random walks, see Section 4.1, Eq. (4.4). Let $(X_k)_{k=0,1,\dots}$ be the position of a random walker that moves 'backwards' through the medium generated by η: Given η, X is a Markov chain; given η and $X_k = x$, the law of the next increment is a function of η in a finite window around the space-time point $(x, -k)$.

Our main result, see Theorem 2 in Section 3, provides a law of large numbers (LLN) and an averaged central limit theorem (CLT) for X. Very broadly speaking we require that the law of an increment of X is close enough to a fixed symmetric finite-range random walk kernel whenever the walk is in a 'good region'; furthermore such good regions are sufficiently frequent in a typical realisation of η, we assume that on suitably coarse-grained space-time scales, the occurrence of good regions can be compared to super-critical oriented percolation. The explicit assumptions are rather technical and we refer to Sections 3.1–3.2 for details.

The reversal of the 'natural' time directions between X and η may seem artificial at first. It is consistent with the interpretation of X_k as the position of the 'ancestor' k generations ago of a particle we picked from position X_0 at time 0. In fact, the spatial embeddings of genealogies in models with fluctuating population sizes and local regulation are (relatively complicated) random walks in a space-time dependent random environment given by the time reversal of the local population size process. In biological applications, they are then often replaced by ordinary

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random walks (without random environments) via an ad-hoc assumption, see for example the discussion and references in [BDE02] and Section 6.4 in [Eth11].

We verify that in a prototypical discrete spatial population model with local regulation, namely logistic branching random walks with Poisson offspring distributions, the assumptions of Theorem 2 are satisfied if the population density in equilibrium is sufficiently large. This allows to formulate in Theorem 3 a LLN and a CLT for the ancestral lineage of an individual sampled from such an equilibrium. Thus we provide at least a partial justification for such ad-hoc assumptions from biology in the sense that here, an ancestral lineage will indeed behave like a random walk when viewed over large space-time scales. This (partly) answers the question posed in [Dep08, Chapter 4] in the affirmative.

As is often the case for random walks in random environments, the main technical tool behind our results is a regeneration construction. The details are somewhat involved; in principle, the medium η can have arbitrary dependence range and its time reversal can in general not be explicitly constructed using local rules. A similar problem was faced in [BČDG13] in the study of a directed random walk on the backbone of an oriented percolation cluster. There, the particular structure of oriented percolation allowed to jointly construct the medium and the walk under the annealed law using suitable space-time local operations (cf. [BČDG13, Sect. 2.1]) and therefrom deduce the regeneration structure. Here, we must use a different approach. Again very broadly speaking, regeneration has now occurred after T steps if the medium η_{-T} in a large window around X_T is 'good' and also the 'local driving randomness' of η in a (large) neighbourhood of the space-time path $\{(X_m, -m) : 0 \le m \le T\}$ has 'good' properties which essentially enforce that everything about η that the random walk path has explored so far is a function of this local randomness. Such a time allows to decouple the past and the future of X conditional on X_T and η_{-T} in a finite window around it. A difficulty stems from the fact that if this regeneration fails at a given time k, this means that we have potentially gained a lot of information about (undesirable) behaviour of η_n at times $n < -k$ which might render successful regeneration at a later time $\ell > k$ much less likely. We address this problem by covering the path and the medium around it by a carefully chosen sequence of eventually nested cones, see Figure 7. We finally express X as an additive functional of a Markov chain which keeps track of the increments between regeneration times and local η configurations at the regeneration points.

Note that random walk in a dynamic random environment generated by various interacting particle systems, in particular also by the contact process in continuous time, has received considerable attention recently, see for example [RV13], [AdSV13], [dHKS14], [AJV14], [MV15]. We also refer to the more detailed discussion in [BČDG13, Remark 1.7]. A fundamental difference to the present set-up lies in the time directions. Traditionally, both the walker and the dynamic environment have the same 'natural' forwards time direction whereas here, forwards time for the walk is backwards time for the medium.

The rest of this text is organised as follows: We first introduce and study in Section 2 a class of random walks which travel through the time-reversal of the discrete time contact process, i.e., η is literally a super-critical contact process (note that unlike the set-up in [BČDG13], here the walk is also allowed to step on sites where η equals 0). We use this simple model to develop and explain our regeneration construction and obtain a LLN and an annealed CLT in the 'p close to 1' regime, see Theorem 1. In Section 3 we develop 'abstract' conditions for spatial models and RWs governed by their time-reversal which allow to implement an analogous regeneration construction on a coarse-grained grid and then in particular prove Theorem 2. Section 4 introduces logistic branching random walks, the class of stochastic spatial population models mentioned above; an ancestral lineage in such a model is a particular RW in dynamic RE, see (4.10). We show that this class provides a family of examples where the abstract conditions from Section 3 can be implemented.

Finally, we note that a natural next step will be to extend our regeneration construction to two random walks on the same realisation of η and to then also deduce a quenched CLT, analogous to [BČDG13]. We defer this to future work.

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2 An auxiliary model

In this section we prove an annealed central limit theorem for a particular type of random walk in random environment. Later, we will explain how the dynamics of ancestral lineages in spatial stochastic population models relates with this particular random walk.

2.1 Definition of the model and results

We define the model first. Let $\omega := \{\omega(x,n) : (x,n) \in \mathbb{Z}^d \times \mathbb{Z}\}$ be a family of independent Bernoulli random variables with parameter $p > 0$. We call a site (x,n) *open* if $\omega(x,n) = 1$ and *closed* if $\omega(x,n) = 0$. Throughout the paper $\|\cdot\|$ denotes sup-norm. For $m \le n$, we say there is an *open path* from (x,m) to (y,n) if there is a sequence $x_m, \dots, x_n \in \mathbb{Z}^d$ such that $x_m = x$, $x_n = y$, $\|x_k - x_{k-1}\| \le 1$ for $k = m+1, \dots, n$ and $\omega(x_k, k) = 1$ for all $k = m, \dots, n$. In this case we write $(x,m) \to^\omega (y,n)$, and in the complementary case $(x,m) \not\to^\omega (y,n)$. For sets $A, B \subseteq \mathbb{Z}^d$ and $m \le n$ we write $A \times \{m\} \to^\omega B \times \{n\}$, if there exist $x \in A$ and $y \in B$ so that $(x,m) \to^\omega (y,n)$. Here, slightly abusing the notation, we use the convention that $\omega(x,m) = \mathbb{1}_A(x)$ while for $k > m$ the $\omega(x,k)$ are i.i.d. Bernoulli as above. With this convention for $A \subset \mathbb{Z}^d$, $m \in \mathbb{Z}$ we define the *discrete time contact process* $\eta^A := (\eta_n^A)_{n \ge m}$ driven by ω as

$$\eta_m^A = \mathbb{1}_A \quad \text{and} \quad \eta_n^A(x) := \mathbb{1}_{\{A \times \{m\} \to^\omega (x,n)\}}, \quad n > m. \tag{2.1}$$

Alternatively $(\eta_n^A)_{n \ge m}$ can be viewed as a Markov chain with $\eta_m^A = \mathbb{1}_A$ and the following local dynamics:

$$\eta_{n+1}^A(x) = \begin{cases} 1 & \text{if } \omega(x,n+1) = 1 \text{ and } \eta_n^A(y) = 1 \text{ for some } y \in \mathbb{Z}^d \text{ with } \|x - y\| \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

For a distribution μ on $\{0,1\}^{\mathbb{Z}^d}$ we write $\eta^\mu = (\eta_n^\mu)_{n \ge m}$ for the discrete time contact process with initial condition η_0^μ distributed according to μ.

Of course the contact process $\eta^A = (\eta_n^A)_{n \ge m}$ is closely related with oriented percolation. In this context A is the set of "wet" sites at time m and the set $\{x \in \mathbb{Z}^d : \eta_n^A(x) = 1\} \times \{n\}$ is the n-th time-slice of the cluster of wet sites. Obviously for any $p < 1$ the Dirac measure on the 0 configuration is a trivial invariant distribution of the discrete time contact process. It is well known that there is a critical percolation probability $p_c \in (0,1)$ such that for $p > p_c$ and any non-empty $A \subset \mathbb{Z}^d$ the process η^A survives with positive probability. Furthermore, in this case there is a unique non-trivial extremal invariant measure ν, referred to as the *upper invariant measure*, such that, starting at any time $m \in \mathbb{Z}$ the distribution of $\eta_n^{\mathbb{Z}^d}$ converges to ν.

We assume $p > p_c$ throughout this section. Given a configuration ω, we define the *stationary discrete time contact process* driven by ω as

$$\eta := (\eta_n)_{n \in \mathbb{Z}} := \{\eta_n(x) : x \in \mathbb{Z}^d, n \in \mathbb{Z}\} \quad \text{with} \quad \eta_n(x) := \mathbb{1}_{\{\mathbb{Z}^d \times \{-\infty\} \to^\omega (x,n)\}}, \tag{2.3}$$

where the event on the right hand side should be understood as $\cap_{m\le n}\big\{\mathbb{Z}^d\times\{m\}\to^\omega (x,n)\big\}$. In the above notation we have $\eta=\eta^{\mathbb{Z}^d}=(\eta_n^{\mathbb{Z}^d})_{n\in\mathbb{Z}}$.

To define the random walk in random environment generated by η, let

$$\kappa := \{\kappa_n(x,y) : n\in\mathbb{Z},\, x,y\in\mathbb{Z}^d\} \tag{2.4}$$

be a family of random transition kernels defined on the same probability space as η, in particular $\kappa_n(x,y)\ge 0$, $\sum_{y\in\mathbb{Z}^d}\kappa_n(x,y)=1$ holds for all $n\in\mathbb{Z}$, $x\in\mathbb{Z}^d$. Given κ, we then consider a $\mathbb{Z}^d$-valued random walk $X:=(X_n)_{n\in\mathbb{Z}_+}$ with $X_0=0$ and transition probabilities given by

$$\mathbb{P}\big(X_{n+1}=y\,\big|\,X_n=x,\kappa\big)=\kappa_n(x,y), \tag{2.5}$$

that is, the random walk at time n takes a step according to the kernel $\kappa_n(x,\cdot)$ if its position at time n is x. We make the following four assumptions on the distribution of κ.

Assumption 2.1 (Locality)**.** The transition kernels in the family κ depend locally on the time-reversal of η, that is for a fixed $R_{\mathrm{loc}}\in\mathbb{N}$

$$\kappa_n(x,\cdot)\ \text{ depends only on }\ \big\{\omega(y,-n),\eta_{-n}(y) : \|x-y\|\le R_{\mathrm{loc}}\big\}. \tag{2.6}$$

Assumption 2.2 (Approximate symmetry on $\eta_{-n}(x)=1$)**.** There is a deterministic symmetric probability measure κ_{ref} on $\mathbb{Z}^d$ with finite range $R_{\mathrm{ref}}\in\mathbb{N}$, i.e., $\kappa_{\mathrm{ref}}(x)=0$ if $\|x\|>R_{\mathrm{ref}}$, and a suitably small $\varepsilon_{\mathrm{ref}}>0$ such that

$$\|\kappa_n(x,x+\cdot)-\kappa_{\mathrm{ref}}(\cdot)\|_{\mathrm{TV}}<\varepsilon_{\mathrm{ref}} \quad\text{whenever}\quad \eta_{-n}(x)=1. \tag{2.7}$$

Here $\|\cdot\|_{\mathrm{TV}}$ denotes the total variation norm.

Assumption 2.3 (Shift invariance and symmetry)**.** The kernels in the family κ are shift-invariant and respect the symmetries of $\mathbb{Z}^d$. That is, using notation $\theta^{z,m}\omega(\,\cdot\,,\,\cdot\,)=\omega(z+\cdot,m+\cdot)$, we have $\kappa_n(x,y)(\omega)=\kappa_{n+m}(x+z,y+z)(\theta^{z,m}\omega)$. Moreover, if ϱ is the (spatial) point reflection operator acting on ω, i.e., $\varrho\omega(x,n)=\omega(-x,n)$ for any $n\in\mathbb{Z}$ and $x\in\mathbb{Z}^d$, then $\kappa_n(0,y)(\omega)=\kappa_n(0,-y)(\varrho\omega)$.

Assumption 2.4 (Finite range)**.** There is $R_\kappa<\infty$ such that a.s.

$$\kappa_n(x,y)=0 \quad\text{whenever}\quad \|y-x\|>R_\kappa. \tag{2.8}$$

Remark 2.5 (Interpretation of the assumptions)**.** The Assumptions 2.1–2.4 are natural as we want to interpret the random walk as the spatial embedding of an ancestral lineage in a spatial population model, see also Section 4 and in particular the discussion around (4.10).

By (2.5) and (2.6), we can and often shall think of creating the walk from η and ω in a local window around the current position and additional auxiliary randomness.

The main result of this section is the following theorem. Its proof is given in Subsection 2.4.

Theorem 1 (LLN and annealed CLT)**.** *Assume that κ satisfies Assumptions 2.1–2.4 with $\varepsilon_{\mathrm{ref}}\ll 1$ and that p is sufficiently close to* 1*. Then X satisfies the law of large numbers with speed* 0 *and an annealed (i.e. when averaging over both ω and the walk) central limit theorem with non-trivial variance.*

Remark 2.6 (Time-reversal of η, oriented percolation interpretation)**.** In [BČDG13], the stationary η was equivalently parametrized via its time reversal $\xi(x,n):=\eta_{-n}(x)$. Then ξ is the (indicator of) the *backbone* of the oriented percolation cluster and was notationally and conceptually convenient to use in [BČDG13] not least because then the "medium" ξ and the walk X had the same positive time direction.

Here, we keep η as our basic datum because we wish to emphasise and in fact later use in Section 3 the interplay between the medium η, interpreted as describing the dynamics of a population, and the walk X, cf. (2.5) above, describing the embedding of an ancestral lineage. Furthermore, in the more general population models, as the one studied in Section 4 for instance, there will be no natural parametrization of the time-reversal of η.

Note that the result in Theorem 1 is in a sense conceptual rather than practical, the proofs in Section 2.3 require $1-p$ to be very small. Situations with $p > p_c$ but also $1-p$ appreciably large require an additional coarse-graining step so that the arguments from Section 3 can be applied.

In order to prove Theorem 1 we will construct suitable regeneration times and show that these regeneration times and the corresponding spatial increments of the walk have finite moments of order b for some $b > 2$. This regeneration construction is rather intricate. The main source of the complications is the fact that in order to construct the random walk X one should know ω and η in the vicinity of its trajectory; cf. Remark 2.5 above. While it is easy to deal with the knowledge of ω's, because they are i.i.d., the knowledge of η's leads to problems. Due to the definition (2.3) of η and Assumption 2.1 on κ, this knowledge gives a non-trivial information about *future behaviour* of X which is not desirable at regeneration times.

In more detail, we need to deal with two types of information on η. The first type, the *negative* information, that is knowing that $\eta_n(x) = 0$ for some n and x is dealt with similarly as in [BČDG13]. The key is the observation that such information is essentially local: to discover that $\eta_n(x) = 0$ one should check that $\mathbb{Z}^d \times \{-\infty\} \not\to^\omega (x, n)$ which requires observing ω's in a layer $\mathbb{Z}^d \times \{n - T, \dots, n\}$ where T is a random variable with exponentially decaying tails.

The second type of information, the *positive one*, that is knowing that $\eta_n(x) = 1$, is then removed by making use of strong coupling properties of the forwards-in-time dynamics of η. When starting from $\eta_{-t} = \mathbb{1}_{x+\{-L,-L+1,\dots,L\}^d}$ there is a substantial chance that every infection, i.e., every '1' of η, inside a growing space-(forwards)time cone with base point $(x, -t)$ can be traced back to $(x + \{-L, -L + 1, \dots , L\}^d) \times \{-t\}$. Furthermore, whether this event has occured can be checked by observing the restriction of η_{-t} to $x+\{-L, -L+1, \dots , L\}^d)$ and the ω's inside a suitably fattened shell of the cone in question, in particular without looking at any $\eta_m(y)$ for $m < -t$; see (2.25) and Lemma 2.12 below. We will construct suitable random times T at which this event occurs for η at the current space-time position $(X_T, -T)$ of the walker and in addition the space-time path of the walk up to T, $\{(X_k, -k) : 0 \le k \le T\}$, is completely covered by that cone. Such a time T allows to regenerate.

For the proof of Theorem 1 we first collect some results on the high density discrete time contact process in Section 2.2. We then rigorously implement the regeneration construction sketched above in Section 2.3.

2.2 Some results about the contact process

This section contains several estimates for the discrete-time contact process η that will be crucial for the regeneration construction. The main results of this section are the estimates given in Lemma 2.10 and Lemma 2.12. We start by recalling two well known results.

Lemma 2.7. *For $p > p_c$ there exist $C(p), \gamma(p) \in (0,\infty)$ such that*

$$\mathbb{P}\big(\mathbb{Z}^d \times \{-n\} \to^\omega (0,0) \;\; \textit{but} \;\; \mathbb{Z}^d \times \{-\infty\} \not\to^\omega (0,0)\big) \le C(p)e^{-\gamma(p)n}, \quad n \in \mathbb{N}. \tag{2.9}$$

Moreover, we have $\limsup_{p\nearrow 1} C(p) < \infty$ *and* $\lim_{p\nearrow 1} \gamma(p) = \infty$.

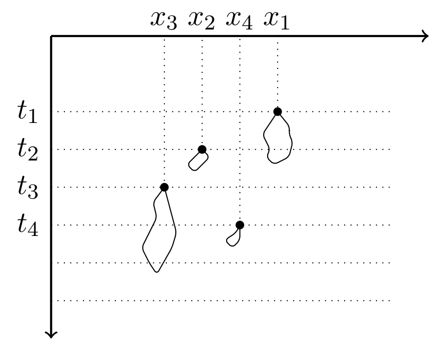

Figure 1: Possibly overlapping finite clusters starting at $V = \{(x_i, t_i)\}$ that appear in Lemma 2.10. Here $k = 4$, $D_1 = 2, D_2 = 1, D_3 = 3, D_4 = 1$, hence $M = 3$, $S_1 = 1, S_2 = 3, S_3 = 6$.

Proof. Due to self-duality of the contact process this is a reformulation of the fact that for $p > p_c$ and $\eta^{\{0\}} = (\eta_n^{\{0\}})_{n\geq 0}$ there exist $C(p), \gamma(p) \in (0, \infty)$ such that

$$\mathbb{P}(\eta^{\{0\}} \text{ dies eventually} \,|\, \eta_n^{\{0\}} \not\equiv 0) \leq C(p)e^{-\gamma(p)n}, \quad n \in \mathbb{N}. \tag{2.10}$$

For a proof we refer to e.g. [Dur84, GH02]; see also Lemma A.1 in [BČDG13]. □

Lemma 2.8. *Let $\eta^\nu = (\eta_n^\nu)_{n\geq 0}$ be the contact process with initial configuration distributed according to the upper invariant measure ν. For p sufficiently close to 1 there exist $s_{\text{coupl}} > 0$, $C < \infty$, $\gamma > 0$ such that*

$$\mathbb{P}(\eta_n^{\{0\}}(x) = \eta_n^\nu(x) \;\; \textit{for all} \;\; \|x\| \leq s_{\text{coupl}} n \,|\, \eta_n^{\{0\}} \not\equiv 0) \geq 1 - Ce^{-\gamma n}, \quad n \in \mathbb{N}. \tag{2.11}$$

Proof. For the contact process in continuous time, this is proved in [DG82], see in particular (33) and (34) in Proposition 6. Although literally, [DG82, Eq. (34)] refers to conditioning on $\{\eta^{\{0\}} \text{ survives}\}$ the result follows by (2.10). □

Remark 2.9. In [FvZ03] it is shown (literally, for the contact process in continuous time) that for any $p > p_c$ and $a > 0$, there is a $C < \infty$ such that $\mathbb{P}(C_n \,|\, \eta^{\{0\}} \text{ survives}) \geq 1 - Cn^{-a}$ and in fact it is believed that exponential decay as in Lemma 2.8 should hold for every $p > p_c$, see the discussion after Thm. 1.3 on p. 767 in [FvZ03].

The first main result of this section is the following lemma on controlling the probabilities of certain *negative* events; cf. Lemma 7 in Section 3 of [Dur92] for a related result.

Lemma 2.10. *For p large enough there exists $\varepsilon(p) \in (0, 1]$ satisfying $\lim_{p\nearrow 1} \varepsilon(p) = 0$ such that for any $V = \{(x_i, t_i) : 1 \leq i \leq k\} \subset \mathbb{Z}^d \times \mathbb{Z}$ with $t_1 > t_2 > \cdots > t_k$, we have*

$$\mathbb{P}\big(\eta_t(x) = 0 \textit{ for all } (x, t) \in V\big) \leq \varepsilon(p)^k. \tag{2.12}$$

Remark 2.11. In (2.12) it is essential that the t_i's are distinct. For a general set $V \subset \mathbb{Z}^d \times \mathbb{Z}$, temporal boundary effects can play a role and the decay will only be stretched exponential in $|V|$.

Proof of Lemma 2.10. An immediate consequence of Lemma 2.7 is that for every $p > p_c$

$$\mathbb{P}\big(\mathbb{Z}^d \times \{-n\} \to^\omega (0, 0) \text{ but } \mathbb{Z}^d \times \{-\infty\} \not\to^\omega (0, 0)\big) \leq e^{-c_1(p)(n+1)}, \quad n = 0, 1, 2, \ldots, \tag{2.13}$$

with $c_1 = c_1(p) > 0$ satisfying $\lim_{p \nearrow 1} c_1(p) = \infty$. To prove (2.12), we now extend the finite sequence $\{t_1, \dots, t_k\}$ to an infinite sequence via $t_{k+j} := t_k - j$, $j \geq 1$, and put

$$D_i := \min \Big\{ \ell \in \mathbb{N} : \mathbb{Z}^d \times \{t_{i+\ell}\} \not\to^{\omega} (x_i, t_i) \Big\}. \tag{2.14}$$

Note that the random variables D_i are upper bounds on the heights of the backwards-clusters of open sites attached to (x_i, t_i) given by (see Figure 1)

$$\{(y, m) \in \mathbb{Z}^d \times \mathbb{Z} : m \leq t_i, (y, m) \to^{\omega} (x_i, t_i)\}.$$

For each $(x_i, t_i) \in V$ we have $\eta_{t_i}(x_i) = 0$ if and only if $D_i < \infty$. Thus the left-hand side of (2.12) satisfies

$$\mathbb{P}\big(\eta_t(x) = 0 \text{ for all } (x, t) \in V\big) = \mathbb{P}\Big(\bigcap_{i=1}^{k} \{D_i < \infty\}\Big). \tag{2.15}$$

On the event $\cap_{i=1}^{k} \{D_i < \infty\}$ define further

$$S_1 = 1, S_2 = S_1 + D_{S_1}, \dots, S_{i+1} = S_i + D_{S_i} \quad \text{as long as } S_i \leq k,$$

and let M be such that $S_{M-1} \leq k < S_M$; see Figure 1. For $i = 1, \dots, M$ we set $\widehat{D}_i := D_{S_i}$ and $\widehat{D}_i := \infty$ for $i > M$. Finally we set

$$\mathcal{I}(m, k) = \{(d_1, \dots, d_m) \in \mathbb{N}^m : d_1 + \cdots + d_{m-1} \leq k < d_1 + \cdots + d_m\}, \tag{2.16}$$

and for $(d_1, \dots, d_m) \in \mathcal{I}(m, k)$ we write $u(1) = 1, u(2) = u(1) + d_1, \dots, u(m) = u(m-1) + d_{m-1}$. Then we have

$$\bigcap_{i=1}^{k} \{D_i < \infty\} \subset \bigcup_{m=1}^{k} \bigcup_{(d_1,\dots,d_m) \in \mathcal{I}(m,k)} \Big\{ \widehat{D}_1 = d_1, \dots, \widehat{D}_m = d_m \Big\}. \tag{2.17}$$

Note that

$$\begin{aligned} &\Big\{ \widehat{D}_1 = d_1, \dots, \widehat{D}_m = d_m \Big\} \\ &\quad = \bigcap_{j=1}^{m} \big\{ \mathbb{Z}^d \times \{t_{u(j)+d_j-1}\} \to^{\omega} (x_{u(j)}, t_{u(j)}) \big\} \cap \big\{ \mathbb{Z}^d \times \{t_{u(j)+d_j}\} \not\to^{\omega} (x_{u(j)}, t_{u(j)}) \big\}. \end{aligned} \tag{2.18}$$

The events in the intersection on the right-hand side depend on ω restricted to disjoint sets and are thus independent. Furthermore we observe that the event

$$\big\{ \mathbb{Z}^d \times \{t_{u(j)+d_j-1}\} \to^{\omega} (x_{u(j)}, t_{u(j)}) \big\} \cap \big\{ \mathbb{Z}^d \times \{t_{u(j)+d_j}\} \not\to^{\omega} (x_{u(j)}, t_{u(j)}) \big\}$$

enforces that $(x_{u(j)}, t_{u(j)})$ is the starting point of a finite (backwards) cluster of height at least $t_{u(j)+d_j-1} - t_{u(j)} \geq d_j - 1$ (when $d_j = 1$ this means that $\omega(x_{u(j)}, t_{u(j)})$ is closed, which also gives a factor $1 - p < 1$). Hence, using (2.13) we obtain

$$\begin{aligned} &\mathbb{P}\big(\widehat{D}_1 = d_1, \dots, \widehat{D}_m = d_m\big) \\ &\quad = \prod_{j=1}^{m} \mathbb{P}\Big(\big\{ \mathbb{Z}^d \times \{t_{u(j)+d_j-1}\} \to^{\omega} (x_{u(j)}, t_{u(j)}), \mathbb{Z}^d \times \{t_{u(j)+d_j}\} \not\to^{\omega} (x_{u(j)}, t_{u(j)}) \big\} \Big) \\ &\quad \leq \prod_{j=1}^{m} e^{-c_1(p) d_j} = e^{-c_1(p) \sum_{j=1}^{m} d_j}. \end{aligned} \tag{2.19}$$

Now (2.19) with (2.17) imply

$$\mathbb{P}\big(\eta_t(x)=0 \text{ for all } (x,t)\in V\big) \le \sum_{m=1}^{k} \sum_{(d_1,\dots,d_m)\in\mathcal{I}(m,k)} e^{-c_1(p)\sum_{j=1}^m d_j}$$

$$= \sum_{m=1}^{k}\sum_{s=k+1}^{\infty} e^{-c_1(p)s}\cdot \#\{(d_1,\dots,d_m)\in\mathcal{I}(m,k): d_1+\dots+d_m=s\}. \quad (2.20)$$

By definition of $\mathcal{I}(m,k)$ for $s\ge k+1$ we have

$$\#\{(d_1,\dots,d_m)\in\mathcal{I}(m,k): d_1+\dots+d_m=s\}$$
$$= \#\{(d_1,\dots,d_m)\in\mathcal{I}(m,k): d_1+\dots+d_m=k+1\} = \binom{k}{m-1} \le 2^k. \quad (2.21)$$

Thus, the right hand side of (2.20) can be bounded by

$$2^k \sum_{m=1}^{k}\sum_{s=k+1}^{\infty} e^{-c_1(p)s} = k2^k \frac{e^{-c_1(p)(k+1)}}{1-e^{-c_1(p)}}, \quad (2.22)$$

yielding the claim of the lemma. □

The second main result of this section, Lemma 2.12 below, is the crucial tool in the construction of a certain coupling which will be useful to forget the positive information about η in the regeneration construction. To state this lemma we need to introduce more notation. For $A\subset\mathbb{Z}^d$ let $\eta^A=(\eta^A_n)_{n\ge 0}$ be the discrete time contact process as defined in (2.1). For positive b,s,h we write (denoting by $\mathbb{Z}_+$ the set non-negative integers)

$$\mathsf{cone}(b,s,h) := \big\{(x,n)\in\mathbb{Z}^d\times\mathbb{Z}_+ :\ \|x\|\le b+sn, 0\le n\le h\big\}. \quad (2.23)$$

for a (truncated upside-down) *cone* with base radius b, slope s, height h and base point $(0,0)$. Furthermore for

$$b_{\mathrm{inn}}\le b_{\mathrm{out}} \quad \text{and} \quad s_{\mathrm{inn}}<s_{\mathrm{out}}, \quad (2.24)$$

we define the *conical shell* with inner base radius b_{inn}, inner slope s_{inn}, outer base radius b_{out}, outer slope s_{out}, and height h by

$$\mathsf{cs}(b_{\mathrm{inn}},b_{\mathrm{out}},s_{\mathrm{inn}},s_{\mathrm{out}},h) := \big\{(x,n)\in\mathbb{Z}^d\times\mathbb{Z} : b_{\mathrm{inn}}+s_{\mathrm{inn}}n\le\|x\|\le b_{\mathrm{out}}+s_{\mathrm{out}}n, 0<n\le h\big\}. \quad (2.25)$$

The conical shell can be thought of as a difference of the *outer cone* $\mathsf{cone}(b_{\mathrm{out}},s_{\mathrm{out}},h)$ and the *inner cone* $\mathsf{cone}(b_{\mathrm{inn}},s_{\mathrm{inn}},h)$ with all boundaries except the bottom boundary of that difference included; see Figure 2.

Let $\eta^{\mathsf{cs}}=(\eta^{\mathsf{cs}}_n)_{n\ge0}$ be the contact process in the infinite conical shell $\mathsf{cs}(b_{\mathrm{inn}},b_{\mathrm{out}},s_{\mathrm{inn}},s_{\mathrm{out}},\infty)$ with initial condition $\eta^{\mathsf{cs}}_0(x)=\mathbb{1}_{\{b_{\mathrm{inn}}\le\|x\|\le b_{\mathrm{out}}\}}$, defined for $n>0$ by

$$\eta^{\mathsf{cs}}_{n+1}(x) = \begin{cases} 1 & \text{if } (x,n+1)\in\mathsf{cs}(b_{\mathrm{inn}},b_{\mathrm{out}},s_{\mathrm{inn}},s_{\mathrm{out}},\infty),\ \omega(x,n+1)=1 \text{ and } \\ & \eta^{\mathsf{cs}}_n(y)=1 \text{ for some } y\in\mathbb{Z}^d \text{ with } \|x-y\|\le 1, \\ 0 & \text{otherwise.}\end{cases}$$

We think of η^{cs} as a version of the contact process where all ω's outside the conical shell $\mathsf{cs}(b_{\mathrm{inn}},b_{\mathrm{out}},s_{\mathrm{inn}},s_{\mathrm{out}},\infty)$ have been set to 0. We say that η^{cs} *survives (in all parts of the conical*

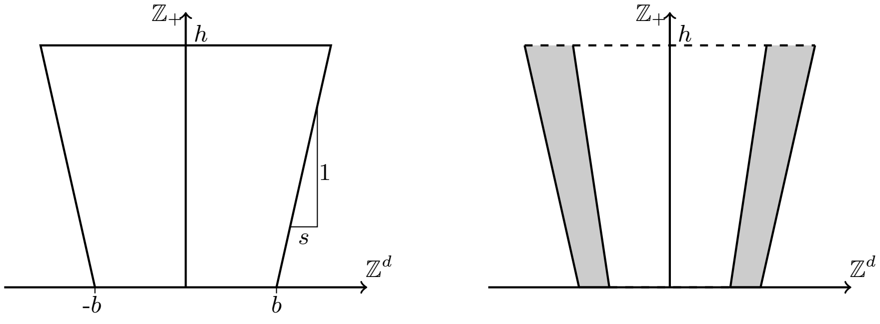

Figure 2: The left figure shows $\mathsf{cone}(b, s, h)$. The grey region in the figure on the right (without the bottom line) shows $\mathsf{cs}(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, h)$ in $\mathbb{Z}^d \times \mathbb{Z}$.

shell) if for all $n \geq 0$ there is $x \in \mathbb{Z}^d$ with $\eta_n^{\mathsf{cs}}(x) = 1$ and in the case $d = 1$ we require additionally that there is $x \in \mathbb{Z}_+$ and $y \in \mathbb{Z}_-$ with $\eta_n^{\mathsf{cs}}(x) = \eta_n^{\mathsf{cs}}(y) = 1$. For a directed path

$$\gamma = \big((x_m, m), (x_{m+1}, m+1), \dots, (x_n, n)\big), \quad m \leq n, \ x_i \in \mathbb{Z}^d \ \text{ with } \ \|x_{i-1} - x_i\| \leq 1 \tag{2.26}$$

we say that γ *crosses* the conical shell $\mathsf{cs}(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, \infty)$ from the outside to the inside if the following three conditions are fulfilled:

(i) the starting point lies outside the outer cone, i.e., $\|x_m\| > b_{\text{out}} + m s_{\text{out}}$,

(ii) the terminal point is inside the inner cone, i.e., $\|x_n\| < b_{\text{inn}} + n s_{\text{inn}}$,

(iii) the remaining points are inside the shell, i.e., $(x_i, i) \in \mathsf{cs}(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, \infty)$ for $i = m+1, \dots, n-1$.

We say that γ *intersects* η^{cs} if there exists $i \in \{m+1, \dots, n-1\}$ with $\eta_i^{\mathsf{cs}}(x_i) = 1$. Finally we say that γ is *open in* $\mathsf{cs}(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, \infty)$ if $\omega(x_i, i) = 1$ for $i = m+1, \dots, n-1$. Note that even if η^{cs} survives, for any $d \geq 1$ it is in principle possible that a path crosses the conical shell $\mathsf{cs}(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, \infty)$ without intersecting η^{cs}. The next lemma states that the probability of that can be made arbitrarily small.

Lemma 2.12. *Assume that the relations in* (2.24) *hold and consider the events*

$$G_1 := \{\eta^{\mathsf{cs}} \textit{ survives}\},$$
$$G_2 := \{\textit{every open path } \gamma \textit{ that crosses } \mathsf{cs}(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, \infty) \textit{ intersects } \eta^{\mathsf{cs}}\}.$$

For any $\varepsilon > 0$ *and* $0 \leq s_{\text{inn}} < s_{\text{out}} < 1$ *one can choose* p *sufficiently close to* 1 *and* $b_{\text{inn}} < b_{\text{out}}$ *sufficiently large so that*

$$\mathbb{P}(G_1 \cap G_2) \geq 1 - \varepsilon.$$

Remark 2.13 (Observations concerning G_1 and G_2)**.** The meaning of the event G_1 is clear. Let us just note that it essential that the relations in (2.24) hold. In particular, in the case $s_{\text{inn}} = s_{\text{out}}$ survival of η^{cs} is only possible in the trivial case $p = 1$.

To understand the importance of the event G_2, observe that if a path γ as in (2.26) crosses $\mathsf{cs}(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, \infty)$, is open in $\mathsf{cs}(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, \infty)$, and also intersects η^{cs} then necessarily we have $\eta_n^{\mathsf{cs}}(x_n) = \omega(x_n, n)$. Thus, on G_2 the values of the contact process η inside the inner cone, that is for (x, n) with $\|x\| < b_{\text{inn}} + n s_{\text{inn}}$, are independent of what happens outside of the shell, see (2.59) and Lemma 2.21.

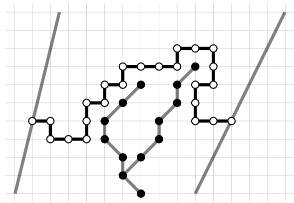

Figure 3: A contour and an open cluster in a piece of a "ray" of a conical shell in the case $d = 1$. Circles $\circ$ and $\bullet$ indicate some closed and open sites respectively. Contours cannot take diagonal steps since they must prevent 'diagonally running' infections.

Proof of Lemma 2.12. The proof consists of four steps. In the first two steps we prove the assertion of the lemma for the case $d = 1$. Then, in the last two steps, we will use the assertion for $d = 1$ to give a proof for $d \geq 2$.

Step 1. Consider the case $d = 1$. We first check via a contour counting argument that the discrete contact process survives with high probability in any "oblique cone" when p is large enough.

For $0 < s_1 < s_2 < 1$, $b \in \mathbb{N}$, set

$$\mathsf{C}_{b,s_1,s_2} := \big\{(x,n) : x \in \mathbb{Z},\ n \in \mathbb{Z}_+,\ s_1 n \leq x \leq s_2 n + b\big\}$$

and let $\tilde{\eta} := (\tilde{\eta}_n)_{n\geq 0}$ be the discrete time contact process in C_{b,s_1,s_2} starting from $\tilde{\eta}_0 = \mathbb{1}_{[0,b]\cap\mathbb{Z}}$ and with all ω's outside C_{b,s_1,s_2} set to 0. We claim that

$$\begin{aligned}&\text{for any } 0 < s_1 < s_2 < 1 \text{ and } \varepsilon > 0 \text{ we can choose } b \text{ large and } p_0 < 1 \text{ such that}\\ &\text{for all } p \geq p_0,\ \tilde{\eta} \text{ survives in } \mathsf{C}_{b,s_1,s_2} \text{ with probability at least } 1 - \varepsilon.\end{aligned} \tag{2.27}$$

If $\tilde{\eta}$ does not survive, there must be a 'contour', that is a non-directed nearest-neighbour path without diagonal steps (see Figure 3), crossing C_{b,s_1,s_2} from left to right which passes only through closed sites. Note that there is a $c_1 = c_1(s_1, s_2) > 0$ such that if the contour starts at height n, i.e. its initial point is $([s_1 n], n)$, it must consist of at least $b + c_1 n$ steps. There are at most $4 \cdot 3^{k-1}$ contours of length $k \geq b + c_1 n$ starting from $([s_1 n], n)$. Thus, the probability that such a closed contour exists is at most

$$\sum_{n=1}^{\infty} \sum_{k=[b+c_1 n]}^{\infty} 4 \cdot 3^{k-1}(1-p)^k \leq \frac{4}{3} \sum_{n=1}^{\infty} \frac{(3(1-p))^{b+[c_1 n]}}{1 - 3(1-p)} \leq \frac{4}{3} \frac{(3(1-p))^b}{1-3(1-p)} \cdot \frac{1}{1-(3(1-p))^{c_1}}$$

which can be made small by choosing b large and $1 - p$ small.

Step 2. Still considering the case $d = 1$ we let G be the event that $\tilde{\eta}$ survives separately inside the regions

$$\begin{aligned}&\big\{(x,n) : x \in \mathbb{Z},\ n \in \mathbb{Z}_+,\ s_{\text{inn}} n + b_{\text{inn}} \leq x \leq (s_{\text{inn}} + \delta)n + b_{\text{inn}} + b'\big\},\\ &\big\{(x,n) : x \in \mathbb{Z},\ n \in \mathbb{Z}_+,\ (s_{\text{out}} - \delta)n + b_{\text{out}} - b' \leq x \leq s_{\text{out}} n + b_{\text{out}}\big\}\end{aligned}$$

and also inside their mirror images obtained by flipping along the time axis. Observe that $G \subset G_1 \cap G_2$. By using the result from Step 1 four times, we see that we can choose $b' \in \mathbb{N}$ and $\delta > 0$ so that G occurs with arbitrarily high probability. This proves the assertion of the lemma for $d = 1$.

Step 3. Consider now $d > 1$. Let

$$\mathsf{bC}(x,n) := \big\{(y,m) \in \mathbb{Z}^d \times \mathbb{Z} : m \leq n, (y,m) \to^{\omega} (x,n)\big\}$$

be the open backwards cluster corresponding to the space-time point (x, n) (viewed on the full space). We claim that (for suitably tuned $b_{\mathrm{inn}}, b_{\mathrm{out}}, s_{\mathrm{inn}}, s_{\mathrm{out}}, b', \delta$ and when p close to 1) for $(x, n) \in \mathsf{cs}(b_{\mathrm{inn}} + b', b_{\mathrm{out}} - b', s_{\mathrm{inn}} + \delta, s_{\mathrm{out}} - \delta, \infty)$

$$\mathbb{P}\big(\mathsf{bC}(x,n) \text{ has height} \geq h \text{ and } \eta^{\mathsf{cs}}_n(x) = 0 \text{ and } \eta^{\mathsf{cs}}_{\lceil \delta n \rceil} \not\equiv 0\big) \leq Ce^{-c_2 h} \quad \text{for all } h \leq \delta n \tag{2.28}$$

with $C < \infty$, $c_2 > 0$ (and in fact c_2 is uniformly bounded away from 0 in the tuning parameters we consider). The idea behind (2.28) is that conditioned on the event $\{\eta^{\mathsf{cs}}_{\lceil \delta n \rceil} \not\equiv 0\}$, η^{cs}_{n-h} will typically 'look like ν' in a (large) box $B_{s_{\mathrm{coupl}} h}(x)$ around x so in particular will have many 1's in there. Furthermore, conditioned on $\{\mathsf{bC}(x,n) \text{ has height} \geq h\}$, with high probability many space-time points from $B_{s_{\mathrm{coupl}} h}(x) \times \{n-h\}$ are connected to (x, n) via open paths (see Lemma 2.8) and it is thus highly unlikely that then $\eta^{\mathsf{cs}}_n(x) = 0$ occurs.

To prove (2.28), fix $(x, n) \in \mathsf{cs}(b_{\mathrm{inn}} + b', b_{\mathrm{out}} - b', s_{\mathrm{inn}} + \delta, s_{\mathrm{out}} - \delta, \infty)$. By self-duality of the contact process and complete convergence results described in Lemma 2.8, there exists $s_{\mathrm{coupl}} > 0$ and $c_3 > 0$ such that

conditional on $\{\text{height of } \mathsf{bC}(x,n) \geq h\}$, the set $\{y : (y, n-h) \in \mathsf{bC}(x,n)\}$ can with probability larger than $1 - e^{-c_3 h}$ be coupled inside $B_{s_{\mathrm{coupl}} h}(x)$ with the set of 1's under the upper invariant measure ν of the (full) contact process. (2.29)

We now want to use the coupling in $d = 1$ described in the last paragraph of Step 2. This requires to compare the d-dimensional contact process in the conical shell with a 1-dimensional process. If $x = (0, \dots, 0, x_k, 0, \dots, 0)$, that is x lies along the k-th coordinate axis, this is straightforward: We simply restrict η to the k-th direction (by setting all ω out of the k-th coordinate direction to 0) and obtain a one-dimensional contact process. For the general case we need more notation: Let $\alpha = (\alpha_i)_{i \in \mathbb{Z}}$ be a self-avoiding nearest neighbour path in $\mathbb{Z}^d$ with $\alpha_0 = 0$ and $\alpha_{\|x\|_1} = x$, making steps only in direction of x, that is $x_j \cdot (\alpha_i - \alpha_{i-1})_j \geq 0$ for all $i \in \mathbb{Z}$ and $j = 1, \dots, d$, staying close to the line connecting x and 0, that is $\{\alpha_i : i \in \mathbb{Z}\} \subset \{tx + z : t \in \mathbb{R}, z \in \mathbb{R}^d, \|z\| \leq 2\}$.

Let $(\eta^{(1)}_n(x))_{x \in \alpha, n \in \mathbb{Z}}$ be an ergodic one-dimensional discrete contact process which "lives" on $\alpha \times \mathbb{Z}$, that is

$$\eta^{(1)}_n(x) = \mathbb{1}_{\{\alpha \times \{-\infty\} \to^\omega (x,n) \text{ by a path lying in } \alpha \times \mathbb{Z}\}}. \tag{2.30}$$

We assume that p is large enough, so that this process is non-trivial. Note that the law of $\eta^{(1)}$ does not depend on the embedding α when α is identified with $\mathbb{Z}$ via the mapping that maps the i-th coordinate of α to $i \in \mathbb{Z}$. Moreover, it is obvious from the construction that $\eta^{(1)}$ is dominated by η as defined in (2.3), in particular the upper invariant measure $\nu^{(1)}$ of $\eta^{(1)}$ is dominated by the upper invariant measure ν of η (when restricted to α). We may assume that p is sufficiently close to 1 so that under $\nu^{(1)}$ the density of 1's is larger than 1/2, say. Using then (2.29) and large deviation estimates for the density of 1's under $\nu^{(1)}$, see [DS88, Thm. 1] (literally, proved there for the continuous-time contact process) we see that

$$\begin{aligned}\mathbb{P}\big(\#\{y \in \alpha : (y, n-h) \in \mathsf{bC}(x,n), \|y - x\| \leq h s_{\mathrm{coupl}}\} \leq h s_{\mathrm{coupl}}/4,& \\ \text{height of } \mathsf{bC}(x,n) \geq h\big) \leq Ce^{-ch}&.\end{aligned} \tag{2.31}$$

For $k \in \mathbb{N}$ and $n \geq k$ let

$$\eta^{\mathsf{cs},k,\alpha}_n(x) = \begin{cases} 1, & \exists\, y \text{ such that } \eta^{\mathsf{cs}}_k(y) = 1 \text{ and } (y,k) \to^\omega (x,n) \text{ by a path} \\ & \text{lying in } (\alpha \times \{k, k+1, \dots\}) \cap \mathsf{cs}(b_{\mathrm{inn}}, b_{\mathrm{out}}, s_{\mathrm{inn}}, s_{\mathrm{out}}, \infty), \\ 0, & \text{else} \end{cases} \tag{2.32}$$

($(\eta^{\mathsf{cs},k,\alpha}_n)_{n \geq k}$ is the contact process in $(\alpha \times \{k, k+1, \dots\}) \cap \mathsf{cs}(b_{\mathrm{inn}}, b_{\mathrm{out}}, s_{\mathrm{inn}}, s_{\mathrm{out}}, \infty)$ with initial condition given by η^{cs}_k restricted to α). The process $\eta^{\mathsf{cs},k,\alpha}$ has the law of a $1+1$-dimensional contact process that lives in a conical shell, analogous to Steps 1–2. By arguments as in the last paragraph of Step 2 we see that on the event

$$\big\{\#\{y \in \alpha : \eta^{\mathsf{cs}}_{\lceil n/2\rceil}(y) = 1\} \geq \delta n\big\}, \tag{2.33}$$

with probability at least $1 - Ce^{-cn}$, $\eta_n^{\mathsf{cs},\lceil n/2\rceil,\alpha}$ can be coupled in $\alpha \cap B_{hs_{\mathrm{coupl}}}(x)$ to $\eta^{(1)}$. When that coupling event occurs, the event in (2.28) implies that

$$\eta^{(1)}(y) = 0 \text{ for all } \{y \in \alpha : (y, n-h) \in \mathsf{bC}(x,n), \|y - x\| \leq hs_{\mathrm{coupl}}\}. \tag{2.34}$$

Note that for any $A \subset \alpha$,

$$\nu^{(1)}(\eta_n^{(1)}(y) = 0 \text{ for all } y \in A) = \mathbb{P}(\eta^{(1),A} \text{ dies eventually}) \leq Ce^{-c\#A}, \tag{2.35}$$

see e.g. [Lig99, Thm. B24].

Finally, note that conditional on $\{\eta^{\mathsf{cs}}_{\lceil \delta n\rceil} \not\equiv 0\}$, the event in (2.33) has probability at least $1 - Ce^{-cn}$. This can be proven by a suitable geometric iteration of the arguments just given. On $\{\eta^{\mathsf{cs}}_{\lceil \delta n\rceil} \not\equiv 0\}$ there is a (suitably large) infected region somewhere from which we can start $\eta^{\mathsf{cs},\lceil \delta n\rceil,\alpha'}$ which (with high probability) survives forever, for some 'discrete direction' α' (which does not necessary 'point towards x'). After $n\delta'$ steps, this $1+1$-dim. slice will have forced η^{cs} to have sufficiently many 1's in a 'fattening around it' so that we can re-start the argument with a new 'direction' α'' that is 'closer' to α than α', etc.

Combining this with (2.31), (2.34) and the observation in (2.35) yields (2.28).

Step 4. We may now conclude the proof of Lemma 2.12. Obviously, the fact that G_1 has high probability can be reduced to the case $d = 1$ in Step 1. Indeed, if η^{cs} survives in a 1+1-dimensional slice of $\mathsf{cs}(b_{\mathrm{inn}}, b_{\mathrm{out}}, s_{\mathrm{inn}}, s_{\mathrm{out}}, \infty)$, then it survives in $d > 1$ as well.

To bound the probability of $G_2^c \cap G_1$, observe that on this event there is an open path γ which crosses $\mathsf{cs}(b_{\mathrm{inn}}, b_{\mathrm{out}}, s_{\mathrm{inn}}, s_{\mathrm{out}}, \infty)$ but does not intersect η^{cs}. This path must hit some space-time point (x, n) inside the cone shell with

$$\frac{b_{\mathrm{inn}} + b_{\mathrm{out}}}{2} + n\frac{s_{\mathrm{inn}} + s_{\mathrm{out}}}{2} - d \leq \|x\| \leq \frac{b_{\mathrm{inn}} + b_{\mathrm{out}}}{2} + n\frac{s_{\mathrm{inn}} + s_{\mathrm{out}}}{2},$$

and $n \geq (b_{\mathrm{out}} - b_{\mathrm{inn}})/2$. Moreover, this path comes from the outside of $\mathsf{cs}(b_{\mathrm{inn}}, b_{\mathrm{out}}, s_{\mathrm{inn}}, s_{\mathrm{out}}, \infty)$, so the backwards cluster $\mathsf{bC}(x,n)$ has height at least δn (with $\delta = \delta(b_{\mathrm{inn}}, b_{\mathrm{out}}, s_{\mathrm{inn}}, s_{\mathrm{out}}) > 0$ suitably chosen). Using (2.28), it follows that

$$P_p(G_2^c \cap G_1) \leq C \sum_{n=(b_{\mathrm{out}}-b_{\mathrm{inn}})/2}^{\infty} \big(n(s_{\mathrm{out}} + s_{\mathrm{inn}}) + b_{\mathrm{out}} + b_{\mathrm{inn}}\big)^{d-1} e^{-c_2 \delta n}$$

which can be made small by tuning the parameters. □

Remark 2.14 (Coupling of η^{cs} and $\eta^{\mathbb{Z}}$ in dimension one)**.** The construction in Step 2 of the proof of Lemma 2.12 actually proves more: On G by the $1+1$-dimensional geometry, the contact processes η^{cs} and $\eta^{\mathbb{Z}}$ which use the same ω are perfectly coupled inside the slightly smaller conical shell $\mathsf{cs}(b_{\mathrm{inn}} + b', b_{\mathrm{out}} - b', s_{\mathrm{inn}} + \delta, s_{\mathrm{out}} - \delta, \infty)$. Put differently, when G occurs then a space-time point in this smaller conical shell is connected to $\mathbb{Z} \times \{0\}$ by an open path if and only if it is connected to the bottom of $\mathsf{cs}(b_{\mathrm{inn}}, b_{\mathrm{out}}, s_{\mathrm{inn}}, s_{\mathrm{out}}, \infty)$ by an open path within this shell. This will be useful later.

2.3 Regeneration construction

In Theorem 1 we claim that the speed of the random walk X is 0. As an intermediate result we will show that the speed is bounded. This will be needed for the regeneration construction.

Lemma 2.15 (A priori bound on the speed of the random walk)**.** *If the the family of kernels κ satisfies Assumptions 2.1–2.4 then there are positive finite constants $s_{\max}$, c and C so that*

$$\mathbb{P}\big(\|X_n\| > s_{\max} n\big) \le Ce^{-cn}, \tag{2.36}$$

in particular $\limsup_{n\to\infty}\|X_n\|/n \le s_{\max}$ almost surely. The bound $s_{\max}$ can be chosen arbitrarily small by taking $\varepsilon_{\text{ref}} \ll 1$ (where ε_{ref} is from Assumption 2.2) and $1-p \ll 1$.

Proof. With the percolation interpretation in mind, we say that a space-time site (x,k) is "wet" if $\eta_k(x)=1$, and "dry" if $\eta_k(x)=0$. Let Γ_n be the set of all n-step paths γ on $\mathbb{Z}^d$ starting from $\gamma_0=0$ with the restriction $\|\gamma_i-\gamma_{i-1}\| \le R_\kappa$, $i=1,\dots,n$, where R_κ is the range of the kernels κ_n from Assumption 2.4. For $\gamma\in\Gamma_n$ and $0\le i_1<i_2\cdots<i_k\le n$ we define

$$\begin{aligned} D^{\gamma}_{i_1,\dots,i_k} &\coloneqq \{\eta_{-\ell}(\gamma_\ell)=0 \text{ for all } \ell\in\{i_1,\dots,i_k\}\},\\ W^{\gamma}_{i_1,\dots,i_k} &\coloneqq \{\eta_{-\ell}(\gamma_\ell)=1 \text{ for all } \ell\in\{1,\dots,n\}\setminus\{i_1,\dots,i_k\}\}. \end{aligned}$$

Let $H_n \coloneqq \#\{0\le i\le n : \eta_{-i}(X_i)=0\}$ be the number of dry sites the walker visits up to time n and set $K\coloneqq \max_{x\in\mathbb{Z}^d}\{\kappa_{\text{ref}}(x)\}+\varepsilon_{\text{ref}}$. For $k\in\{1,\dots,n\}$ by Lemma 2.10 we have

$$\begin{aligned} \mathbb{P}(H_n=k) &= \sum_{0\le i_1<\dots<i_k\le n}\ \sum_{\gamma\in\Gamma_n} \mathbb{P}\big((X_0,\dots,X_n)=\gamma, W^{\gamma}_{i_1,\dots,i_k}, D^{\gamma}_{i_1,\dots,i_k}\big)\\ &\le \sum_{0\le i_1<\dots<i_k\le n}\ \sum_{\gamma\in\Gamma_n} K^{n-k}\mathbb{P}\big(D^{\gamma}_{i_1,\dots,i_k}\big)\\ &\le \sum_{i_1<i_2<\dots<i_k\le n} R_\kappa^{dn}K^{n-k}\varepsilon(p)^k = \binom{n}{k} R_\kappa^{dn}K^{n-k}\varepsilon(p)^k. \end{aligned} \tag{2.37}$$

It follows

$$\begin{aligned} \mathbb{P}(H_n\ge\delta n) &\le \sum_{k=\lfloor n\delta\rfloor}^{n}\binom{n}{k}R_\kappa^{dn}K^{n-k}\varepsilon(p)^k\\ &\le \Big(2R_\kappa^d K\Big)^n \sum_{k=\lfloor n\delta\rfloor}^{\infty}(\varepsilon(p)/K)^k = \Big(2R_\kappa^d K\Big)^n \frac{(\varepsilon(p)/K)^{\delta n}}{1-\varepsilon(p)/K} \le c_1e^{-c_2n} \end{aligned} \tag{2.38}$$

with $c_1,c_2\in(0,\infty)$, when $\delta>0$ is sufficiently small and $p\ge p_0=p_0(\delta,\varepsilon_{\text{ref}})$.

Let $X_n=(X_{n,1},\dots,X_{n,d})$. We can couple $(X_{n,1})_{n\ge0}$ and a one-dimensional random walk $\widetilde{X}=(\widetilde{X}_n)_{n\ge0}$ with transition probabilities given by

$$\mathbb{P}\left(\widetilde{X}_n-\widetilde{X}_{n-1}=x\right) = (1-\varepsilon_{\text{ref}})\sum_{(x_2,\dots,x_d)\in\mathbb{Z}^{d-1}}\kappa_{\text{ref}}\big(0,(x,x_2,\dots,x_d)\big)+\varepsilon_{\text{ref}}\delta_{R_\kappa}(x), \quad x\in\mathbb{Z}$$

(i.e., $\widetilde{X}$ takes with probability $1-\varepsilon_{\text{ref}}$ a step according to the projection of κ_{ref} on the first coordinate and with probability ε_{ref} simply a step of size R_κ to the right) such that for all $n\in\mathbb{N}$

$$X_{n,1}\le \widetilde{X}_n + R_\kappa H_n.$$

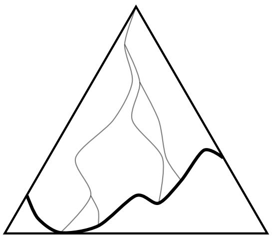

Figure 4: A caricature of the determining triangle $D(x,n)$ with a closed contour. The height of the triangle is $\ell(x,n)+1$.

Then, we have

$$\begin{aligned}\mathbb{P}\left(X_{n,1} > \bar{s}n\right) &\le \mathbb{P}\left(H_n \ge \delta n\right) + \sum_{k=0}^{\lfloor n\delta \rfloor} \mathbb{P}\left(X_{n,1} > \bar{s}n, H_n = k\right) \\ &\le \mathbb{P}\left(H_n \ge \delta n\right) + \sum_{k=0}^{\lfloor n\delta \rfloor} \mathbb{P}\left(\widetilde{X}_n > \bar{s}n - kR_\kappa\right) \\ &\le \mathbb{P}\left(H_n \ge \delta n\right) + \delta n \mathbb{P}\left(\widetilde{X}_n > (\bar{s} - \delta R_\kappa)n\right).\end{aligned} \tag{2.39}$$

The estimates (2.38), (2.39) and standard large deviations bounds for $\widetilde{X}$ show that

$$\mathbb{P}\left(X_{n,1} > \bar{s}n\right) \le c_3 e^{-c_4 n} \quad \text{holds for all } n \in \mathbb{N} \tag{2.40}$$

with $c_3, c_4 \in (0,\infty)$ when $\bar{s} - \delta R_\kappa > \mathbb{E}[\widetilde{X}_1] = \varepsilon_{\mathrm{ref}} R_\kappa$. By symmetry, we have an analogous bound for $\mathbb{P}\left(X_{n,1} < -\bar{s}n\right)$. The same reasoning applies to the coordinates $X_{n,2}, \dots, X_{n,d}$. Thus, we have

$$\mathbb{P}\left(\|X_n\| > \bar{s}n\right) \le \sum_{i=1}^{d} \mathbb{P}\left(|X_{n,i}| > \bar{s}n\right) \le 2dc_3 e^{-c_4 n}. \tag{2.41}$$

In particular, $\limsup_{n\to\infty} \|X_n\|/n \le \bar{s}$ a.s. by the Borel-Cantelli lemma. □

Denote the R_{loc}-tube around the first n steps of the path by

$$\mathsf{tube}_n := \{(y,-k) : 0 \le k \le n, \|y - X_k\| \le R_{\mathrm{loc}}\}. \tag{2.42}$$

For $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}$ let $\ell(x,n)$ be the length of the longest (backwards in time) directed open path starting in (x,n) with the convention $\ell(x,n) = -1$ if $\omega(x,n) = 0$ and $\ell(x,n) = \infty$ if $\eta_n(x) = 1$.

For each (x,n) we define its *determining triangle* by

$$D(x,n) := \begin{cases} \emptyset, & \text{if } \eta_n(x) = 1, \\ \{(y,m) : \|y - x\| \le (n-m),\ n - \ell(x,n) - 1 \le m \le n\}, & \text{if } \eta_n(x) = 0. \end{cases} \tag{2.43}$$

The idea is that if $\eta_n(x) = 0$, i.e. (x,n) is not connected to $-\infty$, then this information can be deduced by inspecting the ω's in $D(x,n)$. By definition of $\ell(x,n)$, in this case there must be a *closed contour* contained in $D(x,n)$ which separates (x,n) from $-\infty$; see Figure 4. Note in particular that $D(x,n) = \{(x,n)\}$ if $\omega(x,n) = 0$.

When constructing the walk X for n steps we must inspect ω and η in tube_n (cf. Remark 2.5). By the nature of η, this yields in principle information on the configuration of η that the walk will find in its future. *Positive information* of the form $\eta_m(y) = 1$ for certain m and y is at this stage harmless because η has positive correlations and in view of Assumption 2.2 this suggests a well-behaved path in the future. On the other hand, *negative information* of the form $\eta_m(y) = 0$ for certain m and y is problematic because this increases the chances to find more 0's of η in the walk's future. In this case Assumption 2.2. In order to "localise" this negative information we "decorate" the tube around the path with the determining triangles for all sites in tube_n (obviously, only those with $\eta = 0$ matter)

$$\mathsf{dtube}_n = \bigcup_{(y,k)\in \mathsf{tube}_n} D(y,k). \tag{2.44}$$

Define

$$D_n := n + \max\big\{\ell(y,-n) + 2 : \|X_n - y\| \le R_{\mathrm{loc}},\, \ell(y,-n) < \infty\big\}. \tag{2.45}$$

Note that D_n is precisely the time (for the walk) at which the reasons for $\eta_{-n}(y) = 0$ for all y from the R_{loc}-neighbourhood of X_n are explored by inspecting all the determining triangles with base points in $B_{R_{\mathrm{loc}}}(X_n) \times \{-n\}$. The information $\eta_{-n}(y) = 0$ does not affect the law of the random walk after time D_n. (Note that the "height" of a non-empty triangle $D(y,-n)$ is $\ell(y,-n)+1$, this is why $\ell(y,-n)+2$ appears in definition (2.45).)

Now, between time n and D_n the random walk might have explored more negative information which in general will be decided after time D_n and will affect the law of the random walk after time D_n. To deal with this *cumulative negative future information* we define recursively a sequence

$$\sigma_0 := 0, \quad \sigma_i := \min\Big\{m > \sigma_{i-1} : \max_{\sigma_{i-1}\le n\le m} D_n \le m\Big\}. \tag{2.46}$$

In words, σ_i is the first time m after σ_{i-1} when the reasons for all the negative information that the random walk explores in the time interval $\sigma_{i-1},\dots,m$ are decided "locally" and thus the law of the random walk after time σ_i does not depend on that negative information. The σ_i are stopping times with respect to the filtration $\mathcal{F} = (\mathcal{F}_n)_{n=0,1,2,\dots}$, where

$$\mathcal{F}_n := \sigma\big(X_j : 0 \le j \le n\big) \vee \sigma\big(\eta_j(y), \omega(y,j) : (y,j) \in \mathsf{tube}_n\big) \vee \sigma\big(\omega(y,j) : (y,j) \in \mathsf{dtube}_n\big). \tag{2.47}$$

Note that by construction we have $\eta_{-\sigma_i}(y) = 1$ for all $y \in B_{R_{\mathrm{loc}}}(X_{\sigma_i})$.

Lemma 2.16. *When p is sufficiently close to 1 there exist finite positive constants c and C so that*

$$\mathbb{P}\big(\sigma_{i+1} - \sigma_i > n \,\big|\, \mathcal{F}_{\sigma_i}\big) \le Ce^{-cn} \quad \text{for all } n = 1,2,\dots,\ i = 0,1,\dots \text{ a.s.}, \tag{2.48}$$

in particular, all σ_i are a.s. finite. Furthermore, we have

$$\mathscr{L}\big((\omega(\cdot,-j-\sigma_i)_{j=0,1,\dots} \,\big|\, \mathcal{F}_{\sigma_i}\big) \succcurlyeq \mathscr{L}\big((\omega(\cdot,-j)_{j=0,1,\dots}\big) \quad \text{for all } i = 0,1,\dots \text{ a.s.}, \tag{2.49}$$

where "$\succcurlyeq$" denotes stochastic domination.

Proof. Throughout the proof we write $\widehat{R}_\kappa := (2R_\kappa+1)^d$ and $\widehat{R}_{\mathrm{loc}} := (2R_{\mathrm{loc}}+1)^d$ for the number of elements in $B_{R_\kappa}(0)$ respectively in $B_{R_{\mathrm{loc}}}(0)$.

Consider first the case $i = 0$ in (2.48). The event $\{\sigma_1 > n\}$ enforces that in the "R_{loc}-vicinity" of the path there are space-time points $(y_j,-j)$ with $\eta_{-j}(y_j) = 0$ for $j = 0,1,\dots,n$. For a fixed choice of the y_j's by Lemma 2.10 the probability of that event is bounded by $\varepsilon(p)^n$. We use a relatively crude estimate to bound the number of relevant vectors $(y_0,y_1,\dots,y_n) \in (\mathbb{Z}^d)^{n+1}$, as follows. There are $\widehat{R}_\kappa^n$ possible n-step paths for the walk. Assume there are exactly k time points along the path, say $0 \le m_1 < \dots < m_k \le n$, when a point $(y_{m_i},-m_i) \in B_{R_{\mathrm{loc}}}(X_{m_i}) \times \{-m_i\}$ with

$\eta_{-i}(y_{m_i}) = 0$ is encountered and hence the corresponding "determining" triangle $D(y_{m_i}, -m_i)$ is not empty (when $n > 1$, we necessarily have $m_1 = 0$ or $m_1 = 1$).

For consistency of notation we write $m_{k+1} = n$. Necessarily, the height of $D(y_{m_i}, -m_i)$ is bounded below by $m_{i+1} - m_i$. For a fixed n-step path of X and fixed $m_1 < \cdots < m_k$, there are at most $\widehat{R}_{\mathrm{loc}}^k$ many choices for the y_{m_i}, $i = 1, \dots, k$, and inside $D(y_{m_i}, -m_i)$ we have at most $\widehat{R}_\kappa^{m_{i+1}-m_i-1}$ choices to pick $y_{m_i+1}, y_{m_i+2} \dots, y_{m_{i+1}-1}$ (start with y_{m_i}, then follow a longest open path which is not connected to $-\infty$, these sites necessarily have $\eta = 0$). Thus, there are at most

$$\widehat{R}_\kappa^n \sum_{k=1}^n \sum_{m_1 < \cdots < m_k \le m_{k+1} = n} \widehat{R}_{\mathrm{loc}}^k \prod_{i=1}^k \widehat{R}_\kappa^{m_{i+1}-m_i-1} = \widehat{R}_\kappa^n \sum_{k=1}^n \binom{n}{k} \widehat{R}_{\mathrm{loc}}^k \widehat{R}_\kappa^{n-k} \le \widehat{R}_\kappa^n \big(\widehat{R}_{\mathrm{loc}}^k + \widehat{R}_\kappa\big)^n$$

possible choices of $(y_0, y_1, \dots, y_n)$ and hence we have

$$\mathbb{P}(\sigma_1 > n) \le \big(\widehat{R}_\kappa(\widehat{R}_{\mathrm{loc}}^k + \widehat{R}_\kappa)\varepsilon(p)\big)^n.$$

The right hand side decays exponentially when p is close to 1 so that $\varepsilon(p)$ is small enough. For general $i > 0$ (2.48) follows by induction, employing (2.49) and the argument for $i = 0$.

In order to verify (2.49) note that the stopping times σ_i are special in the sense that on the one hand, at a time σ_i the "negative information" in $\mathcal{F}_{\sigma_i}$, that is the knowledge that some of the η's were $= 0$ in the R_{loc}-neighbourhood of the path, has been "erased" because the reasons for that are decided by local information contained in $\mathcal{F}_{\sigma_i}$. On the other hand, the "positive information" where η's were $= 1$, which enforces the existence of certain open paths for the ω's, is possibly retained. Thus, (2.49) follows from the FKG inequality for the ω's. □

Corollary 2.17 (Reformulation of Lemma 2.10)**.** *For any $V = \{(x_1, t_1), \dots, (x_k, t_k)\} \subset \mathbb{Z}^d \times \mathbb{N}$ with $t_1 < t_2 < \cdots < t_k$ and $\varepsilon(p)$ as in Lemma 2.10 we have*

$$\mathbb{P}\left(\eta_{-t-\sigma_i}(x + X_{\sigma_i}) = 0 \text{ for all } (x,t) \in V \,\middle|\, \mathcal{F}_{\sigma_i}\right) \le \varepsilon(p)^k. \tag{2.50}$$

Proof. The assertion is an easy consequence of (2.49) and Lemma 2.10. □

For $t \in \mathbb{N}$ we define $R_t := \inf\{i \in \mathbb{Z}_+ : \sigma_i \ge t\}$ and for $m = 1, 2, \dots$ we put

$$\widetilde{\tau}_m^{(t)} := \begin{cases} \sigma_{R_t-m+1} - \sigma_{R_t-m}, & m \le R_t, \\ 0, & \text{else.} \end{cases} \tag{2.51}$$

In words, $\widetilde{\tau}_1^{(t)}$ is the length of the time interval $(\sigma_{i-1}, \sigma_i]$ which contains t and $\widetilde{\tau}_m^{(t)}$ is the length of the $(m-1)$-th interval before it.

Lemma 2.18. *When p is sufficiently close to 1 there exist finite positive constants c and C so that for all $i, n = 0, 1, \dots$*

$$\mathbb{P}\big(\widetilde{\tau}_1^{(t)} \ge n \,\big|\, \mathcal{F}_{\sigma_i}\big) \le Ce^{-cn} \quad \text{a.s. on } \{\sigma_i < t\}, \tag{2.52}$$

and generally

$$\mathbb{P}\big(R_t \ge i + m, \widetilde{\tau}_m^{(t)} \ge n \,\big|\, \mathcal{F}_{\sigma_i}\big) \le Cm^2 e^{-cn} \quad \text{for } m = 1, 2, \dots \text{ a.s. on } \{\sigma_i < t\}. \tag{2.53}$$

Proof. For (2.52), we have

$$
\begin{aligned}
&\mathbb{P}\big(\widetilde{\tau}_1^{(t)} \geq n \,\big|\, \mathcal{F}_{\sigma_i}\big) \\
&= \mathbb{P}\big(\sigma_{i+1} \geq t \vee (n+\sigma_i) \,\big|\, \mathcal{F}_{\sigma_i}\big) + \sum_{j>i} \sum_{\ell=\sigma_i+1}^{t-1} \mathbb{P}\big(\sigma_j = \ell, \sigma_{j+1} \geq t \vee (\ell+n) \,\big|\, \mathcal{F}_{\sigma_i}\big) \\
&\leq Ce^{-cn} + \sum_{\ell=\sigma_i+1}^{t-1} Ce^{-c((t-\ell)\vee n)} \mathbb{P}\big(\exists\, j > i \,:\, \sigma_j = \ell \,\big|\, \mathcal{F}_{\sigma_i}\big) \\
&\leq Ce^{-cn} + \mathbb{1}_{\{\sigma_i \leq t-n-2\}} \sum_{\ell=\sigma_i+1}^{t-n-1} Ce^{-c(t-\ell)} + \mathbb{1}_{\{n+1\leq t\}} \sum_{\ell=t-n}^{t-1} Ce^{-cn} \leq C\big(1 + \tfrac{e^{-c}}{1-e^{-c}} + n\big)e^{-cn}
\end{aligned}
$$

where we used Lemma 2.16 and

$$
\mathbb{P}\big(\sigma_j = \ell, \sigma_{j+1} \geq t \vee (\ell+n) \,\big|\, \mathcal{F}_{\sigma_i}\big) = \mathbb{E}\big[\mathbb{1}_{\{\sigma_j=\ell\}} \mathbb{P}(\sigma_{j+1} - \sigma_j \geq (t-\ell) \vee n \,|\, \mathcal{F}_{\sigma_j}) \big|\, \mathcal{F}_{\sigma_i}\big]
$$

in the first inequality.

Similarly, for $m \geq 2$ (we assume implicitly that $\sigma_i \leq t-n-m-1$ for otherwise the conditional probability appearing on the right-hand side of (2.53) equals 0)

$$
\begin{aligned}
&\mathbb{P}\big(\widetilde{\tau}_m^{(t)} \geq n \,\big|\, \mathcal{F}_{\sigma_i}\big) \\
&\quad= \sum_{j>i} \sum_{k=\sigma_i+1}^{t-m-n} \sum_{\ell=k+n}^{t-m+1} \mathbb{P}\big(\sigma_j = k, \sigma_{j+1} = \ell, \sigma_{j+m-1} < t, \sigma_{j+m} \geq t \,\big|\, \mathcal{F}_{\sigma_i}\big) \\
&\quad\leq \sum_{j>i} \sum_{k=\sigma_i+1}^{t-m-n} \sum_{\ell=k+n}^{t-m+1} \mathbb{P}\big(\sigma_j = k, \sigma_{j+1} = \ell \,\big|\, \mathcal{F}_{\sigma_i}\big) \times (m-1)Ce^{-c(t-\ell)/(m-1)} \\
&\quad\leq C(m-1) \sum_{k=\sigma_i+1}^{t-m-n} \sum_{\ell=k+n}^{t-m+1} e^{-c(t-\ell)/(m-1)} \sum_{j>i} \mathbb{P}\big(\sigma_j = k \,\big|\, \mathcal{F}_{\sigma_i}\big) \times Ce^{-c(\ell-k)} \\
&\quad\leq C^2(m-1) \sum_{k=\sigma_i+1}^{t-m-n} e^{ck-ct/(m-1)} \sum_{\ell=k+n}^{t-m+1} \exp\big(-c\tfrac{m-2}{m-1}\ell\big)
\end{aligned}
$$

where we used in the first inequality that

$$
\big\{\sigma_{j+1} = \ell, \sigma_{j+m-1} < t, \sigma_{j+m} \geq t\big\} \subset \bigcup_{r=j+2}^{j+m} \big\{\sigma_r - \sigma_{r-1} \geq \tfrac{t-\ell}{m-1}\big\}
$$

together with Lemma 2.16 and then argued analogously to the proof of (2.52) for the second inequality. For $m = 2$, the chain of inequalities above yields the bound

$$
\mathbb{P}\big(\widetilde{\tau}_m^{(t)} \geq n \,\big|\, \mathcal{F}_{\sigma_i}\big) \leq C^2 \sum_{k=\sigma_i+1}^{t-n} (t-k-n)e^{-c(t-k)} \leq C^2 e^{-cn} \sum_{\ell=0}^{\infty} \ell e^{-c\ell}
$$

whereas for $m > 2$ we obtain

$$\begin{aligned}
\mathbb{P}\big(\widetilde{\tau}_m^{(t)} \ge n \,\big|\, \mathcal{F}_{\sigma_i}\big) &\le C^2(m-1) \sum_{k=\sigma_i+1}^{t-m-n} e^{ck-ct/(m-1)} \frac{\exp\big(-c\frac{m-2}{m-1}(k+n)\big)}{1-e^{c\frac{m-2}{m-1}}} \\
&= \frac{C^2(m-1)}{1-e^{c\frac{m-2}{m-1}}} \exp\big(-c\tfrac{m-2}{m-1}n - c\tfrac{t}{m-1}\big) \sum_{k=\sigma_i+1}^{t-m-n} e^{ck/(m-1)} \\
&\le \frac{C^2(m-1)}{1-e^{c\frac{m-2}{m-1}}} \exp\big(-c\tfrac{m-2}{m-1}n - c\tfrac{t}{m-1}\big) \frac{e^{c(t-n)/(m-1)}}{e^{c/(m-1)}-1} \\
&= \frac{C^2(m-1)}{1-e^{c\frac{m-2}{m-1}}} \frac{e^{-cn}}{e^{c/(m-1)}-1} \le C'(m-1)^2 e^{-cn}.
\end{aligned}$$

Thus, (2.53) holds (with suitable adaptation of the value of the prefactor). □

As a result of (2.49) and Assumption 2.2, the walk is well-behaved at least along the sequence of stopping times σ_i, we formalize this in the following result.

Lemma 2.19. *When p is sufficiently close to 1 there exist finite positive constants c and C so that for all finite $\mathcal{F}$-stopping times T with $T \in \{\sigma_i : i \in \mathbb{N}\}$ a.s. and all $k \in \mathbb{N}$*

$$\mathbb{P}\big(\|X_k - X_T\| > s_{\max}(k-T) \,\big|\, \mathcal{F}_T\big) \le Ce^{-c(k-T)} \quad \text{a.s. on } \{T < k\} \tag{2.54}$$

and for $j < k$

$$\mathbb{P}\big(\|X_k - X_j\| > (1+\epsilon)s_{\max}(k-j) \,\big|\, \mathcal{F}_T\big) \le Ce^{-c(k-j)} \quad \text{a.s. on } \{T \le j\} \tag{2.55}$$

with $s_{\max}$ as in Lemma 2.15.

Proof. Note that by Lemma A.1, we may assume that $T = \sigma_\ell$ for some $\ell \in \mathbb{Z}_+$, the general case follows by writing $1 = \sum_{\ell=0}^{\infty} \mathbb{1}_{\{T=\sigma_\ell\}}$ (a.s.).

For (2.54), combine (2.49) and in particular Corollary 2.17 with the proof of Lemma 2.15. For (2.55), let $T' := \inf\big(\{\sigma_i : i \in \mathbb{N}\} \cap [j,\infty)\big)$ be the time of the next σ_i after time j. Inequality (2.52) from Lemma 2.18 shows that $\mathbb{P}\big(T' - j > \epsilon(k-j) \,\big|\, \mathcal{F}_T\big)$ is exponentially small in $k-j$. On $\{T' - j \le \epsilon(k-j)\}$ we use (2.54) starting from time T' and simply use the fact that increments are bounded for the initial piece between time j and time T'. □

For $m < n$ we say that n is a (b,s)-*cone time point for the decorated path beyond* m if

$$\begin{aligned}
\big(\mathsf{tube}_n \cup \mathsf{dtube}_n\big) &\cap \big(\mathbb{Z}^d \times \{-n, -n+1, \dots, -m\}\big) \\
&\subset \big\{(x,-j) : m \le j \le n, \|x - X_n\| \le b + s(n-j)\big\}.
\end{aligned} \tag{2.56}$$

In words (see also Figure 5), n is a cone time point for the decorated path beyond m if the space-time path $(X_j, -j)_{j=m,\dots,n}$ together with its R_{loc}-tube and decorations by determining triangles is contained in $\mathsf{cone}(b,s,n-m)$ shifted to the base point $(X_n, -n)$ (recall the definition of $\mathsf{cone}(b,s,h)$ in (2.23)). Note that (2.56) in particular implies

$$\|X_n - X_j\| \le b + s(n-j) \quad \text{for } j = m, \dots, n-1. \tag{2.57}$$

Lemma 2.20. *There exist $b > 0$ and $s > s_{\max}$ such that for all finite $\mathcal{F}$-stopping times T with $T \in \{\sigma_i : i \in \mathbb{N}\}$ a.s. and all $k \in \mathbb{N}$, with $T' := \inf\{\sigma_i : \sigma_i \ge k\}$*

$$\mathbb{P}\big(T' \text{ is a } (b,s)\text{-cone time point for the decorated path beyond } T \,\big|\, \mathcal{F}_T\big) \ge 1 - \varepsilon \tag{2.58}$$

a.s. on $\{T < k\}$. Furthermore $0 < s - s_{\max} \ll 1$ can be chosen small.

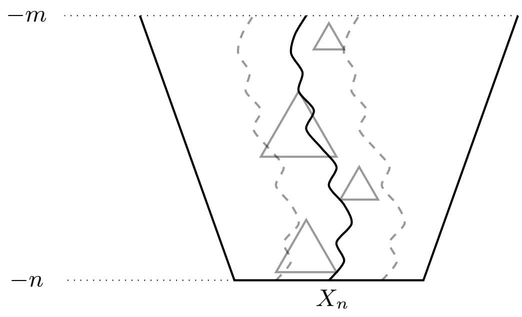

Figure 5: Cone time point n for the decorated path beyond m. Dashed lines indicate the R_{loc} tube around the path and the triangles are the determining triangles. The cone is given by in $\mathsf{cone}(b, s, n-m)$ shifted to the base point $(X_n, -n)$.

Proof. Denote the event in (2.57) (with $m = T$, $n = k$ and b replaced by $b - M$, where $M > 0$ will be tuned later) by $B_{T,k}$. We have

$$1 - \mathbb{P}\big(B_{T,k} \,\big|\, \mathcal{F}_T\big) \leq \sum_{j=T}^{k-1} \mathbb{P}\big(\|X_k - X_j\| > b - M + s(k-j) \,\big|\, \mathcal{F}_T\big)$$
$$\leq \sum_{j=k-m}^{k-1} \mathbb{P}\big(\|X_k - X_j\| > b - M \,\big|\, \mathcal{F}_T\big) + \sum_{j=T}^{k-m-1} \mathbb{P}\big(\|X_k - X_j\| > s(k-j) \,\big|\, \mathcal{F}_T\big).$$

Using (2.55) from Lemma 2.19 we can make the second sum small by choosing m sufficiently large and $s > s_{\max}$. Then we can make the first sum small (or even vanish) by picking $b-M$ sufficiently large.

Recall that R_κ is the range of the random walk X. Inequality (2.52) from Lemma 2.18 implies that $\mathbb{P}\big(T' - k \geq (M - R_{\mathrm{loc}})/R_\kappa \,\big|\, \mathcal{F}_T\big)$ can be made arbitrarily small by choosing M sufficiently large. On $B_{T,k} \cap \{T' - k < (M - R_{\mathrm{loc}})/R_\kappa\}$ (which has high probability under $\mathbb{P}(\cdot \,|\, \mathcal{F}_T)$), we have by construction that

$$\mathsf{tube}_{T'} \cap \big(\mathbb{Z}^d \times \{-T', -T'+1, \dots, -T\}\big)$$
$$\subset \big\{(x, -j) : T \leq j \leq T', \|x - X_{T'}\| \leq b + s(T' - j)\big\},$$

i.e., the path together with its R_{loc}-tube is covered by a suitably shifted cone with base point $(X_{T'}, -T')$.

It remains to verify that under $\mathbb{P}(\cdot \,|\, \mathcal{F}_T)$ with high probability also the decorations (recall (2.43), (2.44)) are covered by the same cone. To show this we may assume $T = \sigma_i$ for notational simplicity; this is justified by Lemma A.1. Let R_k, $\widetilde{\tau}_1^{(k)}, \widetilde{\tau}_2^{(k)}, \dots$ be as defined in and around (2.51) with $t = k$.

Note that $\mathsf{dtube}_{T'}$ is contained in a union of space-time rectangles with heights $\widetilde{\tau}_m^{(k)}$, side lengths $2\widetilde{\tau}_m^{(k)}(R_{\mathrm{loc}} \vee R_\kappa)$ and base points $\big(X_{T' - \widetilde{\tau}_1^{(k)} - \dots - \widetilde{\tau}_m^{(k)}}, -(T' - \widetilde{\tau}_1^{(k)} - \dots - \widetilde{\tau}_m^{(k)})\big)$. A geometric argument shows that on the event

$$B_{T,k} \cap \bigcap_{m=1}^{\infty} \Big(\big\{R_k \geq i + m, \widetilde{\tau}_m^{(t)} < M_1 + \epsilon_1 m\big\} \cup \big\{R_k < i + m\big\}\Big),$$

for ϵ_1, M_1 chosen suitably in relation to b and s, we also have

$$\mathsf{dtube}_{T'} \cap \big(\mathbb{Z}^d \times \{-T', -T'+1, \dots, -T\}\big)$$
$$\subset \big\{(x, -j) : T \leq j \leq T', \|x - X_{T'}\| \leq b + s(T' - j)\big\}.$$

Using inequality (2.53) from Lemma 2.18, we see that

$$\sum_{m=1}^{\infty} \mathbb{P}\big(R_k \geq i+m, \widetilde{\tau}_m^{(k)} \geq M_1 + \epsilon_1 m \,\big|\, \mathcal{F}_T\big) \leq \sum_{m=1}^{\infty} C m^2 e^{-c(M_1+\epsilon_1 m)} \quad \text{a.s. on } \{T < k\}$$

which can be made arbitrarily small when M_1 and ϵ_1 are suitably tuned. This completes the proof of (2.58). □

Note that the σ_i defined in (2.46) are themselves not regeneration times since (2.49) is in general not an equality of laws. We use another layer in the construction with suitably nested cones to forget remaining positive information.

Recall the definition of cones and cone shells from (2.23), (2.25) and Figure 2. The following sets of "good" ω-configurations in conical shells will play a key role in the regeneration construction. Let $G(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, h) \subset \{0,1\}^{\mathsf{cs}(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, h)}$ be the set of all ω-configurations with the property

$$\begin{aligned} &\forall\, \eta_0, \eta_0' \in \{0,1\}^{\mathbb{Z}^d} \text{ with } \eta_0|_{B_{b_{\text{out}}}(0)} = \eta_0'|_{B_{b_{\text{out}}}(0)} \equiv 1 \quad \text{and} \\ &\quad \omega \in \{0,1\}^{\mathbb{Z}^d \times \{1,\dots,h\}} \text{ with } \omega|_{\mathsf{cs}(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, h)} \in G(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, h) \;: \\ &\qquad \eta_n(x) = \eta_n'(x) \text{ for all } (x,n) \in \mathsf{cone}(b_{\text{inn}}, s_{\text{inn}}, h), \end{aligned} \tag{2.59}$$

where η and η' are both constructed from (2.2) with the same ω's. In words, when there are 1's at the bottom of the outer cone, a configuration from $G(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, h)$ guarantees successful coupling inside the inner cone irrespective of what happens outside the outer cone.

Lemma 2.21. *For parameters p, b_{inn}, b_{out}, s_{inn} and s_{out} as in Lemma 2.12,*

$$\mathbb{P}\big(\omega|_{\mathsf{cs}(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, h)} \in G(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, h)\big) \geq 1 - \varepsilon \tag{2.60}$$

uniformly in $h \in \mathbb{N}$.

Proof. The assertion follows from Lemma 2.12 because if the event $G_1 \cap G_2$ defined there occurs, then $\omega|_{\mathsf{cs}(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, h)} \in G(b_{\text{inn}}, b_{\text{out}}, s_{\text{inn}}, s_{\text{out}}, h)$ holds. □

Let us denote the space-time shifts on $\mathbb{Z}^d \times \mathbb{Z}$ by $\Theta^{(x,n)}$, i.e.,

$$\Theta^{(x,n)}(A) = \{(x+y, m+n) : (y,m) \in A\} \quad \text{for } A \subset \mathbb{Z}^d \times \mathbb{Z}. \tag{2.61}$$

An elementary geometric consideration reveals that we can choose a deterministic sequence $t_\ell \nearrow \infty$ with the property that for $\ell \in \mathbb{N}$ and $\|x\| \leq s_{\max} t_{\ell+1}$

$$\Theta^{(0,-t_\ell)}\big(\mathsf{cone}(t_\ell s_{\max} + b_{\text{out}}, s_{\text{out}}, t_\ell)\big) \subset \Theta^{(x,-t_{\ell+1})}\big(\mathsf{cone}(b_{\text{inn}}, s_{\text{inn}}, t_{\ell+1})\big). \tag{2.62}$$

Note that this essentially enforces $t_\ell \approx \rho^\ell$ for a suitable $\rho > 1$. Indeed, a worst case picture (see Figure 6) shows that we need

$$t_{\ell+1} s_{\text{inn}} + b_{\text{inn}} - t_{\ell+1} s_{\max} > t_\ell s_{\max} + b_{\text{out}} + t_\ell s_{\text{out}} \iff t_{\ell+1} > \frac{t_\ell(s_{\text{out}} + s_{\max}) + b_{\text{out}} - b_{\text{inn}}}{s_{\text{inn}} - s_{\max}},$$

that is we can use (for ℓ sufficiently large)

$$t_\ell = \lceil \rho^\ell \rceil \text{ for any } \rho > \frac{s_{\text{out}} + s_{\max}}{s_{\text{inn}} - s_{\max}}. \tag{2.63}$$

Furthermore note that

$$\mathbb{P}\big(\exists\, n \leq t_\ell \,:\, \|X_n\| > s_{\max} t_\ell\big) \leq \sum_{n=\lceil t_\ell s_{\max} \rceil}^{t_\ell} \mathbb{P}\big(\|X_n\| > s_{\max} n\big) \leq C' e^{-c' t_\ell} \tag{2.64}$$

using Lemma 2.15. Since t_ℓ grows exponentially in ℓ, the right hand side is summable in ℓ. Thus, from some random ℓ_0 on, we have $\sup_{n \leq t_\ell} \|X_n\| \leq s_{\max} t_\ell$ for all $\ell \geq \ell_0$, and ℓ_0 has very short tails.

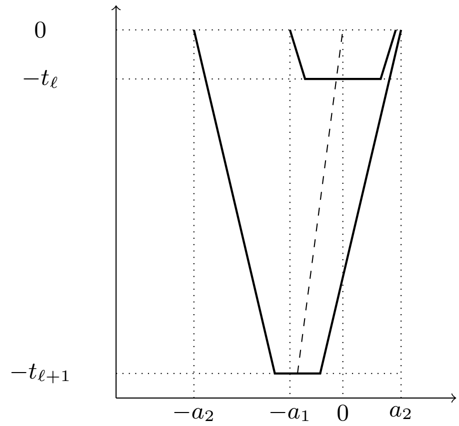

Figure 6: Growth condition for the sequence (t_ℓ): The small inner cone is $\mathsf{cone}(t_\ell s_{\max}+b_{\text{out}}, s_{\text{out}}, t_\ell)$ shifted to the base point $(0, -t_\ell)$. The big outer cone is $\mathsf{cone}(b_{\text{inn}}, s_{\text{inn}}, t_{\ell+1})$ shifted to the base point $(-t_{\ell+1}s_{\max}, -t_{\ell+1})$. The slope of the dashed line is $s_{\max}$. The sequence (t_ℓ) must satisfy $a_1 < a_2$ for $a_1 = s_{\text{out}}t_\ell + b_{\text{out}} + s_{\max}t_\ell$ and $a_2 = s_{\text{inn}}t_{\ell+1} + b_{\text{inn}} - s_{\max}t_{\ell+1}$.

2.4 Proof of Theorem 1

We now have all the ingredients for the regeneration construction, which runs as follows (see also Figure 7):

1. Go to the first σ_i after t_1, check if η in the b_{out}-neighbourhood of $(X_{\sigma_i}, -\sigma_i)$ is $\equiv 1$, the path (together with its tube and decorations) has stayed inside the interior of the corresponding conical shell based at the current space-time position and the ω's in that conical shell are in the good set as defined in (2.59). This has positive (in fact, very high) probability (cf. Lemma 2.20) and if it occurs, we have found the regeneration time.

2. If the event fails, we must try again. We successively check at times t_2, t_3, etc.: If not previously successful, at the ℓ-th step let $\tilde\sigma_\ell$ be the first σ_i after t_ℓ, check if $\tilde\sigma_\ell$ is a cone point for the decorated path beyond $t_{\ell-1}$ with $\|X_{\tilde\sigma_\ell}\| \le s_{\max}\tilde\sigma_\ell$, the η's in the b_{out}-neighbourhood of $(X_{\tilde\sigma_\ell}, -\tilde\sigma_\ell)$ are $\equiv 1$, ω's in the corresponding conical shell are in the good set as defined in (2.59) and the path (with tube and decorations) up to time $t_{\ell-1}$ is contained in the box of diameter $s_{\text{out}}t_{\ell-1} + b_{\text{out}}$ and height $t_{\ell-1}$. If this all holds, we have found the regeneration time.

   (We may assume that $\tilde\sigma_{\ell-1}$ is suitably close to $t_{\ell-1}$, this has very high probability by Lemma 2.18.)

3. The path containment property holds from some finite ℓ_0 on. Given the construction and all the information obtained from it up to the $(\ell-1)$-th step, the probability that the other requirements occur is uniformly high (for the cone time property use Lemma 2.20 with $k = t_\ell$; use (2.49) to verify that the probability to see $\eta \equiv 1$ in a box around $(X_{\tilde\sigma_\ell}, -\tilde\sigma_\ell)$ is high; use Lemma 2.21 to check that conditional on the construction so far the probability that the ω's in the corresponding conical shell are in the good set is high, note that these ω's have not yet been looked at by the construction so far).

4. We will thus at most require a geometric number of t_ℓ's to construct the regeneration time, then shift the space-time origin and start afresh conditioned on seeing configuration $\eta \equiv 1$ in the b_{out}-box around 0.

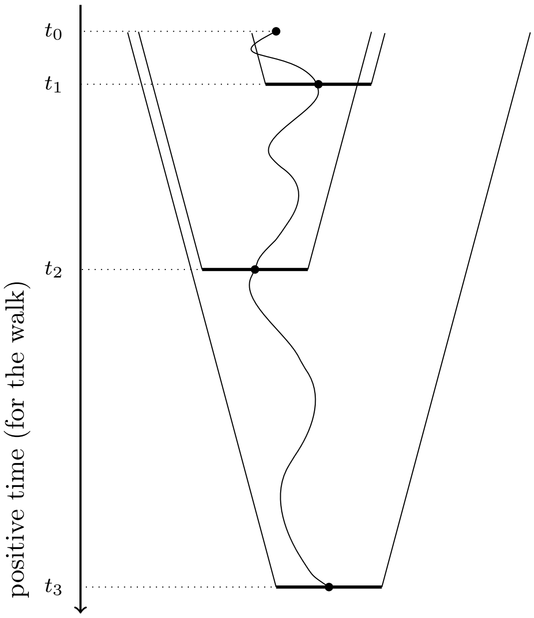

Figure 7: A schematic example: The walk passing through a sequence of cones in an attempt to regenerate. Here, $\tau_1 = \tau_0 + t_3$.

The sequence t_ℓ grows exponentially in ℓ with rate ρ (see (2.63)) and we need to go to at most a random ℓ with geometric distribution with a success parameter $1-\delta$ very close to 1. We thus can enforce a finite very high moment of the regeneration time:

$$\begin{aligned}\mathbb{P}(\text{regeneration after time } n) &\le \mathbb{P}(\text{more than } \log n/\log\rho \text{ steps needed})\\ &\le \delta^{\log n/\log\rho} = n^{-a},\end{aligned} \tag{2.65}$$

where $a = \log(1/\delta)/\log\rho$ can be made large by choosing δ small and ρ close to 1. Both is achieved by choosing p close to 1.

Existence of regeneration times with "enough" moments on the increments implies Theorem 1 by standard arguments. Note that the speed must be 0 by the assumed symmetry; see Assumption 2.3.

Remark 2.22. In the general case without the Assumption 2.3 the above argument yields that there must be a limit speed, its value would be so far given only implicitly as $\mathbb{E}[X_{T_1}]/\mathbb{E}[T_1]$, where T_1 denotes the first regeneration time.

If in Assumption 2.3 we would additionally require symmetries with respect to coordinate permutations and with respect to reflections along coordinate hyperplanes then the limiting law Φ would be a (non-trivial) centred isotropic d-dimensional normal law, cf. the proof of Theorem 1.1 in [BČDG13].

3 A more abstract set-up

The goal of this section is to present an abstract set-up where the renewal construction similar to one of the previous section can be applied. The main motivation of this set-up is, of course, the dynamics of the ancestral lineages in spatial populations, but it can be likely applied for other types of directed random walk in random environment.

In Sections 3.1 and 3.2, we present certain abstract assuptions on the random environment and the associated random walk. These assumptions allow to control the behaviour of the random walk using a regeneration construction that is very similar to the one from Section 2. In particular, they allow to link the model with the directed percolation, using a coarse-graining technique.

We would, however, like to stress that the coarse graining does not convert the presented model to the one of the previous section. In particular, the nature of regenerations is rather different. We will see that the sequence of regeneration times and associated displacements, $(T_{i+1}-T_i, X_{T_{i+1}}-X_{T_i})_{i\geq 2}$ is not i.i.d. but can be generated as a certain function of an irreducible, finite-state Markov chain and additional randomness. This of course, by recurrence of such chains, leads to the same results as previously.

Theorem 2. *Let the random environment η and the random walk X satisfy the assumptions of Sections 3.1 and 3.2 below with sufficiently small parameter ε_U. Then the random walk X satisfies the law of large numbers with speed* 0 *and the annealed central limit theorem.*

A concrete example satisfying the abstract assumptions of Sections 3.1 and 3.2 will be given in Section 4. They can also be verified for the oriented random walk on the backbone of the oriented percolation cluster which was treated in [BČDG13] using simpler, but related, methods.

3.1 Assumptions for the environment

We now formulate two assumptions on the random environment. The first assumption requires that the environment is Markovian (in the positive time direction), and that there is a 'flow construction' for this Markov process, coupling the processes with different starting conditions. The second assumption then allows the coarse graining and the links with the directed percolation.

Formally, let

$$U := \{U(x,n) : x \in \mathbb{Z}^d, n \in \mathbb{Z}\}$$

be an i.i.d. random field, $U(0,0)$ taking values in some Polish space E_U (E_U could be $\{-1,+1\}$, $[0,1]$, a path space, etc.). Furthermore for $R_\eta \in \mathbb{N}$ let $B_{R_\eta} = B_{R_\eta}(0) \subset \mathbb{Z}^d$ be the ball of radius R_η around 0. Let

$$\varphi : \mathbb{Z}_+^{B_{R_\eta}} \times E_U^{B_{R_\eta}} \to \mathbb{Z}_+$$

be a measurable function.

Assumption 3.1 (Markovian, local dynamics, flow construction)**.** We assume that $(\eta_n)_n$ is a Markov chain with values in $\mathbb{Z}_+^{\mathbb{Z}^d}$ (or in $(\mathbb{Z}_+^m)^{\mathbb{Z}^d}$ when thinking of several types) whose evolution is local in the sense that the value $\eta_{n+1}(x)$ depends only on the values $\eta_n(y)$ for y in a finite ball around x. In particular we assume that η can be realised using the "driving noise" U as

$$\eta_{n+1}(x) = \varphi\big(\, \theta^x \eta_n|_{B_{R_\eta}}\,, \theta^x U(\cdot, n+1)|_{B_{R_\eta}}\,\big), \quad x \in \mathbb{Z}^d,\ n \in \mathbb{Z}. \tag{3.1}$$

Here $\theta^x \eta_n|_{B_{R_\eta}}$ is the restriction of the configuration $\theta^x\eta_n$ to the ball $B_{R_\eta} = B_{R_\eta}(0)$, and θ^x is the spatial shift, i.e., $\theta^x\eta(\,\cdot\,) = \eta(\,\cdot\, + x)$.

Note that (3.1) defines a flow, in the sense that given a realisation of U we can construct (η_n) simultaneously for all starting configurations. In most situations we have in mind the constant zero configuration $\underline{0}$ is an equilibrium for (η_n), that is,

$$\varphi\big(\, \underline{0}|_{B_{R_\eta}}\,, \cdot\,\big) \equiv 0,$$

and there is another non-trivial equilibrium. It will be a consequence of our assumptions that the latter is in fact the unique non-trivial ergodic equilibrium.

The second assumption, inspired by [BD07], allow the comparison of (η_n) with a supercritical oriented percolation on suitable space-time scales. Loosely speaking, this assumption states that if we have a good configuration on the bottom of a (suitably big) block and the driving noise inside the blocks is good, too, then the configuration on the top of the block is also good and the good region grows with high probability. Furthermore if we input two good configurations at the bottom of the block then good noise inside the block produces a coupled region at the top of the block.

Formally, let $L_\mathrm{t}, L_\mathrm{s} \in \mathbb{N}$. We use space-time boxes whose "bottom parts" are centred at points in the coarse-grained grid $L_\mathrm{s}\mathbb{Z}^d \times L_\mathrm{t}\mathbb{Z}$. They will be partly overlapping in the spatial direction but not in the temporal direction, and we typically think of $L_\mathrm{t} > L_\mathrm{s} \gg R_\eta$.

For $(\widetilde{x}, \widetilde{n}) \in \mathbb{Z}^d \times \mathbb{Z}$, let

$$\mathsf{block}_m(\widetilde{x}, \widetilde{n}) \coloneqq \big\{(y, k) \in \mathbb{Z}^d \times \mathbb{Z} : \|y - L_\mathrm{s}\widetilde{x}\| \le mL_\mathrm{s}, \widetilde{n}L_\mathrm{t} < k \le (\widetilde{n}+1)L_\mathrm{t}\big\}, \tag{3.2}$$

and $\mathsf{block}(\widetilde{x}, \widetilde{n}) \coloneqq \mathsf{block}_1(\widetilde{x}, \widetilde{n})$; see Figure 8.

For a set $A \subset \mathbb{Z}^d \times \mathbb{Z}$ we denote by $U|_A$ the restriction of the random field U to A. In particular, $U|_{\mathsf{block}_4(\widetilde{x},\widetilde{n})}$ is the restriction of U to $\mathsf{block}_4(\widetilde{x}, \widetilde{n})$ and can be viewed as element of $E_U^{B_{4L_\mathrm{s}}(0)\times\{1,2,\dots,L_\mathrm{t}\}}$.

Assumption 3.2 ("Good" noise configurations and propagation of coupling)**.** There exist a finite set $G_\eta \subset \mathbb{Z}_+^{B_{2L_\mathrm{s}}(0)}$ of "good" (local) configurations and a set of "good" local realisations of the driving noise $G_U \subset E_U^{B_{4L_\mathrm{s}}(0)\times\{1,2,\dots,L_\mathrm{t}\}}$ with the following properties:

- For a suitably small ε_U,

$$\mathbb{P}\big(\, U|_{\mathsf{block}_4(0,0)} \in G_U\big) \ge 1 - \varepsilon_U \tag{3.3}$$

- For any $(\widetilde{x}, \widetilde{n}) \in \mathbb{Z}^d \times \mathbb{Z}$ and any configurations $\eta_{\widetilde{n}L_\mathrm{t}}, \eta'_{\widetilde{n}L_\mathrm{t}} \in \mathbb{Z}_+^{\mathbb{Z}^d}$ at time $\widetilde{n}L_\mathrm{t}$,

$$\begin{aligned} &\eta_{\widetilde{n}L_\mathrm{t}}\big|_{B_{2L_\mathrm{s}}(L_\mathrm{s}\widetilde{x})}, \ \eta'_{\widetilde{n}L_\mathrm{t}}\big|_{B_{2L_\mathrm{s}}(L_\mathrm{s}\widetilde{x})} \in G_\eta \quad \text{and} \quad U|_{\mathsf{block}_4(\widetilde{x},\widetilde{n})} \in G_U \\ &\Rightarrow \quad \eta_{(\widetilde{n}+1)L_\mathrm{t}}(y) = \eta'_{(\widetilde{n}+1)L_\mathrm{t}}(y) \quad \text{for all } y \text{ with } \|y - L_\mathrm{s}\widetilde{x}\| \le 3L_\mathrm{s} \\ &\qquad \text{and} \quad \eta_{(\widetilde{n}+1)L_\mathrm{t}}\big|_{B_{2L_\mathrm{s}}(L_\mathrm{s}(\widetilde{x}+\widetilde{e}))} \in G_\eta \text{ for all } \widetilde{e} \text{ with } \|\widetilde{e}\| \le 1, \end{aligned} \tag{3.4}$$

  and

$$\eta_{\widetilde{n}L_\mathrm{t}}\big|_{B_{2L_\mathrm{s}}(L_\mathrm{s}\widetilde{x})} = \eta'_{\widetilde{n}L_\mathrm{t}}\big|_{B_{2L_\mathrm{s}}(L_\mathrm{s}\widetilde{x})} \quad \Rightarrow \quad \eta_k(y) = \eta'_k(y) \ \text{ for all } (y,k) \in \mathsf{block}(\widetilde{x}, \widetilde{n}), \tag{3.5}$$

  where (η_n) and (η'_n) are given by (3.1) with the same U but possibly different initial conditions.

- There is a fixed (e.g., L_s-periodic or even constant in space) reference configuration $\eta^{\mathrm{ref}} \in \mathbb{Z}_+^{\mathbb{Z}^d}$ such that $\eta^{\mathrm{ref}}\big|_{B_{2L_\mathrm{s}}(L_\mathrm{s}\widetilde{x})} \in G_\eta$ for all $\widetilde{x} \in \mathbb{Z}^d$.

Note in particular that if the event in (3.4)–(3.5) occurs, a coupling of η and η' on $B_{2L_\mathrm{s}}(L_\mathrm{s}\widetilde{x}) \times \{\widetilde{n}L_\mathrm{t}\}$ has propagated to $B_{2L_\mathrm{s}}(L_\mathrm{s}(\widetilde{x}+\widetilde{e})) \times \{(\widetilde{n}+1)L_\mathrm{t}\}$ for $\|\widetilde{e}\| \le 1$ and also the fact that the local configuration is "good" has propagated. The event in (3.4) enforces "propagation of goodness" and can also be viewed as a "contractivity property" of the local dynamics. In other words he flow tends to "merge" local configuations once they are in the "good set".

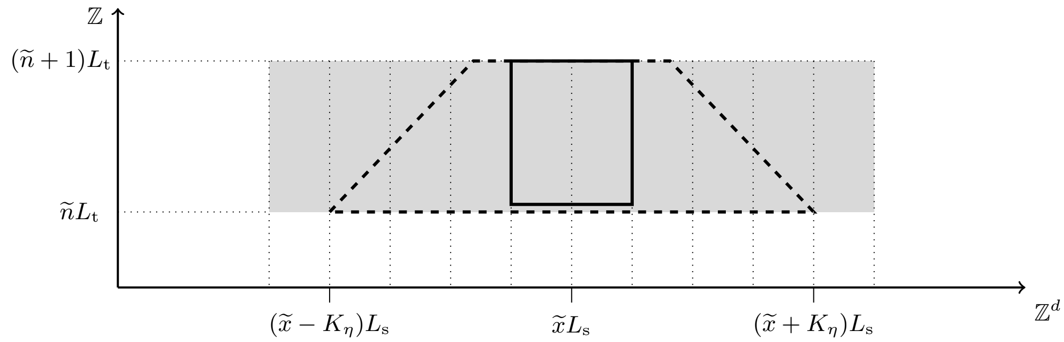

Figure 8: Locality of construction of (η_n) on block level for $d = 1$. In solid lined $\mathsf{block}(\widetilde{x}, \widetilde{n})$ is drawn. If U is known in the grey region and $\eta_{\widetilde{n}L_\mathrm{t}}$ is known on the bottom of the dashed trapezium then the configurations η_k are completely determined inside $\mathsf{block}(\widetilde{x}, \widetilde{n})$.

Remark 3.3 (Locality on the block level)**.** Put

$$K_\eta := R_\eta\big(\lceil \tfrac{L_\mathrm{t}}{L_\mathrm{s}} \rceil + 1\big). \tag{3.6}$$

From the local construction of (η_n) given in (3.1) it follows easily (see Figure 8) that for fixed $(\widetilde{x}, \widetilde{n}) \in \mathbb{Z}^d \times \mathbb{Z}$ the values $\eta_n(x)$ for $(x, n) \in \mathsf{block}(\widetilde{x}, \widetilde{n})$ are completely determined by $\eta_{\widetilde{n}L_\mathrm{t}}$ restricted to $B_{K_\eta L_\mathrm{s}}(\widetilde{x}L_\mathrm{s})$ and U restricted to $\cup_{\|\widetilde{y}\| \le K_\eta} \mathsf{block}(\widetilde{x} + \widetilde{y}, \widetilde{n})$.

Using the above assumptions, it is fairly standard to couple η to an oriented percolation. Recall the notation in Section 2.1 and in particular the definition of the stationary discrete time contact process in (2.3).

Lemma 3.4 (Coupling with oriented percolation)**.** *Put*

$$\widetilde{U}(\widetilde{x}, \widetilde{n}) := \mathbb{1}_{\{U|_{\mathsf{block}_4(\widetilde{x}, \widetilde{n})} \in G_U\}}, \quad (\widetilde{x}, \widetilde{n}) \in \mathbb{Z}^d \times \mathbb{Z}. \tag{3.7}$$

If ε_U is sufficiently small, we can couple $\widetilde{U}(\widetilde{x}, \widetilde{n})$ to an i.i.d. Bernoulli random field $\widetilde{\omega}(\widetilde{x}, \widetilde{n})$ with $\mathbb{P}(\widetilde{\omega}(\widetilde{x}, \widetilde{n}) = 1) \ge 1 - \varepsilon_{\widetilde{\omega}}$ such that $\widetilde{U} \ge \widetilde{\omega}$, and $\varepsilon_{\widetilde{\omega}}$ can be chosen small (how small depends on ε_U, of course).

Moreover, the process η then has a unique non-trivial ergodic equilibrium and one can couple a stationary process $(\eta_n)_{n \in \mathbb{Z}}$ with η_0 distributed according to that equilibrium with $\widetilde{\omega}$ so that

$$\widetilde{G}(\widetilde{x}, \widetilde{n}) := \widetilde{U}(\widetilde{x}, \widetilde{n}) \mathbb{1}_{\{\eta_{\widetilde{n}L_\mathrm{t}}|_{B_{2L_\mathrm{s}}(L_\mathrm{s}\widetilde{x})} \in G_\eta\}} \ge \widetilde{\xi}(\widetilde{x}, \widetilde{n}), \quad (\widetilde{x}, \widetilde{n}) \in \mathbb{Z}^d \times \mathbb{Z} \tag{3.8}$$

where $\widetilde{\xi} := \{\widetilde{\xi}(\widetilde{x}, \widetilde{n}) : \widetilde{x} \in \mathbb{Z}^d, \widetilde{n} \in \mathbb{Z}\}$ is the discrete time contact process defined by

$$\widetilde{\xi}(\widetilde{x}, \widetilde{n}) := \mathbb{1}_{\{\mathbb{Z}^d \times \{-\infty\} \to^{\widetilde{\omega}} (x,n)\}}. \tag{3.9}$$

Proof. The first part is standard: Note that the $\widetilde{U}(\widetilde{x}, \widetilde{n})$'s are i.i.d. in the $\widetilde{n}$-coordinate, with finite range dependence in the $\widetilde{x}$-coordinate. Using (3.3), (3.4) and (3.5), we can employ e.g. the Liggett-Schonman-Stacey device ([LSS97] or [Lig99, Thm. B26]).

For the second part consider for each $k \in \mathbb{N}$ the process $\eta^{(k)} = (\eta^{(k)}_n)_{n \ge -kL_\mathrm{t}}$ which starts from $\eta^{(k)}_{-kL_\mathrm{t}} = \eta^{\mathrm{ref}}$ and evolves according to (3.1) for $n \ge -kL_\mathrm{t}$, using given $\widetilde{U}$'s which are coupled to $\widetilde{\omega}$'s as above so that $\widetilde{U} \ge \widetilde{\omega}$ holds. We see from the coupling properties guaranteed by Assumption 3.2 and Lemma 3.11 below that the law of $\eta^{(k)}$ restricted to any finite space-time window converges.

By a diagonal argument we can take a subsequence $k_m \nearrow \infty$ such that $\eta_n(x) := \lim_{m\to\infty} \eta_n^{(k_m)}(x)$ exists a.s. for all $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}$, then (3.8) and (3.9) hold by construction.

The fact that the law of limit is the unique non-trivial ergodic equilibrium can be proved analogously to [BD07, Cor. 4]. □

Remark 3.5 (Clarification about the relation between $\widetilde{\xi}$ and η)**.** The contact process $\widetilde{\xi}$ is defined here with respect to $\widetilde{\omega}$ analogously to the definition of the discrete time contact process η with respect to ω in (2.3). The rationale behind this change of notation is that throughout the paper η is a stationary population process (contact process in Section 2 and logistic BRW in Section 4) and the random walk X is interpreted as an ancestral lineage of an individual from that population. The coarse grained contact process $\widetilde{\xi}$ plays a different role. In particular, the knowledge of $\widetilde{\xi}$ does not determine X; cf. Section 3.11.

Finally, we need the following technical assumption which is enough for our purposes but can be relaxed presumably.

Assumption 3.6 (Irreducibility on G_η)**.** On $\widetilde{G}(\widetilde{x},\widetilde{n})$, conditioned on seeing a particular local configuration $\chi \in \mathbb{Z}_+^{B_{2L_\mathrm{s}}(L_\mathrm{s}\widetilde{x})} \cap G_\eta$ at the bottom of the space-time box [time coordinate $\widetilde{n}L_\mathrm{t}$], every configuration $\chi' \in G_\eta$ has a uniformly positive chance of appearing at the top of the space-time box [time coordinate $(\widetilde{n}+1)L_\mathrm{t}$].

Remark 3.7. For the (discrete time) contact process the above assumptions can be checked easily in the case $d = 1$ when p is sufficiently close to 1. For G_η we could for instance take configurations η with

$$|\{\|x\| \le L_s/2, \eta(x) = 1\}| \ge \frac{2}{3} L_s.$$

For G_U we could take configurations of ω's for which this property propagates to the top of the block and its neighbours irrespective of the positions of the 1's in the initial configuration (cf. construction in the proof of Lemma 2.12). For $d \ge 2$ one can reduce the argument to the one-dimensional case.

3.2 Assumptions for random walk on η

We now state the assumptions for the random walk $X = (X_k)_{k=0,1,\dots}$ in the random environment η. To this end let $\hat{U} := (\hat{U}(x,k) : x \in \mathbb{Z}^d, k \in \mathbb{Z}_+)$ be an independent space-time i.i.d. field of uniform([0, 1]) random variables, and let $\varphi_X : \mathbb{Z}_+^{B_{R_X}} \times \mathbb{Z}_+^{B_{R_X}} \times [0,1] \to B_{R_X}$ a measurable function, where $R_X \in \mathbb{N}$ is an upper bound on the jump size as well as on the dependence range. Given η, let $X_0 = 0$ and put

$$X_{k+1} := X_k + \varphi_X\big(\, \theta^{X_k}\eta_{-k}\big|_{B_{R_X}}, \theta^{X_k}\eta_{-k-1}\big|_{B_{R_X}}, \hat{U}(X_k,k)\big), \quad k = 0,1,\dots. \tag{3.10}$$

Note that, as usual, forwards time direction for X is backwards time direction for η.

Assumption 3.8 (Closeness to SRW while on $\widetilde{G} = 1$)**.** A walker with dynamics (3.10) starting from the middle half of the top of a box with $\widetilde{G}(\widetilde{x},\widetilde{n}) = 1$ stays inside the box with high probability:

$$\min_{z\,:\,\|z-\widetilde{x}\|\le L_\mathrm{s}/2} \mathbb{P}\Big(\max_{(n-1)L_\mathrm{t}<k\le nL_\mathrm{t}} \|X_k - z\| \le \frac{L_\mathrm{s}}{4} \,\Big|\, X_{(n-1)L_\mathrm{t}} = z, \widetilde{G}(\widetilde{x},\widetilde{n}) = 1, \eta\Big) \ge 1-\varepsilon. \tag{3.11}$$

Remark 3.9. (a) Note that (3.11) translates into the upper bound $\varepsilon R_X + L_\mathrm{s}/(4L_\mathrm{t})$ for the speed of the walk X on a block satisfying $\widetilde{G}(\widetilde{x},\widetilde{n}) = 1$. The factor $\frac{1}{4}$ in $\frac{L_\mathrm{s}}{4}$ inside (3.11) is somewhat arbitrary. Depending on ε_U and the ratio of L_s to L_t one could use a different factor.

(b) The simple Assumption 3.8 allows to obtain a rough a priori bound on the speed of the walk and suffices for our purposes here, a more elaborate version could require successful couplings of the coordinates of X with true random walks with a small drift while on the box, similar to the proof of Lemma 2.15.

Assumption 3.10 (Symmetry of φ_X w.r.t. point reflection)**.** Let ϱ be the (spatial) point reflection operator acting on η, i.e., $\varrho\eta_k(x) = \eta_k(-x)$ for any $k \in \mathbb{Z}$ and $x \in \mathbb{Z}^d$. We assume

$$\varphi_X\big(\varrho\eta_0|_{B_{R_X}}, \varrho\eta_{-1}|_{B_{R_X}}, \hat{U}(0,0)\big) = -\varphi_X\big(\eta_0|_{B_{R_X}}, \eta_{-1}|_{B_{R_X}}, \hat{U}(0,0)\big). \tag{3.12}$$

Note that (3.12) guarantees that the averaged speed of X will be 0.

3.3 The determining cluster of a block

We now explain how Theorem 2 can be proved using similar ideas as in Section 2. In order to avoid repetitions and to keep the lenght of the paper acceptable, we only explain the major differences to the previous proof.

The main change that should be dealt with is the fact that the construction of the random walk X requires not only the knowledge of the coarse-grained oriented percolation $\widetilde{\xi}$, but also of the underlying random environment η. This additional information on η should be controlled at regeneration times. To tackle this problem, Assumption 3.2 and Lemma 3.4 play the key role. By this lemma, the value of $\eta(x,n)$ can be reconstructed by looking only on the driving noise U in certain finite set 'below' (x,n).

Formally, for $(\widetilde{x},\widetilde{n}) \in \mathbb{Z}^d \times \mathbb{Z}$ we define its the determining cluster $\mathsf{DC}(\widetilde{x},\widetilde{n})$ by the following recursive algorithm:

1. Initially, put $\widetilde{k} := \widetilde{n}$, $\mathsf{DC}(\widetilde{x},\widetilde{n}) := \{(\widetilde{x},\widetilde{n})\}$.
2. If $\widetilde{\xi}(\widetilde{y},\widetilde{k}) = 1$ for all $(\widetilde{y},\widetilde{k}) \in \mathsf{DC}(\widetilde{x},\widetilde{n})$: Stop.
3. Otherwise, for all blocks where this condition fails, add every block one time layer below that could have influenced it (cf. Remark 3.3), that is replace

$$\mathsf{DC}(\widetilde{x},\widetilde{n}) := \mathsf{DC}(\widetilde{x},\widetilde{n}) \cup \big\{(\widetilde{z},\widetilde{k}-1) : \|\widetilde{z}-\widetilde{y}\| \le K_\eta \text{ for some } \widetilde{y} \text{ with } \widetilde{\xi}(\widetilde{y},\widetilde{k}) = 0\big\}, \tag{3.13}$$

put $\widetilde{k} := \widetilde{k}-1$ and go back to Step 2.

Lemma 3.11. *For ε_U small enough, the height (and thus diameter) of $\mathsf{DC}(\widetilde{x},\widetilde{n})$, defined as*

$$\text{height}(\mathsf{DC}(\widetilde{x},\widetilde{n})) := \max\big\{\widetilde{n}-\widetilde{k} : (\widetilde{y},\widetilde{k}) \in \mathsf{DC}(\widetilde{x},\widetilde{n})\big\}, \tag{3.14}$$

is finite a.s. with exponential tail bounds.

Proof. Cf. proof of Lemma 2.10; alternatively see Lemma 7 of [Dur92], or proof of Lemma 14 in [BD07]. □

Remark 3.12. On $\{\widetilde{\xi}(\widetilde{x},\widetilde{n}) = 1\}$, $\eta|_{\mathsf{block}(\widetilde{x},\widetilde{n})}$ is a function of local randomness, in fact it is then determined by $U|_{\mathsf{block}_5(\widetilde{x},\widetilde{n}) \cup \mathsf{block}_5(\widetilde{x},\widetilde{n}-1)}$.

Thus, η on $\mathsf{block}(\widetilde{x},\widetilde{n})$ is determined by "wet boundary" plus local randomness in (a slightly "thickened" version of) $\mathsf{DC}(\widetilde{x},\widetilde{n})$. $\mathsf{DC}(\widetilde{x},\widetilde{n})$ is the analogue of the "determining triangle" $D(x,n)$ from (2.43) in this coarse-grained context.

Proof. Consider the system $(\eta'_n : (\widetilde{n}-1)L_{\rm t} \le n \le (\widetilde{n}+1)L_{\rm t})$ which starts from $\eta'_{(\widetilde{n}-1)L_{\rm t}} = \eta^{\rm ref}$ and uses the fixed boundary condition $\eta'_n(y) = \eta^{\rm ref}(y)$ for $\|y - L_{\rm s}\widetilde{x}\| > 5L_{\rm s}$ and $(\widetilde{n}-1)L_{\rm t} < n \le (\widetilde{n}+1)L_{\rm t}$; for $(y,n) \in \mathsf{block}_5(\widetilde{x},\widetilde{n}) \cup \mathsf{block}_5(\widetilde{x},\widetilde{n}-1)$; $\eta'_n(y)$ is computed using (3.1) with the same realisations of U as the true system η.

Note that $\widetilde{\xi}(\widetilde{x},\widetilde{n}) = 1$ implies that $U|_{\mathsf{block}_4(\widetilde{x},\widetilde{n})} \in G_U$ and

$$\max_{\|\widetilde{e}\|\le 1} \mathbb{1}_{G_U}\big(U|_{\mathsf{block}_4(\widetilde{x}+\widetilde{e},\widetilde{n}-1)}\big) \mathbb{1}_{G_\eta}\big(\eta_{(\widetilde{n}-1)L_{\rm t}}|_{B_{2L_{\rm s}}(L_{\rm s}(\widetilde{x}+\widetilde{e}))}\big) = 1$$

Now use (3.4) to see that $\eta'_{\widetilde{n}L_{\rm t}}$ and $\eta_{\widetilde{n}L_{\rm t}}$ agree on $B_{2L_{\rm s}}(L_{\rm s}\widetilde{x})$, the use this and (3.4)–(3.5) to verify that η' and η agree on $\mathsf{block}(\widetilde{x},\widetilde{n})$. $\square$

3.4 A regeneration structure

In this section we construct regeneration times similar to those constructed in Section 2.3. First we need to introduce the analogue of the "tube around the path" and its "decoration with determining triangles"; cf. equations (2.42), (2.43) and (2.44). We set

$$\widetilde{V}_{\widetilde{m}} := \{\widetilde{x} : \exists k, (\widetilde{m}-1)L_{\rm t} \le k \le \widetilde{m}L_{\rm t}, \|X_k - \widetilde{x}L_{\rm t}\| \le L_{\rm s} + R_X\}, \tag{3.15}$$

$$\mathsf{Tube}_{\widetilde{n}} := \bigcup_{\widetilde{m}\le\widetilde{n}} \widetilde{V}_{\widetilde{m}} \times \{\widetilde{m}\}, \tag{3.16}$$

$$\mathsf{DTube}_{\widetilde{n}} := \bigcup_{(\widetilde{x},\widetilde{\jmath})\in\mathsf{Tube}_{\widetilde{n}}} \mathsf{DC}(\widetilde{x},\widetilde{\jmath}). \tag{3.17}$$

We define the coarse-graining function $\widetilde{\pi} : \mathbb{Z}^d \to \mathbb{Z}^d$ by

$$\widetilde{\pi}(x) = \widetilde{\pi}(x_1,\dots,x_d) = (\widetilde{x}_1,\dots,\widetilde{x}_d) := \Big(\Big\lceil \frac{x_1}{L_{\rm s}} - \frac{1}{2}\Big\rceil, \dots, \Big\lceil \frac{x_d}{L_{\rm s}} - \frac{1}{2}\Big\rceil\Big), \tag{3.18}$$

and denote by $\widetilde{\rho}(x)$ the relative position of x inside the block centred at $\widetilde{x}L_{\rm s}$, i.e. we set

$$\widetilde{\rho}(x) := x - \widetilde{x}L_{\rm s}. \tag{3.19}$$

We define the *coarse grained random walk* $\widetilde{X} = (\widetilde{X}_{\widetilde{n}})_{\widetilde{n}=0,1,\dots}$ and – to preserve the Markovian structure – we also take into account the *relative positions* $\widetilde{Y} = (\widetilde{Y}_{\widetilde{n}})_{\widetilde{n}=0,1,\dots}$ by

$$\widetilde{X}_{\widetilde{n}} := \widetilde{\pi}(X_{\widetilde{n}L_{\rm t}}) \quad \text{and} \quad \widetilde{Y}_{\widetilde{n}} := \widetilde{\rho}(X_{\widetilde{n}L_{\rm t}}). \tag{3.20}$$

Between the original random walk and the coarse grained components just defined we have the following relation:

$$X_{\widetilde{n}L_{\rm t}} = \widetilde{X}_{\widetilde{n}}L_{\rm s} + \widetilde{Y}_{\widetilde{n}}.$$

We define the filtration $\widetilde{\mathcal{F}} := (\widetilde{\mathcal{F}}_{\widetilde{n}})_{\widetilde{n}=0,1,\dots}$ by

$$\widetilde{\mathcal{F}}_{\widetilde{n}} := \sigma\big((\widetilde{X}_{\widetilde{\jmath}},\widetilde{Y}_{\widetilde{\jmath}}) : 0 \le \widetilde{\jmath} \le \widetilde{n}\big) \vee \sigma\big(\widetilde{\omega}(\widetilde{y},\widetilde{\jmath}), \widetilde{\xi}(\widetilde{y},\widetilde{\jmath}), U|_{\mathsf{block}_4(\widetilde{y},\widetilde{\jmath})} : (\widetilde{y},\widetilde{\jmath}) \in \mathsf{DTube}_{\widetilde{n}}\big). \tag{3.21}$$

To mimic the proofs of Section 2 for the model considered here we need the following ingredients:

1. We have (as in Lemma 2.15): There exist $\widetilde{s}_{\max}$ (that is close to $\frac{1}{4}$ under our assumptions) and C, c such that

$$\mathbb{P}\big(\|\widetilde{X}_{\widetilde{n}}\| > \widetilde{s}_{\max}\widetilde{n}\big) \le Ce^{-c\widetilde{n}}. \tag{3.22}$$

2. We need stopping times (analogous to σ's in (2.46)). We set

$$\widetilde{D}_{\widetilde{n}} = \widetilde{n} + \max\{\mathrm{height}(\mathsf{DC}(\widetilde{x},\widetilde{n})) : \widetilde{x} \in \widetilde{V}_{\widetilde{n}}\} \tag{3.23}$$

$$\widetilde{\sigma}_0 := 0, \quad \widetilde{\sigma}_i := \min\Big\{\widetilde{m} > \widetilde{\sigma}_{i-1} : \max_{\widetilde{\sigma}_{i-1} \le \widetilde{n} \le \widetilde{m}} \widetilde{D}_{\widetilde{n}} \le \widetilde{m}\Big\}. \tag{3.24}$$

Lemma 3.13. *When $1-\varepsilon_{\widetilde{\omega}}$ is sufficiently close to 1 there exist finite positive constants c and C so that*

$$\mathbb{P}\big(\widetilde{\sigma}_{i+1} - \widetilde{\sigma}_i > \widetilde{n} \,\big|\, \widetilde{\mathcal{F}}_{\widetilde{\sigma}_i}\big) \le Ce^{-c\widetilde{n}} \quad \textit{for all } \widetilde{n} = 1,2,\dots,\ i = 0,1,\dots \ \textit{a.s.}, \tag{3.25}$$

in particular, all $\widetilde{\sigma}_i$ are a.s. finite. Furthermore,

$$\mathscr{L}\big((\widetilde{\omega}(\cdot, -\widetilde{\jmath} - \widetilde{\sigma}_i)_{\widetilde{\jmath}=0,1,\dots} \,\big|\, \widetilde{\mathcal{F}}_{\widetilde{\sigma}_i}\big) \succcurlyeq \mathscr{L}\big((\widetilde{\omega}(\cdot, -\widetilde{\jmath})_{\widetilde{\jmath}=0,1,\dots}\big) \quad \textit{for every } i = 0,1,\dots \ \textit{a.s.}, \tag{3.26}$$

where "$\succcurlyeq$" denotes stochastic domination.

Proof. Analogous to the proof of Lemma 2.16 (see also Lemma 3.11). □

We say that $\widetilde{n}$ is a (b,s)-cone time point for the decorated path beyond $\widetilde{m}$ (with $\widetilde{m} < \widetilde{n}$) if (cf. (2.56))

$$\begin{aligned} &\mathsf{DTube}_{\widetilde{n}} \cap \big(\mathbb{Z}^d \times \{-\widetilde{n}, -\widetilde{n}+1, \dots, -\widetilde{m}\}\big) \\ &\qquad \subset \big\{(\widetilde{x}, -\widetilde{\jmath}) : \widetilde{m} \le \widetilde{\jmath} \le \widetilde{n}, \|\widetilde{x} - \widetilde{X}_{\widetilde{n}}\| \le b + s(\widetilde{n} - \widetilde{\jmath})\big\}. \end{aligned} \tag{3.27}$$

In words, (as in Section 2.3) $\widetilde{n}$ is a cone time point for the decorated path beyond $\widetilde{m}$ if the space-time path $(\widetilde{X}_{\widetilde{\jmath}}, -\widetilde{\jmath})_{\widetilde{\jmath}=\widetilde{m},\dots,\widetilde{n}}$ together with its "tilde"-decorations is contained in the cone with base radius b, slope s and base point $(\widetilde{X}_{\widetilde{n}}, -\widetilde{n})$.

Lemma 3.14. *There exist suitable b and s ($s > \widetilde{s}_{\max}$ but $0 < s - \widetilde{s}_{\max} \ll 1$ can be chosen small) such that for all finite $\widetilde{\mathcal{F}}$-stopping times $\widetilde{T}$ with $\widetilde{T} \in \{\widetilde{\sigma}_i : i \in \mathbb{N}\}$ a.s. (i.e., $\widetilde{T} = \widetilde{\sigma}_J$ for a suitable random index J) and all $\widetilde{k} \in \mathbb{N}$, with $\widetilde{T}' := \inf\{\widetilde{\sigma}_i : \widetilde{\sigma}_i \ge \widetilde{k}\}$*

$$\mathbb{P}\big(\widetilde{T}' \textit{ is a } (b,s)\textit{-cone time point for the decorated path beyond } \widetilde{T} \,\big|\, \widetilde{\mathcal{F}}_{\widetilde{T}}\big) \ge 1 - \varepsilon \tag{3.28}$$

a.s. on $\{\widetilde{T} < \widetilde{k}\}$.

Proof. Analogous to the proof of Lemma 2.20. Intermediate results, that is Lemma 2.18 and Lemma 2.19, can be adapted to the present situation. □

We now define "good configurations" of $\widetilde{\omega}$'s (analogous to (2.59)). Recall the definition of a cone shell in (2.25). Let $\widetilde{G}(b_{\mathrm{inn}}, b_{\mathrm{out}}, s_{\mathrm{inn}}, s_{\mathrm{out}}, h) \subset \{0,1\}^{\mathsf{cs}(b_{\mathrm{inn}}, b_{\mathrm{out}}, s_{\mathrm{inn}}, s_{\mathrm{out}}, h)}$ be the set of possible $\widetilde{\omega}$-configurations in $\mathsf{cs}(b_{\mathrm{inn}}, b_{\mathrm{out}}, s_{\mathrm{inn}}, s_{\mathrm{out}}, h)$ with the property

$$\begin{aligned} &\forall\, \widetilde{\xi}(\cdot,0), \widetilde{\xi}'(\cdot,0) \in \{0,1\}^{\mathbb{Z}^d} \text{ with } \widetilde{\xi}(\cdot,0)|_{B_{b_{\mathrm{out}}}(0)} = \widetilde{\xi}'(\cdot,0)|_{B_{b_{\mathrm{out}}}(0)} \equiv 1 \quad \text{and} \\ &\quad \widetilde{\omega} \in \{0,1\}^{\mathbb{Z}^d \times \{1,\dots,h\}} \text{ with } \widetilde{\omega}|_{\mathsf{cs}(b_{\mathrm{inn}}, b_{\mathrm{out}}, s_{\mathrm{inn}}, s_{\mathrm{out}}, h)} \in \widetilde{G}(b_{\mathrm{inn}}, b_{\mathrm{out}}, s_{\mathrm{inn}}, s_{\mathrm{out}}, h) : \\ &\qquad \widetilde{\xi}(\widetilde{x},\widetilde{n}) = \widetilde{\xi}'(\widetilde{x},\widetilde{n}) \text{ for all } (\widetilde{x},\widetilde{n}) \in \mathsf{cone}(b_{\mathrm{inn}}, s_{\mathrm{inn}}, h) \end{aligned} \tag{3.29}$$

where $\widetilde{\xi}$ and $\widetilde{\xi}'$ are both constructed from time 0 using the same $\widetilde{\omega}$'s, i.e. when A and A' are subsets of $\mathbb{Z}^d$ with $\mathbb{1}_A = \widetilde{\xi}(\cdot,0)$ and $\mathbb{1}_{A'} = \widetilde{\xi}'(\cdot,0)$ then (cf. (2.1))

$$\widetilde{\xi}(\cdot,n) = \{\widetilde{x} \in \mathbb{Z}^d : A \times \{0\} \to^{\widetilde{\omega}} (\widetilde{x},\widetilde{n})\} \quad \text{and} \quad \widetilde{\xi}'(\cdot,n) = \{\widetilde{x} \in \mathbb{Z}^d : A' \times \{0\} \to^{\widetilde{\omega}} (\widetilde{x},\widetilde{n})\}.$$

Observe that when $\widetilde{\xi}(\widetilde{x},0)=1$ in ball $B_{b_{\rm out}}(0)$ and $\widetilde{\omega}|_{\mathsf{cs}(b_{\rm inn},b_{\rm out},s_{\rm inn},s_{\rm out},h)} \in \widetilde{G}(b_{\rm inn},b_{\rm out},s_{\rm inn},s_{\rm out},h)$ then

$$\{\eta_n(x) : (x,n) \in \mathsf{block}(\widetilde{x},\widetilde{n}),\ (\widetilde{x},\widetilde{n}) \in \mathsf{cone}(b_{\rm inn},s_{\rm inn},h)\}$$

is a function of $\eta_0(y)$, $\|y\| \le b_{\rm out}L_{\rm s}$ and $U|_{\mathsf{block}_4(\widetilde{x},\widetilde{n})}$, $(\widetilde{x},\widetilde{n}) \in \mathsf{cone}(b_{\rm inn},s_{\rm inn},h)$. In particular, if we start with different η'_0 and U' with $\eta'_0(y)=\eta_0(y)$, $\|y\| \le b_{\rm out}L_{\rm s}$ and $U'|_{\mathsf{block}_4(\widetilde{x},\widetilde{n})} = U|_{\mathsf{block}_4(\widetilde{x},\widetilde{n})}$, $(\widetilde{x},\widetilde{n}) \in \mathsf{cone}(b_{\rm inn},s_{\rm inn},h)$ then

$$\eta_n(x)=\eta'_n(x) \text{ for all } (x,n) \in \mathsf{block}(\widetilde{x},\widetilde{n}), (\widetilde{x},\ \widetilde{n}) \in \mathsf{cone}(b_{\rm inn},s_{\rm inn},h).$$

Proof sketch for Theorem 2. We now have all the ingredients for the regeneration construction, to imitate the proof of Theorem 1.

First we choose a sequence $t_0,t_1,\dots$ with $t_\ell \uparrow \infty$ such that (2.62) is satisfied with $\widetilde{s}_{\max}$ replacing $s_{\max}$ and parameters $b_{\rm out}$, $s_{\rm out}$, $b_{\rm inn}$ and $s_{\rm inn}$ adapted from Lemma 3.14.

1. Go to the first $\widetilde{\sigma}_i$ after t_1, check if $\widetilde{\xi}$ in the $b_{\rm out}$-neighbourhood of $(\widetilde{X}_{\widetilde{\sigma}_i},-\widetilde{\sigma}_i)$ is $\equiv 1$, the path (together with its tube and decorations) has stayed inside the interior of the corresponding conical shell based at the current space-time position and the $\widetilde{\omega}$'s in that conical shell are in the good set as defined in (3.29). This has positive (in fact, very high) probability (cf. Lemma 3.14) and if it occurs, we have found the "regeneration time".
2. If the event fails, we must try again. We successively check at times t_2, t_3, etc.: If not previously successful, at the ℓ-th step let $\widetilde{\sigma}_{J(\ell)}$ be the first $\widetilde{\sigma}_i$ after t_ℓ, check if $\widetilde{\sigma}_{J(\ell)}$ is a cone point for the decorated path beyond $t_{\ell-1}$ with $\|\widetilde{X}_{\widetilde{\sigma}_{J(\ell)}}\| \le \widetilde{s}_{\max}\widetilde{\sigma}_{J(\ell)}$, the η's in the $b_{\rm out}$-neighbourhood of $(X_{\tilde{\sigma}_\ell},-\widetilde{\sigma}_\ell)$ are $\equiv 1$, $\widetilde{\omega}$'s in the corresponding conical shell are in the good set as defined in (3.29) and the path (with tube and decorations) up to time $t_{\ell-1}$ is contained in the box of diameter $s_{\rm out}t_{\ell-1}+b_{\rm out}$ and height $t_{\ell-1}$. If this all holds, we have found the regeneration time.

   (We may assume that $\widetilde{\sigma}_{J(\ell-1)}$ is suitably close to $t_{\ell-1}$, this has very high probability by an adaptation of Lemma 2.18.)
3. The path containment property holds from some finite ℓ_0 on. Given the construction and all the information obtained from it up to the $(\ell-1)$-th step, the probability that the other requirements occur is uniformly high: For the cone time property use Lemma 3.14 with $\widetilde{k}=t_\ell$; use (3.26) to verify that the probability to see $\widetilde{\xi}\equiv 1$ in a box around $(\widetilde{X}_{\widetilde{\sigma}_{J(\ell)}},-\widetilde{\sigma}_{J(\ell)})$ is high; use (a notational adaptation of) Lemma 2.21 to check that conditional on the construction so far the probability that the $\widetilde{\omega}$'s in the corresponding conical shell are in the good set $\widetilde{G}(b_{\rm inn},b_{\rm out},s_{\rm inn},s_{\rm out},t_\ell)$ is high. Note that these $\widetilde{\omega}$'s have not yet been looked at.
4. We thus construct a random time $\widetilde{R}_1$ with the following properties:
   (i) $\widetilde{\xi}(\widetilde{X}_{\widetilde{R}_1}+\widetilde{y},\widetilde{R}_1)=1$ for all $\|\widetilde{y}\| \le b_{\rm out}$;
   (ii) the decorated path up to time $\widetilde{R}_1$ is in $\mathsf{cone}(b_{\rm inn},s_{\rm inn},\widetilde{R}_1)$ centred at $(\widetilde{X}_{\widetilde{R}_1},\widetilde{R}_1)$;
   (iii) after centring the cone at base point $(\widetilde{X}_{\widetilde{R}_1},\widetilde{R}_1)$, $\widetilde{\omega}|_{\mathsf{cs}(b_{\rm inn},b_{\rm out},s_{\rm inn},s_{\rm out},\widetilde{R}_1)}$ lies in the good set $\widetilde{G}(b_{\rm inn},b_{\rm out},s_{\rm inn},s_{\rm out},\widetilde{R}_1)$.

   We will thus at most require a geometric number of t_ℓ's to construct the $\widetilde{R}_1$. As in step 4 in the proof of Theorem 1 we obtain

   $$\mathbb{P}(\widetilde{R}_1 \ge \widetilde{n}) \le \mathbb{P}(\text{more than } \log\widetilde{n}/\log c \text{ steps needed}) \le \delta^{\log\widetilde{n}/\log c} = \widetilde{n}^{-a},$$

   where again a can be chosen large when p is close to 1.

5. Set

$$\widehat{\eta}_1 := (\eta_{-L_t \widetilde{R}_1}(x + \widetilde{X}_{\widetilde{R}_1}) : \|x\| \le b_{\text{out}} L_s)$$
$$\widehat{Y}_1 := \widetilde{Y}_{\widetilde{R}_1}$$

Now we shift the space-time origin to $(\widetilde{X}_{\widetilde{R}_1}, \widetilde{R}_1)$ (on coarse grained level). Then we start afresh conditioned on seeing

(i) configuration $\widetilde{\xi} \equiv 1$ in the b_{out}-box around 0 (on the coarse grained level);

(ii) $\widehat{\eta}_1$ on the $b_{\text{out}} L_s$ box (on the "fine" level);

(iii) Displacement of the walker on the fine level relative to the centre of the corresponding coarse graining box given by $\widehat{Y}_1$.

6. We iterate the above construction to obtain a sequence of random times $\widetilde{R}_i$, positions $\widetilde{X}_{\widetilde{R}_i}$, relative displacements $\widehat{Y}_i$ and local configurations $\widehat{\eta}_i$. By construction

$$\big(\widetilde{X}_{\widetilde{R}_i} - \widetilde{X}_{\widetilde{R}_{i-1}}, \widetilde{R}_i - \widetilde{R}_{i-1}, \widehat{Y}_i, \widehat{\eta}_i\big)_{i \in \mathbb{N}}$$

is a Markov chain. Furthermore, $\big(\widehat{Y}_i, \widehat{\eta}_i\big)_{i \in \mathbb{N}}$ is itself a Markov chain (with a finite state space) and the increments $\big(\widetilde{X}_{\widetilde{R}_{i+1}} - \widetilde{X}_{\widetilde{R}_i}, \widetilde{R}_{i+1} - \widetilde{R}_i\big)$ depend only on $\big(\widehat{Y}_i, \widehat{\eta}_i\big)$.

The regeneration structure implies Theorem 2 by fairly standard arguments. Note that along the random times $L_t \widetilde{R}_n$,

$$X_{L_t \widetilde{R}_n} = \widehat{Y}_i + \sum_{i=1}^{n} L_s\big(\widetilde{X}_{\widetilde{R}_i} - \widetilde{X}_{\widetilde{R}_{i-1}}\big)$$

is an additive functional of a well-behaved Markov chain (with exponential mixing properties) and

$$\mathbb{E}\big[(\widetilde{R}_{i+1} - \widetilde{R}_i)^a \,\big|\, \widehat{Y}_i, \widehat{\eta}_i\big] < \infty, \quad \mathbb{E}\big[\|\widetilde{X}_{\widetilde{R}_{i+1}} - \widetilde{X}_{\widetilde{R}_i}\|^a \,\big|\, \widehat{Y}_i, \widehat{\eta}_i\big] < \infty$$

for some $a > 2$ uniformly in $\widehat{Y}_i, \widehat{\eta}_i$ (cf. Step 4).

Note that the speed must be 0 by the assumed symmetry (see (3.12)). □

4 Example: An ancestral lineage in logistic branching random walks

In this section we consider a concrete stochastic model for a locally regulated, spatially distributed population that was introduced and studied in [BD07] and we refer the reader to that paper for a more detailed description, interpretation, context and properties. We call this logistic branching random walk because the function f in (4.1), which describes the dynamics of the population means over one generation, is a 'spatial relative' of the classical logistic function $x \mapsto x(1-x)$ which appears in many (deterministic) models for the growth of populations under limited resources.

4.1 Ancestral lineages in a locally regulated model

Let $p = (p_{xy})_{x,y\in\mathbb{Z}^d} = (p_{y-x})_{x,y\in\mathbb{Z}^d}$ be a symmetric aperiodic stochastic kernel with finite range $R_p \geq 1$. Furthermore let $\lambda = (\lambda_{xy})_{x,y\in\mathbb{Z}^d}$ be a non-negative symmetric kernel satisfying $0 \leq \lambda_{xy} = \lambda_{0,y-x}$ and having finite range R_λ. We set $\lambda_0 \coloneqq \lambda_{00}$ and for a configuration $\zeta \in \mathbb{R}_+^{\mathbb{Z}^d}$ and $x \in \mathbb{Z}^d$ we define

$$f(x;\zeta) \coloneqq \zeta(x)\big(m - \lambda_0\zeta(x) - \sum_{z\neq x}\lambda_{xz}\zeta(z)\big)^+. \tag{4.1}$$

We consider a population process $\eta \coloneqq (\eta_n)_{n\in\mathbb{Z}}$ with values in $\mathbb{Z}_+^{\mathbb{Z}^d}$, where $\eta_n(x)$ is the number of individuals at time $n \in \mathbb{Z}$ at site $x \in \mathbb{Z}^d$. Before giving a formal definition of η let us describe the dynamics informally: Given the configuration η_n in generation n, each individual at x (if any individuals present) has a Poisson distributed number of offspring with mean $f(x;\eta_n)/\eta_n(x)$, independent of everything else. Offspring then take an independent random walk step according to the kernel p from the location of their mother, the offspring of all individuals together form the next generation $n+1$. For obvious reasons p and λ are referred to as *migration* and *competition* kernels respectively. Note that in the case $\lambda \equiv 0$ the process η is literally a branching random walk.

Let us now give a formal construction of η. Let

$$U \coloneqq \{U_n^{(y,x)} : n \in \mathbb{Z},\ x, y \in \mathbb{Z}^d, \|x-y\| \leq R_p\} \tag{4.2}$$

be a collection of independent Poisson processes on $[0,\infty)$ with intensity measures of $U_n^{(y,x)}$ given by $p_{yx}\,dt$. The natural state space for each $U_n^{(y,x)}$ is

$$\widetilde{\mathcal{D}} \coloneqq \big\{\psi : [0,\infty) \to \mathbb{Z}_+ :\ \psi \text{ càdlàg, piece-wise constant, only jumps of size } 1\big\}, \tag{4.3}$$

which is a Polish space as a closed subset of the (usual) Skorokhod space $\mathcal{D}$. For given $\eta_n \in \mathbb{Z}_+^{\mathbb{Z}^d}$, define $\eta_{n+1} \in \mathbb{Z}_+^{\mathbb{Z}^d}$ via

$$\eta_{n+1}(x) \coloneqq \sum_{y\,:\,\|x-y\|\leq R_p} U_n^{(y,x)}\big(f(y;\eta_n)\big), \quad x \in \mathbb{Z}^d. \tag{4.4}$$

Note that for each x, the right-hand side of (4.4) is a finite sum of (conditionally) Poisson random variables with finite means bounded by $\|f\|_\infty$. Thus, (4.4) is well defined for any initial condition – in this discrete time scenario, no growth condition at infinity, etc. is necessary. Furthermore we note that by well known properties of Poisson distribution processes η_{n+1} is a family of independent random variables with

$$\eta_{n+1}(x) \sim \mathrm{Pois}\Big(\sum_{y\in\mathbb{Z}^d} p_{yx} f(y;\eta_n)\Big), \quad x \in \mathbb{Z}^d. \tag{4.5}$$

For $-\infty < m < n$ set

$$\mathcal{G}_{m,n} \coloneqq \sigma(U_k^{(x,y)} : m \leq k < n,\ x, y \in \mathbb{Z}^d). \tag{4.6}$$

By iterating (4.4), we can define a random family of $\mathcal{G}_{m,n}$-measurable mappings

$$\Phi_{m,n} : \mathbb{Z}_+^{\mathbb{Z}^d} \to \mathbb{Z}_+^{\mathbb{Z}^d},\ -\infty < m < n \quad \text{such that} \quad \eta_n = \Phi_{m,n}(\eta_m). \tag{4.7}$$

To this end define $\Phi_{m,m+1}$ as in (4.4) via

$$(\Phi_{m,m+1}(\zeta))(x) \coloneqq \sum_{y\,:\,\|x-y\|\le R_p} U_m^{(y,x)}\big(f(y;\zeta)\big) \quad \text{for } y \in \mathbb{Z}^d \text{ and } \zeta \in \mathbb{Z}_+^{\mathbb{Z}^d} \tag{4.8}$$

and then put

$$\Phi_{m,n} \coloneqq \Phi_{n-1,n} \circ \cdots \circ \Phi_{m,m+1}. \tag{4.9}$$

Using these mappings we can define the dynamics of $(\eta_n)_{n=m,m+1,\dots}$ *simultaneously* for all initial conditions $\eta_m \in \mathbb{Z}^{\mathbb{Z}^d}$ for any $m \in \mathbb{Z}$.

Let us for a moment consider the process $\eta = (\eta_n)_{n=0,1,\dots}$. Obviously, the configuration $\zeta \equiv 0$ is an absorbing state for η. Thus, the Dirac measure in this configuration is a trivial invariant distribution of η. In [BD07] it is shown that for certain parameter regions, in particular $m \in (1,4)$ and suitable λ, the population survives with positive probability. For $m \in (1,3)$ (and again suitable λ) the existence and uniqueness of non-trivial invariant distribution is proven. We recall the relevant results for $m \in (1,3)$.

Proposition 4.1 (Survival and complete convergence, [BD07]). *Assume $m \in (1,3)$ and let p and λ be as above.*

(i) There are $\lambda_0^* = \lambda_0^*(m,p) > 0$ and $a^* = a^*(m,p) > 0$ such that if $\lambda_0 \le \lambda_0^*$ and $\sum_{x\neq 0}\lambda_{0x} \le a^*\lambda_0$ then the process $(\eta_n)_{n=0,1,\dots}$ survives with positive probability (if survival for one step has positive probability) and has a unique non-trivial invariant extremal distribution $\bar{\nu}$.

(ii) Conditioned on non-extinction, η_n converges in distribution in the vague topology to $\bar{\nu}$.

Since we are only interested in the regime when the corresponding deterministic system, cf. (4.14) below, is well controlled (and in particular, Prop. 4.1 guarantees that a non-trivial invariant extremal distribution $\bar{\nu}$ exists) we consider the following general assumption.

Assumption 4.2. 1. With the notation from Proposition 4.1 we assume $m \in (1,3)$ and $\sum_{x\neq 0}\lambda_{0x} \le a^*\lambda_0$.

2. $\gamma \coloneqq \sum_x \lambda_{0x}$ is sufficiently small.

Note that a^* is determined by d, m, p and a renormalised $\widetilde{\lambda}$ by the requirement that the left-hand side of (4.19) at $\zeta \equiv m^*$ must be strictly smaller than 1, see Section 4.2 below.

Under this assumption we can (and do from now on) consider the stationary process $\eta = (\eta_n)_{n\in\mathbb{Z}}$ with η_n distributed according to $\bar{\nu}$. From the informal description (after (4.1)) above and the formal definition (4.4) it is clear that the model can be easily enriched with genealogical information; see e.g. Chapter 4 in [Dep08]. Put

$$p_\eta(k;x,y) \coloneqq \frac{p_{yx} f(y;\eta_{-k-1})}{F(x;\eta_{-k-1})} \quad x,y \in \mathbb{Z}^d,\ k \in \mathbb{Z}_+ \tag{4.10}$$

with some arbitrary convention if the denominator is 0, where $F(x;\eta_{-k-1}) \coloneqq \sum_z p_{zx} f(z;\eta_{-k-1})$. For a given η, conditioned on $\eta_0(0) > 0$, let $(X_k)_{k=0,1,2,\dots}$ be a time-inhomogeneous Markov chain with

$$X_0 = 0, \quad \text{and} \quad \mathbb{P}(X_{k+1} = y \,|\, X_k = x, \eta) = p_\eta(k;x,y). \tag{4.11}$$

This is the dynamics of the space-time embedding of the ancestral lineage of an individual sampled at random from the (space-time) origin at stationarity, conditioned on the (full) space-time configuration η. Note that given η, we see from (4.4) that the number of offspring coming from y in generation $-k-1$ that moved to x, $U_{-k}^{(y,x)}\big(f(y;\eta_{-k-1})\big)$, is $\mathrm{Pois}\big(p_{yx}f(y;\eta_{-k-1})\big)$-distributed conditional on the sum over all y in the neighbourhood of x being equal to $\eta_{-k}(x)$. Since a vector of independent Poisson random variables conditioned on its total sum has a multinomial distribution we see that the dynamics of the ancestral lineage are indeed given by (4.11).

Our main result in this section is the following theorem.

Theorem 3 (LLN and averaged CLT)**.** *Assume $d \geq 1$, let the Assumption 4.2 be satisfied and let $\eta = (\eta_n)_{n\in\mathbb{Z}}$ be the stationary process conditioned on $\eta_0(0) > 0$. For the random walk $(X_k)_{k=0,1,\dots}$ defined in (4.11) we have*

$$P_\eta\Big(\frac{1}{k}X_k \to 0\Big) = 1 \quad \textit{for} \quad \mathbb{P}\big(\cdot\,|\,\eta_0(0) > 0\big)\text{-a.a. } \eta, \tag{4.12}$$

and for any $g \in C_b(\mathbb{R}^d)$

$$\mathbb{E}\Big[g\left(X_k/\sqrt{k}\right)\,\Big|\,\eta_0(0) > 0\Big] \xrightarrow{n\to\infty} \Phi(g), \tag{4.13}$$

where Φ is a non-trivial d-dimensional normal law and $\Phi(g) := \int g(x)\,\Phi(dx)$.

Proof. The assertion of the theorem follows from a combination of Proposition 4.6 and Theorem 2. □

4.2 Deterministic dynamics

For comparison, we consider the dynamical system (also called a coupled map lattice)

$$\zeta_n(x) := \sum_{y\in\mathbb{Z}^d} p_{yx} f(y;\zeta_{n-1}), \quad x \in \mathbb{Z}^d,\ n \in \mathbb{N} \tag{4.14}$$

with f from (4.1), with arbitrary initial condition $\zeta_0 \in [0,\infty)^{\mathbb{Z}^d}$ (cf. [BD07, Eq. (5)]). It is easily seen from (4.1) that with

$$m^* = m^*(\lambda) = \frac{m-1}{\sum_z \lambda_{0,z}}, \tag{4.15}$$

$\zeta^*(\cdot) \equiv m^*$ is an equilibrium of the dynamical system $(\zeta_n)_{n=0,1,\dots}$. Furthermore, setting

$$\gamma := \sum_z \lambda_{0,z}, \quad \widetilde{\lambda}_{xy} := \lambda_{xy}/\gamma \quad \text{and} \quad \widetilde{\zeta}_n(x) := \gamma\zeta_n(x)$$

we see from (4.1) that $(\widetilde{\zeta}_n)_{n=0,1,\dots}$ solves

$$\widetilde{\zeta}_n(x) = \sum_{y\in\mathbb{Z}^d} p_{yx}\widetilde{\zeta}_{n-1}(y)\big(m - \widetilde{\lambda}_0\widetilde{\zeta}_{n-1}(y) - \sum_{z\neq x}\widetilde{\lambda}_{yz}\widetilde{\zeta}_{n-1}(z)\big)^+, \tag{4.16}$$

i.e., (4.14) with λ in the function f replaced by $\widetilde{\lambda}$. Thus, we can and shall assume for the rest of this subsection that $\gamma = 1$.

Lemma 4.3. *There exist $\alpha_0 < \alpha < m^* < \beta$, $\varepsilon = \varepsilon(m,\lambda) > 0$, R_0, k_0, N_0 and s_0 such that for all $R \geq R_0$ the following two assertions hold:*

(i) If $\zeta_0(y) \in [\alpha, \beta]$ for all $y \in B_R(x)$ then

$$\zeta_n(y) \in \big[(1+\varepsilon)\alpha, \beta/(1+\varepsilon)\big] \text{ for all } n \geq N_0,\ \|y-x\| \leq R + s_0(n-N_0), \tag{4.17}$$

and

$$\zeta_n(y) \geq \alpha_0 \text{ for all } n \geq 1,\ \|y-x\| \leq R - k_0 + s_0 n. \tag{4.18}$$

(ii) For $(\zeta(y))_{y \in B_{R_\lambda}(x)} \in [\alpha, \beta]^{B_{R_\lambda}(x)}$ we have

$$\sum_{y \in B_{R_\lambda}(x)} \left| \frac{\partial}{\partial \zeta(y)} f(x;\zeta) \right| < 1 - \frac{\varepsilon}{2}. \tag{4.19}$$

Furthermore, $\beta - \alpha > 0$ can be chosen arbitrarily close to 0.

Proof. Assertions (i) and (iii) follow from Lemma 11 and 12 (and arguments in their proofs) in [BD07]. For (ii) see proof of Lemma 13 and in particular Eq. (40) in [BD07]. □

Remark 4.4 (Interpretation of Lemma 4.3)**.** Assertion (i) in the above lemma means that if ζ_0 in the neighbourhood of x is in the interval $[\alpha, \beta]$ around m^* then the regions around x where ζ_n is bounded away from 0 and where it is close to m^* grow at positive speed (after finite number of steps). Assertion (ii) means that the equilibrium $\zeta^*(\cdot) \equiv m^*$ is attracting.

4.3 Coupling reloaded

Remark 4.5 (Initial/boundary conditions on certain space-time regions)**.** Note that for any $n \in \mathbb{N}$, $\Phi_{0,n}$ as defined in (4.8) can be viewed as a function of $(U_m^{(x,y)} : 0 \leq m < n,\ x, y \in \mathbb{Z}^d)$.

Let $L \in \mathbb{N}$, R_p the range of p, put

$$\mathsf{cone}(L, R_p) := \big\{(x,n) \in \mathbb{Z}^d \times \mathbb{Z}_+ : \ \|x\| \leq L + R_p n\big\} \tag{4.20}$$

(recalling (2.23), we have $\mathsf{cone}(L, R_p) = \cup_{h>0} \mathsf{cone}(L, R_p, h)$). For given values of $\eta_k(x)$, $(x,k) \in \big((\mathbb{Z}^d \times \mathbb{Z}_+) \setminus \mathsf{cone}(L, R_p)\big) \cup ([-L, L]^d \times \{0\})$ (we can view the latter set as a "space-time boundary" of $\mathsf{cone}(L, R_p)$), we can define η_n consistently inside $\mathsf{cone}(L, R_p)$ through (4.4).

In fact, we can think of constructing the space-time field η in a two-step procedure: First, generate the values outside $\mathsf{cone}(L, R_p)$ (in any way consistent with the model), then, conditionally on their outcome, use (4.4) inside.

Proposition 4.6. *Let Assumption 4.2 1. be fulfilled. For any $\varepsilon > 0$ we can find γ^* and such that if $\gamma := \sum_x \lambda_{0x} \leq \gamma^*$ there exists a spatial scale L_s and a temporal scale L_t, a set of good configurations G_η and a set of good Poisson process realisations $G_U \subset \widetilde{\mathcal{D}}^{B_{4L_\mathrm{s}}(0) \times \{1,2,\dots,L_\mathrm{t}\}}$ with $\mathbb{P}\big(\,U|_{\mathsf{block}_4(0,0)} \in G_U\big) \geq 1 - \varepsilon$ such that the contraction and coupling conditions (3.4), (3.5) from Section 3 are fulfilled. Furthermore the random walk defined in (4.11) satisfies (3.11) in Section 3.*

Proof. The crucial idea is that using the flow version (4.4) we can augment the coupling argument in Lemma 13 in [BD07] to work with a set of (good) initial conditions

$$\{\eta_0^{(i)} : i \in I\} = \big\{\eta \in \mathbb{Z}_+^{\mathbb{Z}^d} : \alpha/\gamma \leq \eta(x) \leq \beta/\gamma \text{ for } x \in B_{2L_\mathrm{s}}(0)\big\} \tag{4.21}$$

with α, β from (4.17) and the (uncountable) index set I being defined implicitly here.

The proof consists of 6 steps.

For parameters $K_\mathrm{t}' \gg K_\mathrm{s} \gg K_\mathrm{t}''$ to be suitably tuned below, we set

$$\begin{aligned} L_\mathrm{s} &= \lceil K_\mathrm{s} \log(1/\gamma) \rceil \\ L_\mathrm{t} &= L_\mathrm{t}' + L_\mathrm{t}'' \quad \text{with} \quad L_\mathrm{t}' = \lceil K_\mathrm{t}' \log(1/\gamma) \rceil,\ L_\mathrm{t}'' = \lceil K_\mathrm{t}'' \log(1/\gamma) \rceil. \end{aligned}$$

In the first step, we use the propagation properties of the deterministic system as described in Lemma 4.3 together with the fact that for small γ, the relative fluctuations of the driving Poisson processes are typically small to ensure that after time L_t', the "good region" has increased sufficiently.

In the second step we use the flow version (4.4) and its contraction properties to ensure that in a subregion, after L_t'' steps, coupling has occured with high probability.

Several copies of such subregions are then glued together in Steps 3 and 4, in Step 5 we use the fact that in a good region, the relative fluctuations of η are small so that $p_\eta(k; x, y)$ is close to the deterministic kernel p_{xy}; this ensures (3.11).

Finally, in the last step we collect the requirements on the various constants that occured before and verify that they can be fulfilled consistently.

Step 1. Let

$$\mathcal{X}_1 := \Big\{ \max_{\|x\|, \|y\| \le 5L_s, p_{xy} > 0, 0 < n \le L_\mathrm{t}'} \sup_{u \ge \alpha_0/\gamma} \Big| \frac{U_n^{(x,y)}(u)}{p_{xy} u} - 1 \Big| \le \delta \Big\}$$

with α_0 from Lemma 4.3. By standard large deviation estimates for Poisson processes, we have

$$\mathbb{P}(\mathcal{X}_1) \ge 1 - (10 L_\mathrm{s} R_p)^d L_\mathrm{t}' \exp(-c\alpha_0/\gamma) \tag{4.22}$$

(for some fixed constant $c > 0$) which can be made arbitrarily close to 1 by choosing γ small.

By iterating (4.17) in combination with (4.18) we see that

$$\mathcal{X}_1 \cap \big\{ \eta_0(x) \in [\alpha/\gamma, \beta/\gamma] \text{ for } x \in B_{2L_\mathrm{s}}(0) \big\} \subset \big\{ \eta_{L_\mathrm{t}'}(y) \in [\alpha/\gamma, \beta/\gamma] \text{ for } y \in B_{5L_\mathrm{s}}(0) \big\} \tag{4.23}$$

if the ratio $L_\mathrm{t}' : L_\mathrm{s}$ is chosen sufficiently large. To verify this note that we can consider η_n as a perturbation of the deterministic system ζ_n from (4.14) and on $\mathcal{X}_1$ the relative size of the perturbation is small when γ is small (cf. [BD07, Eq. (13) and the proof of Lemma 7]).

Step 2. Let $G_0 \subset \widetilde{\mathcal{D}}^{B_{3L_\mathrm{s}}(0) \times \{1, \dots, L_\mathrm{t}''\}}$ be the set of Poisson process path configurations in the space-time box $B_{3L_\mathrm{s}}(0) \times \{1, \dots, L_\mathrm{t}''\}$ with the property

$$\begin{aligned} \eta_0|_{B_{2L_\mathrm{s}}(0)} \in [\alpha/\gamma, \beta/\gamma]^{B_{2L_\mathrm{s}}(0)},\ U|_{B_{3L_\mathrm{s}}(0) \times \{1, \dots, L_\mathrm{t}''\}} \in G_0 \\ \implies \quad \big(\Phi_{1, L_\mathrm{t}''}(\eta_0)\big)(x) = \big(\Phi_{1, L_\mathrm{t}''}(\eta^\mathrm{ref})\big)(x) \text{ for } \|x\| \le L_\mathrm{s} \end{aligned} \tag{4.24}$$

with $\Phi_{1, L_\mathrm{t}''}$ as in (4.9) and $\eta^\mathrm{ref} \equiv \lceil m^* \rceil$.

Observe that for $x \in \mathbb{Z}^d$, $n \in \mathbb{Z}_+$

$$\begin{aligned} \sup_{i \in I} \eta_n^{(i)}(x) - \inf_{i \in I} \eta_n^{(i)}(x) &= \sup_{i \in I} \sum_y U_{n-1}^{(y,x)}\big(f(y; \eta_{n-1}^{(i)})\big) - \inf_{i \in I} \sum_y U_{n-1}^{(y,x)}\big(f(y; \eta_{n-1}^{(i)})\big) \\ &\le \sum_y \Big(\sup_{i \in I} U_{n-1}^{(y,x)}\big(f(y; \eta_{n-1}^{(i)})\big) - \inf_{i \in I} U_{n-1}^{(y,x)}\big(f(y; \eta_{n-1}^{(i)})\big) \Big) \\ &= \sum_y U_{n-1}^{(y,x)}\Big(\sup_{i \in I} f(y; \eta_{n-1}^{(i)}) \Big) - U_{n-1}^{(y,x)}\Big(\inf_{i \in I} f(y; \eta_{n-1}^{(i)}) \Big). \end{aligned} \tag{4.25}$$

hence

$$\mathbb{E}\Big[\sup_{i\in I}\eta_n^{(i)}(x)-\inf_{i\in I}\eta_n^{(i)}(x)\Big|\mathcal{F}_{n-1}\Big]\le\sum_y p_{yx}\Big(\sup_{i\in I}f(y;\eta_{n-1}^{(i)})-\inf_{i\in I}f(y;\eta_{n-1}^{(i)})\Big) \tag{4.26}$$

and we can now use contraction properties of f near its fixed point, analogous to the proof of (44), (45) in Lemma 13 in [BD07]. Note that if all $\eta_{n-1}^{(i)}(y)$, $i\in I$ are (locally around x) in the neighbourhood $[\alpha,\beta]$ of m^* (as required for (4.19)), there are $\tilde\eta_{y,z}\in[\alpha,\beta]^{B_{R_\lambda}(y)}$ such that

$$\sup_{i\in I}f(y;\eta_{n-1}^{(i)})-\inf_{i\in I}f(y;\eta_{n-1}^{(i)})\le\sum_{z\in B_{R_\lambda}(y)}\big|\tfrac{\partial}{\partial\eta(z)}f(y;\tilde\eta_{y,z})\big|\Big(\sup_{i\in I}\eta_{n-1}^{(i)}(z)-\inf_{i\in I}\eta_{n-1}^{(i)}(z)\Big). \tag{4.27}$$

Put

$$\psi_R(\eta):=\mathbb{1}_{\{\eta(x)\in[\alpha/\gamma,\beta/\gamma]\text{ for }\|x\|\le R\}}. \tag{4.28}$$

We have on $\{\inf_{i\in I}\psi_{R+R_p+R_\lambda}(\eta_{n-1}^{(i)})=1\}$

$$\begin{aligned}
&\frac{1}{|B_R(0)|}\sum_{x\in B_R(0)}\mathbb{E}\Big[\Big(\sup_{i\in I}\psi_R(\eta_n^{(i)})\eta_n^{(i)}(x)-\inf_{i\in I}\psi_R(\eta_n^{(i)})\eta_n^{(i)}(x)\Big)\big|\mathcal{F}_{n-1}\Big]\\
&\quad\le\frac{1}{|B_R(0)|}\sum_{x\in B_R(0)}\mathbb{E}[\sup_{i\in I}\eta_n^{(i)}(x)-\inf_{i\in I}\eta_n^{(i)}(x)\big|\mathcal{F}_{n-1}]\\
&\quad\le\frac{1}{|B_R(0)|}\sum_{x\in B_R(0)}\sum_{y\in B_{R_p}(x)}p_{yx}\sum_{z\in B_{R_\lambda}(y)}|\nabla_z f(y;\tilde\eta_z)|\Big(\sup_{i\in I}\eta_{n-1}^{(i)}(z)-\inf_{i\in I}\eta_{n-1}^{(i)}(z)\Big)\\
&\quad\le\sum_{z\in B_{R+R_p+R_\lambda}(0)}\Big(\sup_{i\in I}\eta_{n-1}^{(i)}(z)-\inf_{i\in I}\eta_{n-1}^{(i)}(z)\Big)\frac{1}{|B_R(0)|}\sum_{y\in B_{R_\lambda}(z)}|\nabla_z f(y;\tilde\eta_z)|\sum_{x\in B_R(0)}p_{xy}\\
&\quad\le\frac{|B_{R+R_p+R_\lambda}(0)|}{|B_R(0)|}\Big(1-\frac{\varepsilon}{2}\Big)\frac{1}{|B_{R+R_p+R_\lambda}(0)|}\sum_{z\in B_{R+R_p+R_\lambda}(0)}\Big(\sup_{i\in I}\eta_{n-1}^{(i)}(z)-\inf_{i\in I}\eta_{n-1}^{(i)}(z)\Big)\\
&\quad\le c(\varepsilon)\frac{1}{|B_{R+R_p+R_\lambda}(0)|}\sum_{z\in B_{R+R_p+R_\lambda}(0)}\Big(\sup_{i\in I}\eta_{n-1}^{(i)}(z)-\inf_{i\in I}\eta_{n-1}^{(i)}(z)\Big),
\end{aligned} \tag{4.29}$$

where we used (4.26) and (4.27) in the second inequality and assume that R is so large that

$$\frac{|B_{R+R_p+R_\lambda}(0)|}{|B_R(0)|}\Big(1-\frac{\varepsilon}{2}\Big)\le c(\varepsilon)<1. \tag{4.30}$$

Note that the factor $(1-\frac{\varepsilon}{2})$ comes from (4.19).

We can iterate (4.29) for $n=L_{\mathrm{t}}'',L_{\mathrm{t}}''-1,\dots,1$ to obtain on $\mathcal{X}_1$ (which in particular implies $\psi_{L_{\mathrm{s}}+k(R_p+R_\lambda)}(\eta_{n-k})=1$ for $k=1,2,\dots,n-1$) that

$$\begin{aligned}
&\frac{1}{|B_{L_{\mathrm{s}}}(0)|}\sum_{x\in B_{L_{\mathrm{s}}}(0)}\mathbb{E}\Big[\Big(\sup_{i\in I}\psi_{L_{\mathrm{s}}}(\eta_{L_{\mathrm{t}}''}^{(i)})\eta_{L_{\mathrm{t}}''}^{(i)}(x)-\inf_{i\in I}\psi_{L_{\mathrm{s}}}(\eta_{L_{\mathrm{t}}''}^{(i)})\eta_{L_{\mathrm{t}}''}^{(i)}(x)\Big)\big|\mathcal{F}_0\Big]\\
&\qquad\le c(\varepsilon)^{L_{\mathrm{t}}''}\frac{1}{|B_{L_{\mathrm{s}}+L_{\mathrm{t}}''(R_p+R_\lambda)}(0)|}\sum_{z\in B_{L_{\mathrm{s}}+L_{\mathrm{t}}''(R_p+R_\lambda)}(0)}\Big(\sup_{i\in I}\eta_0^{(i)}(z)-\inf_{i\in I}\eta_0^{(i)}(z)\Big)\\
&\qquad\le c(\varepsilon)^{L_{\mathrm{t}}''}\frac{\beta-\alpha}{\gamma}.
\end{aligned} \tag{4.31}$$

(4.31) yields via Markov inequality on $\{\inf_{i\in I}\psi_{R+L''_{\mathrm{t}}(R_p+R_\lambda)}(\eta_0^{(i)})=1\}$

$$\mathbb{P}\Big(\max_{\|x\|\le L_{\mathrm{s}}}\big(\sup_{i\in I}\psi_{L_{\mathrm{s}}}(\eta^{(i)}_{L''_{\mathrm{t}}})\eta^{(i)}_{L''_{\mathrm{t}}}(x)-\inf_{i\in I}\psi_{L_{\mathrm{s}}}(\eta^{(i)}_{L''_{\mathrm{t}}})\eta^{(i)}_{L''_{\mathrm{t}}}(x)\big)\ge 1\,\Big|\,\mathcal{F}_0\Big)\le |B_{L_{\mathrm{s}}}(0)|c(\varepsilon)^{L''_{\mathrm{t}}}\frac{\beta-\alpha}{\gamma} \tag{4.32}$$

which can be made as small as we like by choosing γ small.

Hence for $\mathcal{X}_2 := \{U|_{B_{3L_{\mathrm{s}}}(0)\times\{1,\dots,L''_{\mathrm{t}}\}}\in G_0\}$ we have

$$\mathbb{P}(\mathcal{X}_2)\ge 1-|B_{L_{\mathrm{s}}}(0)|c(\varepsilon)^{L''_{\mathrm{t}}}\frac{\beta-\alpha}{\gamma}. \tag{4.33}$$

Step 3. Let $\mathcal{X}_2(y,k)$ be the event that $\mathcal{X}_2$ occurs in the space-time box whose "bottom" is centred at (y,k), i.e. $\mathcal{X}_2(y,k)=\{U|_{B_{3L_{\mathrm{s}}}(y)\times\{k+1,\dots,k+L''_{\mathrm{t}}\}}\in G_0\}$.

By construction, on

$$\mathcal{X}_3 := \mathcal{X}_1\cap\bigcap_{\substack{j\in\{-2,-1,\dots,2\},\\ k=1,\dots,d}}\mathcal{X}_2(jL_{\mathrm{s}}e_k,L'_{\mathrm{t}}) \tag{4.34}$$

we have

$$\eta_0(x)\in[\alpha/\gamma,\beta/\gamma]\text{ for }x\in B_{2L_{\mathrm{s}}}(0)\quad\Longrightarrow\quad \eta_{L_{\mathrm{t}}}(y)=\big(\Phi_{1,L_{\mathrm{t}}}(\eta^{\mathrm{ref}})\big)(y)\text{ for }\|y\|\le 3L_{\mathrm{s}}, \tag{4.35}$$

i.e. (3.4) holds. In particular

$$\mathbb{P}(\mathcal{X}_3)\ge\mathbb{P}(\mathcal{X}_1)-5^d(1-\mathbb{P}(\mathcal{X}_2)) \tag{4.36}$$

Step 4. On

$$\mathcal{X}_4 := \mathcal{X}_3\cap\bigcap_{j=0,\dots,\lceil L'_{\mathrm{t}}/L''_{\mathrm{t}}\rceil}\mathcal{X}_2(0,jL''_{\mathrm{t}}), \tag{4.37}$$

(3.5) holds as well, and we have

$$\mathbb{P}(\mathcal{X}_4)\ge\mathbb{P}(\mathcal{X}_3)-\lceil L'_{\mathrm{t}}/L''_{\mathrm{t}}\rceil(1-\mathbb{P}(\mathcal{X}_2)) \tag{4.38}$$

Step 5. Note that on $\mathcal{X}_4$ we have

$$\eta_0(x)\in[\alpha/\gamma,\beta/\gamma]\text{ for }x\in B_{2L_{\mathrm{s}}}(0)\quad\Longrightarrow\quad \eta_n(y)\in[\alpha/\gamma,\beta/\gamma]\text{ for }\|y\|\le 2L_{\mathrm{s}},n=1,\dots,L_{\mathrm{t}}.$$

Note that then (4.10) implies

$$\frac{\alpha}{\beta}p_{xy}\le p_\eta(k;x,y)\le\frac{\beta}{\alpha}p_{xy}\quad\text{ for }x,y\in B_{2L_{\mathrm{s}}}(0),\ k=1,\dots,L_{\mathrm{t}},$$

hence the total variation distance between $p_\eta(k;x,\cdot)$ and $p_{x,\cdot}$ is uniformly inside this space-time block at most $(1-\frac{\alpha}{\beta})\vee(\frac{\beta}{\alpha}-1)$. We use Lemma 4.3, *(iii)* to make this so small that coupling arguments as in the proof of Lemma 2.15 (with a comparison random walk that has a deterministic drift $d_{\max}\ll L_{\mathrm{s}}/(L'_{\mathrm{t}}+L''_{\mathrm{t}})$ show (3.11).

Step 6. Finally, we verify that the constants $K_{\mathrm{s}},K'_{\mathrm{t}},K''_{\mathrm{t}}$ can be chosen consistently so that all intermediate requirements are fulfilled.

1. The right-hand side of (4.22) can be chosen arbitrarily close to 1 for any choice of $K_\mathrm{s}, K'_\mathrm{t}, K''_\mathrm{t}$ by making γ small.

2. (4.23) requires that $K'_\mathrm{t} > \frac{3}{s_0} K_\mathrm{s}$ with s_0 from (4.17), (4.18).

3. (4.31), which uses (4.30) L''_t times, requires that $L_\mathrm{s} - (R_p + R_\lambda) L''_\mathrm{t}$ is large. This is achieved when $K_\mathrm{s} \gg (R_p + R_\lambda) K''_\mathrm{t}$ (and γ is small).

4. The right-hand side of (4.33) can be made close to 1 if $K''_\mathrm{t} > (-\log c(\varepsilon))^{-1}$ (and γ is small). This also implies that the right-hand side of (4.36) can be chosen arbitrarily close to 1.

5. For (4.38) note that $\frac{L'_\mathrm{t}}{L''_\mathrm{t}} \approx \frac{K'_\mathrm{t}}{K''_\mathrm{t}}$ is a fixed ratio when γ is small, and $1 - \mathbb{P}(\mathcal{X}_2)$ can be made small by choosing γ small.

We see that for $\gamma \leq \gamma^*$ for some $\gamma^* > 0$, all requirements can be fulfilled e.g. by choosing $K''_\mathrm{t} := 2/(-\log c(\varepsilon))$, $K_\mathrm{s} := C(R_p + R_\lambda) K''_\mathrm{t}$ with some large C and $K'_\mathrm{t} := \frac{6}{s_0} K_\mathrm{s}$. □

A An auxiliary result

The following result should be standard, we give here a brief argument for completeness' sake and for lack of a precise point reference.

Lemma A.1. *Let $\mathcal{F} = (\mathcal{F}_n)_{n=0,1,\dots}$ be a filtration, T, T' finite $\mathcal{F}$-stopping times, and Y a bounded random variable. We have*

$$\mathbb{E}[Y \mid \mathcal{F}_T] \, \mathbb{1}_{\{T=T'\}} = \mathbb{E}[Y \mid \mathcal{F}_{T'}] \, \mathbb{1}_{\{T=T'\}} \quad \text{a.s.} \tag{A.1}$$

Proof. Note that $\{T = T'\} \in \mathcal{F}_T \cap \mathcal{F}_{T'}$ because

$$\{T = T'\} \cap \{T = n\} = \{T = T'\} \cap \{T' = n\} = \{T = n\} \cap \{T' = n\} \in \mathcal{F}_n, \quad n = 0, 1, \dots.$$

Furthermore we have

$$A \in \mathcal{F}_T \cup \mathcal{F}_{T'} \quad \Rightarrow \quad A \cap \{T = T'\} \in \mathcal{F}_T \cap \mathcal{F}_{T'}.$$

To see this note that for $A \in \mathcal{F}_T$

$$A \cap \{T = T'\} \cap \{T' = n\} = (A \cap \{T = n\}) \cap \{T' = n\} \in \mathcal{F}_n, \quad n = 0, 1, \dots.$$

Thus, we obtain $A \cap \{T = T'\} \in \mathcal{F}_{T'}$ and a similar argument for the other case shows the assertion. By approximation arguments we find that

$$Z \text{ is } \mathcal{F}_T\text{-measurable} \quad \Rightarrow \quad Z \mathbb{1}_{\{T=T'\}} \text{ is } (\mathcal{F}_T \cap \mathcal{F}_{T'})\text{-measurable}.$$

Let Z be a version of $\mathbb{E}\left[Y \mathbb{1}_{\{T=T'\}} \mid \mathcal{F}_T\right] = \mathbb{1}_{\{T=T'\}} \mathbb{E}[Y \mid \mathcal{F}_T]$, i.e., Z is $\mathcal{F}_T$-measurable, $\mathbb{E}[Z \mathbb{1}_A] = \mathbb{E}\left[Y \mathbb{1}_{\{T=T'\}} \mathbb{1}_A\right]$ for all $A \in \mathcal{F}_T$. We may assume that $Z = Z \mathbb{1}_{\{T=T'\}}$. Then, Z is also a version of $\mathbb{E}\left[Y \mathbb{1}_{\{T=T'\}} \mid \mathcal{F}_{T'}\right] = \mathbb{1}_{\{T=T'\}} \mathbb{E}[Y \mid \mathcal{F}_{T'}]$. Furthermore $Z = Z \mathbb{1}_{\{T=T'\}}$ is also $\mathcal{F}_{T'}$-measurable and for $A' \in \mathcal{F}_{T'}$,

$$\mathbb{E}[Z \mathbb{1}_{A'}] = \mathbb{E}\left[Z \mathbb{1}_{A' \cap \{T=T'\}}\right] \mathbb{E}\left[Y \mathbb{1}_{\{T=T'\}} \mathbb{1}_{A' \cap \{T=T'\}}\right] = \mathbb{E}\left[Y \mathbb{1}_{\{T=T'\}} \mathbb{1}_{A'}\right].$$

This concludes the proof of the lemma. □

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