

Directed random walk on the backbone of an oriented percolation cluster*

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Abstract

We consider a directed random walk on the backbone of the infinite cluster generated by supercritical oriented percolation, or equivalently the space-time embedding of the “ancestral lineage” of an individual in the stationary discrete-time contact process. We prove a law of large numbers and an annealed central limit theorem (i.e., averaged over the realisations of the cluster) using a regeneration approach. Furthermore, we obtain a quenched central limit theorem (i.e. for almost any realisation of the cluster) via an analysis of joint renewals of two independent walks on the same cluster.

Keywords: Random walk, dynamical random environment, oriented percolation, supercritical cluster, central limit theorem in random environment .

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1 Introduction and main results

In mathematical population genetics it is often important to understand the ancestral relationship of individuals to deduce information about the genetic variability in the population, see e.g. [8]. In spatial population models the ancestry of a collection of individuals can in general be described by a collection of coalescing random walks in a random environment. Depending on the forwards in time evolution of the population such random environments can be rather complicated.

In the present paper we study the ancestral line, that is, the evolution of the positions of the parents, of a single individual in a simple model allowing for locally varying population sizes. More precisely we consider a discrete-time variant of the contact process: a $\{0, 1\}^{\mathbb{Z}^d}$ -valued Markov chain $(\eta_n)_n$ (see below for precise definitions) where $\eta_n(x) = 1$ is interpreted as the event that the site $x \in \mathbb{Z}^d$ is inhabited by a particle in

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generation n . We can view this contact process as a “toy example” of a spatial stochastic population model with fluctuating population sizes and local dispersal: Sites x can have *carrying capacity* 0 or 1 in a given generation n , and in order for a particle at site x to be present not only must the corresponding site be *open* (i.e. have the carrying capacity 1) but there must also have been a particle in the neighbourhood of x in the previous generation $n - 1$ who put her *offspring* there. If there was more than one particle in the neighbourhood of x in generation $n - 1$, we think of randomly assigning one of them to put an offspring to site x . Note that this implicitly models a density-dependent population regulation because particles in sparsely-populated regions will now have a higher chance of actually leaving an offspring.

We will let the carrying capacities be i.i.d. Bernoulli random variables, and consider the process η in the stationary regime. In this regime every living particle at generation 0, say, has an infinite line of ancestors. The question of interest is the distribution of the spatial location of distant-in-time ancestors.

One can of course interpret the discrete-time contact process as a process describing spread of an infection and interpret the carrying capacities as *susceptible for the infection* or *immune* if the Bernoulli random variable at the corresponding site is 1 respectively 0. Then our question of interest is the spatial location of the distant-in-time carriers of the infection from which the infection propagated to a given individual.

By reversing the time direction, the problem has the following equivalent description. We consider a simple directed random walk on the “backbone” of the infinite cluster of the oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}$. The backbone of the cluster, we denote it by \mathcal{C} , is a collection of all sites in $\mathbb{Z}^d \times \mathbb{Z}$ that are connected to infinity by an open path. The “time-slices” of the cluster \mathcal{C} can be seen to be equal in distribution to the time-reversal of the (non-trivial) stationary discrete-time contact process $(\eta_n)_n$, and the directed walk on \mathcal{C} can be interpreted as the spatial embedding of the ancestral lineage of one individual drawn from the equilibrium population. The question posed in the previous paragraph thus amounts to understanding the long time behaviour of this random walk.

In this formulation, the model is of independent interest in the context of the theory of random media: The directed random walk on an oriented percolation cluster can be viewed as a random walk in a Markovian dynamical random environment. The investigation of such random walks is an active research area with a lot of recent progress. The random walk we consider however does not satisfy the usual independence or mixing conditions that appear in the literature; see Remark 1.7 below. In fact, in our case the evolution of the environment as a process in time is rather complicated.

On the other hand, as a random walk on a random cluster, the model is very natural. The investigation of random walks on percolation clusters is a very active research area as well. An important difference to our model is that usually, the walk can move in all (open) directions, whereas we consider a *directed* random walk.

We now give a precise definition of the model. Let $\omega := \{\omega(x, n) : (x, n) \in \mathbb{Z}^d \times \mathbb{Z}\}$ be a family of independent Bernoulli random variables (representing the carrying capacities) with parameter $p > 0$ on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We call a site (x, n) *open* if $\omega(x, n) = 1$ and *closed* if $\omega(x, n) = 0$. We say that there is an *open path* from (y, m) to (x, n) for $m \leq n$ if there is a sequence x_m, \dots, x_n such that $x_m = y$, $x_n = x$, $\|x_k - x_{k-1}\| \leq 1$ for $k = m + 1, \dots, n$ and $\omega(x_k, k) = 1$ for all $k = m, \dots, n$. In this case we write $(x, m) \rightarrow (y, n)$. Here $\|\cdot\|$ denotes the sup-norm.

Given a set $A \subset \mathbb{Z}^d$ we define the *discrete time contact process* $(\eta_n^A)_{n \geq m}$ starting at time $m \in \mathbb{Z}$ from the set A as

$$\eta_m^A(y) = \mathbb{1}_A(y), \quad y \in \mathbb{Z}^d,$$

and for $n \geq m$

$$\eta_{n+1}^A(x) = \begin{cases} 1 & \text{if } \omega(x, n+1) = 1 \text{ and } \eta_n^A(y) = 1 \text{ for some } y \in \mathbb{Z}^d \text{ with } \|x - y\| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\eta_n^A(y) = 1$ if and only if there is an open path from (x, m) to (y, n) for some $x \in A$ (where we use in this definition the convention that $\omega(x, m) = \mathbb{1}_A(x)$ while for $k > m$ the $\omega(x, k)$ are i.i.d. Bernoulli as above). Often we will identify the configuration η_n^A with the set $\{x \in \mathbb{Z}^d : \eta_n^A(x) = 1\}$. Taking $m = 0$, we set

$$\tau^A := \inf\{n \geq 0 : \eta_n^A = \emptyset\}, \tag{1.1}$$

and in the case $A = \{\mathbf{0}\}$ we write τ^0 .

It is well known, see e.g. Theorem 1 in [10], that there is a critical value $p_c \in (0, 1)$ such that $\mathbb{P}(\tau^0 = \infty) = 0$ for $p \leq p_c$ and $\mathbb{P}(\tau^0 = \infty) > 0$ for $p > p_c$. In the following we consider only the supercritical case $p > p_c$. In this case the law of $\eta_n^{\mathbb{Z}^d}$ converges weakly to the upper invariant measure which is the unique non-trivial extremal invariant measure of the discrete-time contact process. By taking $m \rightarrow -\infty$ while keeping $A = \mathbb{Z}^d$ one obtains the stationary process

$$\eta := (\eta_n)_{n \in \mathbb{Z}} := (\eta_n^{\mathbb{Z}^d})_{n \in \mathbb{Z}}. \tag{1.2}$$

We interpret the process η as a population process, where $\eta_n(x) = 1$ means that the position x is occupied by an individual in generation n . We are interested in the behaviour of the ‘‘ancestral lines’’ of individuals. Note that because of the discrete time, there can in principle be several individuals alive in the previous generation that could be ancestors of a given individual at site y , namely all those at some y' with $\|y' - y\| \leq 1$. In that case, we stipulate that one of these potential ancestors is chosen uniformly at random to be the actual ancestor, independently of everything else in the model.

Due to time stationarity, we can focus on ancestral lines of individuals living at time 0. It will be notationally convenient to time-reverse the stationary process η and consider the process $\xi := (\xi_n)_{n \in \mathbb{Z}}$ defined by $\xi_n(x) = 1$ if $(x, n) \rightarrow \infty$ (i.e. there is an infinite directed open path starting at (x, n)) and $\xi_n(x) = 0$ otherwise. Note that indeed $\mathcal{L}((\xi_n)_{n \in \mathbb{Z}}) = \mathcal{L}((\eta_{-n})_{n \in \mathbb{Z}})$. We will from now on consider the forwards evolution of ξ as the ‘‘positive’’ time direction.

On the event $B_0 := \{\xi_0(\mathbf{0}) = 1\}$ there is an infinite path starting at $(\mathbf{0}, 0)$. We define the oriented cluster by

$$\mathcal{C} := \{(x, n) \in \mathbb{Z}^d \times \mathbb{Z} : \xi_n(x) = 1\}$$

and let

$$U(x, n) := \{(y, n+1) : \|x - y\| \leq 1\} \tag{1.3}$$

be the neighbourhood of the site (x, n) in the next generation. On the event B_0 we may define a \mathbb{Z}^d -valued random walk $X := (X_n)_{n \geq 0}$ starting from $X_0 = \mathbf{0}$ with transition probabilities

$$\mathbb{P}(X_{n+1} = y | X_n = x, \xi) = \begin{cases} |U(x, n) \cap \mathcal{C}|^{-1} & \text{when } (y, n+1) \in U(x, n) \cap \mathcal{C}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

Note that $(X_n, n)_{n \geq 0}$ is a directed random walk on the percolation cluster \mathcal{C} , and X can be also viewed as a random walk in a (dynamical) random environment, where the environment is given by the process ξ . As the environment ξ is the time-reversal of the

stationary Markov process η , it is itself Markovian and stationary, the invariant measure being the upper invariant measure of the discrete-time contact process η . While the evolution of η is easy to describe forwards in time by local rules, η is not reversible, and the time evolution of its reversal ξ seems complicated. The transition probabilities for ξ cannot be described by local functions: For example, when viewed as a function of $a = (a_x)_{x \in \mathbb{Z}^d} \in \{0, 1\}^{\mathbb{Z}^d}$, $f(a) := \mathbb{P}(\xi_{n+1}(\mathbf{0}) = 1 | \xi_n = a)$, there will be no finite set $K \subset \mathbb{Z}^d$ such that f depends only on $(a_x)_{x \in K}$ (this can for example be seen by considering various a 's that have a "sea of 0's around the origin"). Presently it is not at all clear to us how to describe the forwards in time evolution of ξ in a more tangible way. Note however, that the process ξ does form a finite-range Markov field in the larger space $\mathbb{Z}^d \times \mathbb{Z}$ because this is true for η , but it is unclear at the moment what use we could make of that fact.

The complicated nature of ξ disallows checking many of the usual conditions that appear in the literature on random walks in dynamical random environment. Some of such conditions (like e.g. ellipticity) are obviously violated by our model. To our best knowledge, the random walk in a (dynamic) random environment that we consider here is not contained in one of the classes studied in the literature so far; see Remark 1.7 below.

Our first result shows the law of large numbers, and a central limit theorem for X when averaging over both the randomness of the walk's steps and the environment ω . We write P_ω for the conditional law of \mathbb{P} , given ω , and E_ω for the corresponding expectation. With this notation we have $\mathbb{P}(X_{n+1} = y | X_n = x, \xi) = P_\omega(X_{n+1} = y | X_n = x)$.

Theorem 1.1 (LLN & annealed CLT). *For any $d \geq 1$ we have*

$$P_\omega\left(\frac{1}{n}X_n \rightarrow \mathbf{0}\right) = 1 \quad \text{for } \mathbb{P}(\cdot | B_0)\text{-a.a. } \omega, \tag{1.5}$$

and for any $f \in C_b(\mathbb{R}^d)$

$$\mathbb{E}\left[f\left(X_n/\sqrt{n}\right) \mid B_0\right] \xrightarrow{n \rightarrow \infty} \Phi(f), \tag{1.6}$$

where $\Phi(f) := \int f(x) \Phi(dx)$ with Φ a non-trivial centred isotropic d -dimensional normal law.

We prove this theorem by exhibiting a regeneration structure for X and ξ , and then showing that the second moments of temporary and spatial increments of X at regeneration times are finite (in fact we will prove existence of exponential moments).

Remark 1.2. *The covariance matrix of Φ in (1.6) is σ^2 times the d -dimensional identity matrix. It is clear from the construction below (see Subsection 2.2) that*

$$\sigma^2 = \sigma^2(p) = \frac{\mathbb{E}[Y_{1,1}^2]}{\mathbb{E}[\tau_1]} \in (0, \infty) \tag{1.7}$$

where τ_1 is the first regeneration time (see (2.10)) of the random walk X and $Y_{1,1}$ is the first coordinate of X_{τ_1} , the position of the random walk at this regeneration time. A simple truncation and coupling argument shows that $p \mapsto \sigma^2(p)$ is continuous on $(p_c, 1]$; see Remark 2.6. The behaviour of $\sigma^2(p)$ as $p \downarrow p_c$ appears to be an interesting problem that merits further research.

It is natural to study also two (or even more) walkers on the same cluster. On the one hand, this allows to obtain information on the long-time behaviour in a multi-type situation. Neuhauser in [17] and more recently Valesin in [23] employed this for

the (continuous-time) contact process. It is also very natural from the modelling of ancestral lineages point of view, where it corresponds to jointly describing the space-time embedding of the ancestry of a sample of size two (or more) individuals when the walks start from different sites. On the other hand, good control of the behaviour of two or more “replicas” of X on the same cluster allows us to strengthen the annealed CLT (1.6) to the quenched version.

Theorem 1.3 (Quenched CLT). *For any $d \geq 1$ and $f \in C_b(\mathbb{R}^d)$*

$$E_\omega \left[f \left(X_n / \sqrt{n} \right) \right] \xrightarrow{n \rightarrow \infty} \Phi(f) \quad \text{for } \mathbb{P}(\cdot | B_0)\text{-a.a. } \omega, \tag{1.8}$$

where Φ is the same non-trivial centred isotropic d -dimensional normal law as in (1.6).

Let us conclude the introduction by mentioning some generalisations of the random walk that we consider here.

Remark 1.4 (More general neighbourhoods). *For simplicity, we defined $U(x, n)$ as in (1.3), but the proofs go through for any finite, symmetric neighbourhood (by “symmetric” we mean that $y \in U(x, n)$ if and only if $-y \in U(x, n)$). In this case the resulting law Φ will in general not be isotropic, see the end of the proof of Theorem 1.1.*

Note also, that for sake of clarity, all figures in this paper are drawn with $U(x, n)$ set to $\{(x + 1, n + 1), (x - 1, n + 1)\}$ for $d = 1$.

Remark 1.5 (Functional central limit theorem). *For the random walk X one can also deduce corresponding annealed and quenched functional limit theorems; see also Remark 3.11.*

Remark 1.6 (Contact process with fluctuating population size). *Let $K(x, n)$, $(x, n) \in \mathbb{Z}^d \times \mathbb{Z}$ be i.i.d. $\mathbb{N} = \{1, 2, \dots\}$ -valued random variables and let us define the discrete time contact process with fluctuating population size, $\hat{\eta} := (\hat{\eta}_n)_{n \in \mathbb{Z}}$, by*

$$\hat{\eta}_n(x) := \eta_n(x)K(x, n), \tag{1.9}$$

and its reversal $\hat{\xi} := (\hat{\xi}_n)_{n \in \mathbb{Z}}$ by

$$\hat{\xi}_n(x) := \xi_n(x)K(x, n). \tag{1.10}$$

One can interpret $K(x, n)$ as a random “carrying capacity” of the site (x, n) . Now conditioned on $\hat{\xi}_0(\mathbf{0}) \geq 1$ the ancestral random walk, we call it X as before, can be defined by $X_0 = \mathbf{0}$ and

$$\mathbb{P}(X_{n+1} = y | X_n = x, \hat{\xi}) = \begin{cases} \frac{\hat{\xi}_{n+1}(y)}{\sum_{(y', n+1) \in U(x, n)} \hat{\xi}_{n+1}(y')} & \text{if } (y, n + 1) \in U(x, n), \\ 0 & \text{otherwise.} \end{cases} \tag{1.11}$$

For such random walks in random environments our arguments can also be adapted and the same results as above can be obtained; see also Remark 2.4.

Remark 1.7 (Relation to other approaches to RWDRE in the literature). *Random walks in dynamic random environments (RWDRE) are currently an active area of research (they can of course by explicitly including the “time” coordinate in principle be expressed in the context of random walk in random (non-dynamic) environments, but the often more complicated structure of the law of the environment does make them somewhat special inside this general class). To the best of our knowledge, the walk (1.4) and our results in Theorems 1.1 and 1.3 are not covered by approaches in the literature. Here is a list of corresponding results so far together with a very brief discussion:*

- Dolgopyat et al. [5] obtain a CLT under “abstract” conditions on the environment process, that appear very hard to verify explicitly for ξ , in particular the absolute continuity condition for ξ viewed relative to the walk w.r.t. the a priori law on ξ .
- Joseph and Rassoul-Agha consider in [13] environments that are “refreshed in each step” (i.e. time-slices are i.i.d.), this does not apply to (ξ_n) .
- Individual coordinates $(\xi_n(x))_{n \in \mathbb{N}}$ with $x \in \mathbb{Z}^d$ fixed are not (independent) Markov chains, in contrast to the set-up in [6] by Dolgopyat and Liverani.
- (ξ_n) does not fulfil the required uniform coupling conditions employed by Redig and Völlering in [19].
- (ξ_n) does not fulfil the cone mixing condition considered in Avena et al. [1]. Intuitively, this stems from the fact that the supercritical contact process has two extremal invariant distributions, the upper invariant measure (which we consider here) and the trivial measure (which concentrates on $\xi \equiv 0$). Thus, at an arbitrarily large level n an atypically high density of zeros around the origin can be achieved by conditioning on large enough islands of zeros below at level 0, an event with positive probability.
- den Hollander et al. [4] weaken the cone-mixing condition from [1] to a conditional cone-mixing condition and obtain a LLN for a class of (continuous-time) random walks in dynamic random environments with this (and some further technical) assumptions. Further research is required to investigate whether a similar condition can be established for ξ but note that presently, the approach in [4] does not yield a CLT.

The rest of the paper is organised as follows. In Section 2, we define the regeneration structure for a single walk. To define the regeneration times, we first give an alternative construction of the walk, using some “external randomness”. Theorem 1.1 then follows by standard arguments, once we show that the regeneration times have finite exponential moments, see Lemma 2.5. In Section 3, we define a joint regeneration structure for two walks on the same cluster, and we compare their joint distribution with the distribution of two walks in two independent copies of the environment, which allows us to prove Theorem 1.3. Finally, in Appendix A we prove a “folklore” result (that we could not find in this form in the literature), namely that the height of a finite cluster in the oriented percolation has, for all supercritical parameters, an exponential tail.

2 Regeneration structure for a single random walk

In this section we describe and study a regeneration structure of the random walk X conditioned on the event B_0 . We adapt arguments from [14] and [17] and show that the regeneration times have some exponential moments and consequently finite second moments. From that Theorem 1.1 follows by standard arguments. The corresponding proof is given after Lemma 2.5.

The construction of the regeneration structure is lengthy but not difficult. Its goal is to build a trajectory of X using rules that are “local”, i.e. which use only local ω 's (and some additional local randomness), but not ξ 's, as to know ξ 's we need to know the “whole future” of the environment ω . Of course, this is not possible in general, but the regeneration times which we will construct are exactly the times when the locally constructed trajectory coincides with the trajectory of X . We note that a similar but non-randomised construction was used in [20] to analyse the collection of rightmost paths in a directed percolation cluster.

2.1 Local construction of random walk on \mathcal{C}

We will need some additional randomness for the construction: For every $(x, n) \in \mathbb{Z}^d \times \mathbb{Z}$ let $\tilde{\omega}(x, n)$ be a uniformly chosen permutation of $U(x, n)$, independently distributed for all (x, n) 's, defined also on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We denote the whole family of these permutations by $\tilde{\omega}$.

For every $(x, n) \in \mathbb{Z}^d \times \mathbb{Z}$ let $\ell(x, n) = \ell_\infty(x, n)$ be the length of the longest directed open path starting at (x, n) ; we set $\ell(x, n) = -1$ when (x, n) is closed. (Recall that a path $(x_0, n), (x_1, n + 1), \dots, (x_k, n + k)$ of length k with $\|x_i - x_{i-1}\| \leq 1$ is open if $\omega(x_0, n) = \omega(x_1, n + 1) = \dots = \omega(x_k, n + k) = 1$.) For every $k \in \{0, 1, \dots\}$ let $\ell_k(x, n) := \ell(x, n) \wedge k$ be the length of the longest directed open path of length at most k starting from (x, n) . Observe that $\ell_k(x, n)$ is measurable with respect to the σ -algebra \mathcal{G}_n^{n+k+1} , where

$$\mathcal{G}_n^m := \sigma(\omega(y, i), \tilde{\omega}(y, i) : y \in \mathbb{Z}^d, n \leq i < m), \quad n < m. \tag{2.1}$$

For $k \in \{0, \dots, \infty\}$, we define $M_k(x, n) \subseteq U(x, n)$ to be the set of sites which maximise ℓ_k over $U(x, n)$, i.e.

$$M_k(x, n) := \left\{ y \in U(x, n) : \ell_k(y) = \max_{z \in U(x, n)} \ell_k(z) \right\}, \tag{2.2}$$

and for convenience we set $M_{-1}(x, n) = U(x, n)$. Observe that we have

$$M_0(x, n) = \{y \in U(x, n) : y \text{ is open}\}, \tag{2.3}$$

$$M_\infty(x, n) = U(x, n) \cap \mathcal{C}, \tag{2.4}$$

$$M_k(x, n) \supseteq M_{k+1}(x, n), \quad k \geq -1. \tag{2.5}$$

Let $m_k(x, n) \in M_k(x, n)$ be the element of $M_k(x, n)$ that appears as the first in the permutation $\tilde{\omega}(x, n)$.

Given (x, n) , k , ω and $\tilde{\omega}$, we define a path $\gamma_k = \gamma_k^{(x, n)}$ of length k via

$$\begin{aligned} \gamma_k(0) &= (x, n), \\ \gamma_k(j + 1) &= m_{k-j-2}(\gamma_k(j)), \quad j = 0, \dots, k - 1. \end{aligned} \tag{2.6}$$

In words, at every step, γ_k checks the neighbours of its present position and picks randomly (using the random permutation $\tilde{\omega}$) one of those where it can go further on open sites, but inspecting only the state of sites in the time-layers $\{n, \dots, n + k - 1\}$. Consequently, the construction of $\gamma_k^{(x, n)}$ is measurable with respect to the σ -algebra \mathcal{G}_n^{n+k} (recall (2.1)). See Figure 1 for an illustration.

The paths $\gamma_k^{(x, n)}$ have the following properties.

Lemma 2.1. Assume that ω and $(x, n) \in \mathcal{C}$ are fixed.

- (a) The law of $(\gamma_\infty^{(x, n)}(j))_{j \geq 0}$ is the same as the law of the random walk $(X_j, n + j)_{j \geq 0}$ on \mathcal{C} started from (x, n) .

If in addition $\tilde{\omega}$ is fixed, then

- (b) $\omega(\gamma_k(m)) = 1$ for all $0 \leq m < k$.
- (c) (stability in k) If the end point of γ_k is open, i.e. $\omega(\gamma_k(k)) = 1$, then the path γ_{k+1} restricted to the first k steps equals γ_k .
- (d) (fixation on \mathcal{C}) Assume that $\gamma_k(j) \in \mathcal{C}$ for some $k \geq 0, j \leq k$. Then, $\gamma_m(j) = \gamma_k(j)$ for all $m > k$.

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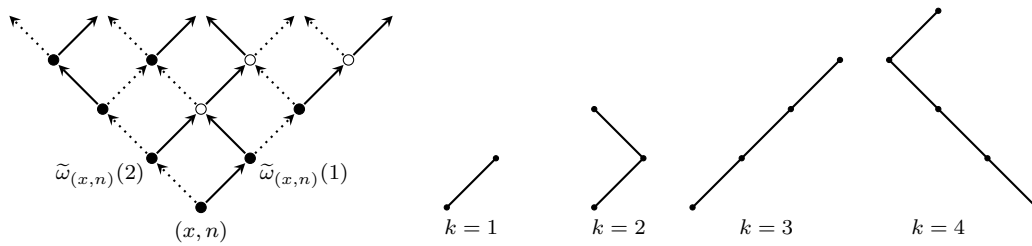


Figure 1: The paths $\gamma_k^{(x,n)}$ from (2.6) based on ω 's and $\tilde{\omega}$'s. Black and white circles represent open sites, i.e. $\omega(\text{site}) = 1$, and closed sites, i.e. $\omega(\text{site}) = 0$, respectively. Solid arrows from a site point to $\tilde{\omega}_{(\text{site})}(1)$ and dotted to $\tilde{\omega}_{(\text{site})}(2)$. On the right the sequence of paths $\gamma_k^{(x,n)}(\cdot)$ for $k = 1, 2, 3, 4$ is shown. For sake of clarity, we used $U(x, n) = \{(x + 1, n + 1), (x - 1, n + 1)\}$, cf. Remark 1.4.

(e) (exploration of finite branches) If $\gamma_k(k - 1) \in \mathcal{C}$ and $\gamma_k(k) \notin \mathcal{C}$ for some k , then $\gamma_j(k) = \gamma_k(k)$ for all $k \leq j \leq k + \ell(\gamma_k(k)) + 1$ and $\gamma_{k+\ell(\gamma_k(k))+2}(k) \neq \gamma_k(k)$.

Proof. Claim (a) follows directly from (2.4), the fact that $m_\infty(\cdot)$ is a uniformly chosen element of $M_\infty(\cdot)$, and the definition of the path γ_∞ . For claim (b), it is sufficient to observe that when $(x, n) \in \mathcal{C}$, there is an open path of length $k - 1$ starting at (x, n) which γ_k will follow. For (c), if $\gamma_k(k)$ is open, then $m_{k-j-2}(\gamma_k(j)) \in M_{k-j-1}(\gamma_k(j))$ for every $0 \leq j < k$, and thus, using the inclusion (2.5), $m_{k-j-1}(\gamma_k(j)) = m_{k-j-2}(\gamma_k(j))$, for $0 \leq j < k$. For (d), if $\gamma_k(j)$ is on \mathcal{C} , then $\gamma_k(j) \in M_m(\gamma_k(j - 1))$ for every $m > k$ by (2.4), and thus $\gamma_k(j) = \gamma_m(j)$. Finally, (e) follows by observing that when $\gamma_k(k) = m_{-1}(\gamma_k(k - 1)) \notin \mathcal{C}$, that is $\ell(\gamma_k(k)) < \infty$, then $\gamma_k(k) = m_j(\gamma_k(k - 1)) \in M_j(\gamma_k(k - 1))$ for all $0 \leq j \leq \ell(\gamma_k(k))$ but $\gamma_k(k) \notin M_{\ell(\gamma_k(k))+1}(\gamma_k(k - 1))$. \square

Remark 2.2. Lemma 2.1(a) allows to couple the random variables $\omega, \tilde{\omega}$ with the random walk (X_k, k) started from $(0, 0)$ by setting

$$(X_k, k) = \gamma_\infty^{(0,0)}(k) = \lim_{j \rightarrow \infty} \gamma_j^{(0,0)}(k). \tag{2.7}$$

(Note that the limit on the right-hand side exists by Lemma 2.1(d).) From now on, we will assume that this coupling is in place.

Remark 2.3. 1. This construction can a priori be used to extend the definition of the random walk X for starting points that are not on the infinite cluster \mathcal{C} . It is sufficient to use similar arguments as in the previous lemma to show that for every (x, n) , ω , and $\tilde{\omega}$, $(X_k, n + k) = \lim_{j \rightarrow \infty} \gamma_j^{(x,n)}(k)$ exists a.s. for every k and is a directed path, which remains on \mathcal{C} once it hits it. Actually, in this way we obtain a coalescing flow on $\mathbb{Z}^d \times \mathbb{Z}$.
2. In analogy with the construction in Section 2 from [17] we can think of the path γ_k defined in (2.6) as leading to $\gamma_k(k)$, the first potential ancestor k generations ago of the particle at (x, n) . The construction of γ_k can easily be extended to a random ordering of all paths of length k , thus yielding an ordered sequence of all potential ancestors.

We will not need these extensions in the present paper.

Remark 2.4 (The construction in the case of fluctuating local population size). In the case of fluctuating local population size as in Remark 1.6, the same construction can be performed. To this end it is only necessary to replace the uniform random permutation

$\tilde{\omega}(x, n)$ by a “weighted” random permutation with distribution

$$\mathbb{P}(\tilde{\omega}(x, n) = (y_1, \dots, y_{|U(x, n)|}) | K) = \frac{1}{Z(x, n)} \prod_{\ell=1}^{|U(x, n)|} K(y_\ell), \tag{2.8}$$

where $(y_1, \dots, y_{|U(x, n)|})$ run over all permutations of $U(x, n)$ and $Z(x, n)$ is the normalisation factor.

2.2 Regeneration times

We can now introduce the regeneration times which will be used to show all main results of the present paper. We consider the random walk (X_n, n) started at $(\mathbf{0}, 0)$ as defined in (2.7) and write γ_k for $\gamma_k^{(\mathbf{0}, 0)}$. We define a sequence $T_j, j \geq 0$, by

$$T_0 := 0 \quad \text{and} \quad T_j := \inf \{k > T_{j-1} : \xi(\gamma_k(k)) = 1\}, \quad j \geq 1. \tag{2.9}$$

(Here and later we use the notation $\xi(y) := \xi_n(x)$ when $y = (x, n) \in \mathbb{Z}^d \times \mathbb{Z}$.)

At times T_j the local construction of the path finds a “real ancestor” of $(\mathbf{0}, 0)$ in the sense that for any $m > T_j, \gamma_m(T_j) = \gamma_{T_j}(T_j)$, by Lemma 2.1(d). Therefore, the local construction “discovers the trajectory of X up to time T_j ”. More precisely we know that $(X_m, m) = \gamma_{T_j}(m)$ for all $0 \leq m \leq T_j$, cf. Lemma 2.1(a) and Remark 2.2.

For $i = 1, 2, \dots$ we set

$$\tau_i := T_i - T_{i-1} \quad \text{and} \quad Y_i := X_{T_i} - X_{T_{i-1}}. \tag{2.10}$$

The strong law of large numbers as well as the (averaged) central limit theorem are consequences of the following lemma.

Lemma 2.5 (Independence and exponential tails for regeneration increments). *Conditioned on B_0 the sequence $((Y_i, \tau_i))_{i \geq 1}$ is i.i.d. and Y_1 is symmetrically distributed. Furthermore, there exist constants $C, c \in (0, \infty)$, such that*

$$\mathbb{P}(\|Y_1\| > n | B_0) \leq C e^{-cn} \quad \text{and} \quad \mathbb{P}(\tau_1 > n | B_0) \leq C e^{-cn}. \tag{2.11}$$

Proof of Theorem 1.1. By symmetry and (2.11), we have $\mathbb{E}[Y_1 | B_0] = \mathbf{0}$. Non-triviality of Φ follows since T_1 and Y_1 are not deterministic multiples of each other and Y_1 is not concentrated on a subspace which follows from the fact that $\mathbb{P}(Y_1 = x, \tau_1 = n | B_0) > 0$ for all $x \in \mathbb{Z}^d$ and $n \geq \|x\|$. To see this we observe that the configuration of the ω ’s in a space-time box of side-length n around the origin consisting only of closed sites except for the origin itself and two disjoint “rays” of open sites, the first connecting $(\mathbf{0}, 0)$ to $(-x, n - 1)$ and ending there, the second connecting $(\mathbf{0}, 0)$ up to (x, n) , has positive probability.

The rest of the proof is standard, see e.g. the proof of Corollary 1 in [14], or the proof of Theorem 4.1. in [22].

Note that since the “basic neighbourhood” $U(\cdot, \cdot)$ is symmetric, Φ is isotropic, i.e. its covariance matrix (Σ_{ij}) is a multiple of the d -dimensional identity matrix: Because Φ is invariant under permutation of coordinates, we must have $\Sigma_{ii} = \Sigma_{jj} = \sigma^2 \in (0, \infty)$, $\Sigma_{ij} = s \in \mathbb{R}$ for all $1 \leq i \neq j \leq d$. Furthermore, the law Φ then also inherits invariance under sign flips of individual coordinates, hence we must have $s = -s = 0$. \square

2.3 Proof of Lemma 2.5

Symmetry of the law of Y_1 follows from the symmetry of the construction. Note that $\|Y_1\| \leq \tau_1$ a.s. Thus, the first bound in (2.11) follows from the second, which we prove now.

The value of T_1 can be obtained by gradually constructing γ_k and checking whether $\xi(\gamma_k(k)) = 1$ for $k = 1, 2, \dots$. We abbreviate $\tilde{\mathbb{P}}(\cdot) := \mathbb{P}(\cdot|B_0)$. Let σ be a $(\mathcal{G}_0^k)_{k \in \mathbb{N}}$ -stopping time, denote the σ -past by \mathcal{G}_0^σ , let W be a \mathcal{G}_0^σ -measurable \mathbb{Z}^d -valued random variable. An application of the FKG inequality yields

$$\tilde{\mathbb{P}}(\xi((W, \sigma)) = 1 | \mathcal{G}_0^\sigma) \geq \mathbb{P}(B_0), \tag{2.12}$$

as follows: For any $n \in \mathbb{N}$, B_0 can be written as the finite disjoint union

$$B_0 = \bigcup_{S \subset B_n(\mathbf{0})} \left(\{(\mathbf{0}, 0) \Rightarrow S \times \{n\}\} \cap \left(\bigcup_{y \in S} \{(y, n) \rightarrow \infty\} \right) \right)$$

where $B_n(\mathbf{0})$ denotes the $\|\cdot\|$ -ball of radius n around 0 in \mathbb{Z}^d and $\{(\mathbf{0}, 0) \Rightarrow S \times \{n\}\} \in \mathcal{G}_0^n$ denotes the event that the set of sites $y \in \mathbb{Z}^d$ with the property that (y, n) can be reached from $(\mathbf{0}, 0)$ via a directed nearest neighbour path whose steps begin on open sites equals exactly S .

Now pick $A \in \mathcal{G}_0^\sigma$. We have

$$\begin{aligned} \mathbb{P}(\{\xi((W, \sigma)) = 1\} \cap A \cap B_0) &= \sum_{w,n} \mathbb{P}(\{\sigma = n, W = w\} \cap A \cap \{(w, n) \rightarrow \infty\} \cap B_0) \\ &= \sum_{w,n} \sum_{S \subset B_n(\mathbf{0})} \mathbb{P}(\{\sigma = n, W = w\} \cap A \cap \{(\mathbf{0}, 0) \Rightarrow S \times \{n\}\} \\ &\quad \cap \{(w, n) \rightarrow \infty\} \cap \bigcup_{y \in S} \{(y, n) \rightarrow \infty\}) \\ &= \sum_{w,n} \sum_{S \subset B_n(\mathbf{0})} \mathbb{P}(\{\sigma = n, W = w\} \cap A \cap \{(\mathbf{0}, 0) \Rightarrow S \times \{n\}\}) \\ &\quad \times \mathbb{P}(\{(w, n) \rightarrow \infty\} \cap \bigcup_{y \in S} \{(y, n) \rightarrow \infty\}) \\ &\geq \sum_{w,n} \sum_{S \subset B_n(\mathbf{0})} \mathbb{P}(\{\sigma = n, W = w\} \cap A \cap \{(\mathbf{0}, 0) \Rightarrow S \times \{n\}\}) \\ &\quad \times \mathbb{P}(\bigcup_{y \in S} \{(y, n) \rightarrow \infty\}) \times \mathbb{P}(\{(w, n) \rightarrow \infty\}) \\ &= \sum_{w,n} \sum_{S \subset B_n(\mathbf{0})} \mathbb{P}(\{\sigma = n, W = w\} \cap A \cap \{(\mathbf{0}, 0) \Rightarrow S \times \{n\}\} \cap \bigcup_{y \in S} \{(y, n) \rightarrow \infty\}) \mathbb{P}(B_0) \\ &= \mathbb{P}(A \cap B_0) \mathbb{P}(B_0) \end{aligned}$$

where we used the FKG inequality in the fourth line and the fact that \mathcal{G}_0^n and \mathcal{G}_n^∞ are independent in the third and the fifth lines. Since $A \in \mathcal{G}_0^\sigma$ is arbitrary, this proves (2.12).

Applying (2.12) with $\sigma = \sigma_0 := 1$ and $W = \gamma_{\sigma_0}(\sigma_0)$ yields

$$\tilde{\mathbb{P}}(T_1 = 1) = \tilde{\mathbb{P}}(T_1 = 1 | \mathcal{G}_0^1) = \tilde{\mathbb{P}}(\xi(\gamma_1(1)) = 1 | \mathcal{G}_0^1) \geq \mathbb{P}(B_0). \tag{2.13}$$

When $\gamma_1(1) \notin \mathcal{C}$, we should wait for the local construction to discover this fact, i.e. for the finite directed cluster starting at $\gamma_1(1)$ to die out. More precisely, on the event $\mathcal{B}_1 = \{\gamma_1(1) \notin \mathcal{C}\}$, $\ell(\gamma_1(1))$, the length of the longest directed open path starting at $\gamma_1(1)$ is finite and the local construction discovers this fact at time $\ell(\gamma_1(1)) + \sigma_0 + 1$ when the longest paths gets stuck, i.e. when it runs into a “dead end” produced by closed sites. Thus, σ_1 , defined by $\sigma_1 = \ell(\gamma_1(1)) + 2 + \sigma_0$ is a stopping time w.r.t. the filtration $(\mathcal{G}_0^m)_{m=1,2,\dots}$ and $\mathcal{B}_1 \in \mathcal{G}_0^{\sigma_1}$. On \mathcal{B}_1 , by Lemma 2.1(e) with $k = 1$, $\xi(\gamma_m(1)) = 0$ and hence also $\xi(\gamma_m(m)) = 0$ for all $m < \sigma_1$. Thus, we have

$$\mathcal{B}_1 = \mathcal{B}_1 \cap \{\xi(\gamma_m(m)) = 0, \forall 1 \leq m < \sigma_1\} \in \mathcal{G}_0^{\sigma_1}$$

and

$$\mathbb{1}_{\mathcal{B}_1} \tilde{\mathbb{P}}(T_1 = \sigma_1 | \mathcal{G}_0^{\sigma_1}) = \mathbb{1}_{\mathcal{B}_1} \tilde{\mathbb{P}}(\xi(\gamma_{\sigma_1}(\sigma_1)) = 1 | \mathcal{G}_0^{\sigma_1}) \geq \mathbb{1}_{\mathcal{B}_1} \mathbb{P}(B_0) \tag{2.14}$$

Directed random walk on oriented percolation cluster

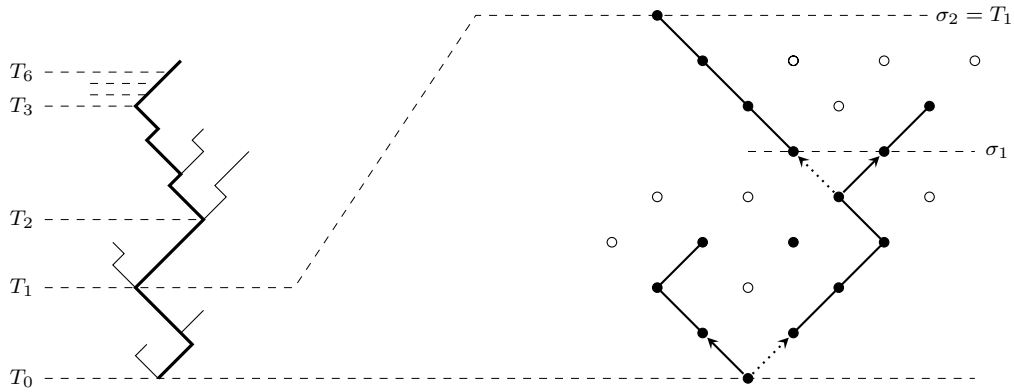


Figure 2: “Discovering” of the trajectory of X between the regeneration times T_0 and T_6 in case $U = \{-1, 1\}$ is shown on the left-hand side of the figure. On the right-hand side we zoom into the evolution between T_0 and T_1 . On the two “relevant sites” we show the realisation of $\tilde{\omega}$ ’s using the same conventions as in Figure 1 (in particular, $U(x, n) = \{(x + 1, n + 1), (x - 1, n + 1)\}$, cf. Remark 1.4).

by (2.12).

Let $\mathcal{B}_2 = \{\gamma_{\sigma_1}(\sigma_1) \notin \mathcal{C}\}$. On $\mathcal{B}_1 \cap \mathcal{B}_2^c$ we know that $T_1 = \sigma_1$, otherwise we define the stopping time $\sigma_2 = \sigma_1 + \ell(\gamma_{\sigma_1}(\sigma_1)) + 2$. By a similar reasoning as before, noting that $\mathcal{B}_2 \in \mathcal{G}_0^{\sigma_2}$, we find

$$\mathbb{1}_{\mathcal{B}_1 \cap \mathcal{B}_2} \tilde{\mathbb{P}}(T_1 = \sigma_2 | \mathcal{G}_0^{\sigma_2}) \geq \mathbb{1}_{\mathcal{B}_1 \cap \mathcal{B}_2} \mathbb{P}(B_0). \tag{2.15}$$

By repeating the same argument, setting recursively

$$\mathcal{B}_{k+1} = \{\gamma_{\sigma_k}(\sigma_k) \notin \mathcal{C}\}, \quad \sigma_{k+1} = \sigma_k + \ell(\gamma_{\sigma_k}(\sigma_k)) + 2 \quad \text{on } \mathcal{B}_1 \cap \mathcal{B}_2 \cap \dots \cap \mathcal{B}_{k+1}, \tag{2.16}$$

we then get

$$\mathbb{1}_{\mathcal{B}_1 \cap \mathcal{B}_2 \cap \dots \cap \mathcal{B}_{k+1}} \mathbb{P}(T_1 = \sigma_{k+1} | \mathcal{G}_0^{\sigma_{k+1}}) \geq \mathbb{1}_{\mathcal{B}_1 \cap \mathcal{B}_2 \cap \dots \cap \mathcal{B}_{k+1}} \mathbb{P}(B_0). \tag{2.17}$$

The number of repetitions needed to find the value of T_1 is thus dominated by a geometric random variable with success probability $\mathbb{P}(B_0) > 0$. Moreover, by Lemma A.1, the random variables $\sigma_{k+1} - \sigma_k = \ell(\gamma_{\sigma_k}(\sigma_k)) + 2$ have exponential tails. By elementary considerations, this implies that T_1 satisfies the desired second inequality in (2.11). See Figure 2 for an illustration of the construction of the T_i .

Finally we should prove that $((\tau_i, Y_i))_{i \geq 1}$ is an i.i.d. sequence. Let $\theta_x, x \in \mathbb{Z}^d \times \mathbb{Z}$ be the standard shift operator on Ω , $(\theta_x \omega)(y) = \omega(x + y)$. We will show that the only information we have about the future of the environment after T_1 (that is about $\omega(x, n)$, $x \in \mathbb{Z}^d, n \geq T_1$) at the instant when we discover T_1 is that $\xi(\gamma_{T_1}(T_1)) = 1$.

To formalise this let $\mathcal{F}_k, k \geq 0$, be the sigma-algebra generated by \mathcal{G}_0^k and the random variables $\xi(\gamma_j(j)), 0 \leq j \leq k$, in particular, \mathcal{F}_k contains the information about the environments ω and $\tilde{\omega}$ that the construction discovers by the time of checking whether $T_1 = k$. Note that T_1 is a stopping time w.r.t. the filtration (\mathcal{F}_k) , write \mathcal{F}_{T_1} for the T_1 -past.

We check that for any event $A \in \mathcal{G}_0^\infty$,

$$\tilde{\mathbb{P}}(\theta_{\gamma_{T_1}(T_1)}(A) | \mathcal{F}_{T_1}) = \tilde{\mathbb{P}}(A). \tag{2.18}$$

Indeed, if this is the case, then (τ_2, Y_2) have the same distribution as (τ_1, Y_1) and they are independent. Proceeding by induction then implies the i.i.d. property.

To prove (2.18) one argues similarly as before: Pick $A' \in \mathcal{F}_{T_1}$, w.l.o.g. assume $A' \subset B_0$ (otherwise consider $A' \cap B_0$). Fix $(z, n) \in \mathbb{Z}^d \times \mathbb{N}$. When $n \geq 2$, by construction,

$$\begin{aligned} \{\gamma_{T_1}(T_1) = (z, n)\} &= \bigcup_{k=1}^{\lfloor n/2 \rfloor} \{T_1 = \sigma_k = n, \gamma_n(n) = (z, n)\} \\ &= \bigcup_{k=1}^{\lfloor n/2 \rfloor} \left(\mathcal{B}_1 \cap \dots \cap \mathcal{B}_k \cap \{\sigma_k = n, \gamma_n(n) = (z, n)\} \cap \{(z, n) \rightarrow \infty\} \right); \end{aligned}$$

for $n = 1$, $\{\gamma_{T_1}(T_1) = (z, 1)\} = \{\gamma_1(1) = (z, 1)\} \cap \{(z, 1) \rightarrow \infty\}$. In particular, there exists $A'_{(z,n)} \in \mathcal{G}_0^n$ such that

$$A' \cap \{\gamma_{T_1}(T_1) = (z, n)\} = A'_{(z,n)} \cap \{(z, n) \rightarrow \infty\} \quad (\subset B_0).$$

Thus

$$\begin{aligned} \mathbb{P}(\theta_{\gamma_{T_1}(T_1)}(A) \cap A' \cap B_0) &= \sum_{(z,n)} \mathbb{P}(\theta_{(z,n)}(A) \cap A' \cap \{\gamma_{T_1}(T_1) = (z, n)\}) \\ &= \sum_{(z,n)} \mathbb{P}(\theta_{(z,n)}(A) \cap \{(z, n) \rightarrow \infty\} \cap A'_{(z,n)}) \\ &= \sum_{(z,n)} \mathbb{P}(\theta_{(z,n)}(A) \cap \{(z, n) \rightarrow \infty\}) \mathbb{P}(A'_{(z,n)}) \tag{2.19} \\ &= \sum_{(z,n)} \mathbb{P}(\theta_{(z,n)}(A) \mid \{(z, n) \rightarrow \infty\}) \mathbb{P}(A'_{(z,n)}) \mathbb{P}(\{(z, n) \rightarrow \infty\}) \\ &= \tilde{\mathbb{P}}(A) \sum_{(z,n)} \mathbb{P}(A'_{(z,n)} \cap \{(z, n) \rightarrow \infty\}) = \tilde{\mathbb{P}}(A) \mathbb{P}(A') = \tilde{\mathbb{P}}(A) \mathbb{P}(A' \cap B_0), \end{aligned}$$

i.e. $\tilde{\mathbb{P}}(\theta_{\gamma_{T_1}(T_1)}(A) \cap A') = \tilde{\mathbb{P}}(A) \tilde{\mathbb{P}}(A')$, where we used the independence of \mathcal{G}_0^n and \mathcal{G}_n^∞ in the third and in the fifth lines and translation invariance in the fifth line. Since $A' \in \mathcal{F}_{T_1}$ is arbitrary, this proves (2.18) and concludes the proof of Lemma 2.5. \square

Remark 2.6 (Continuity of $\sigma(p)$, cf. Remark 1.2). *The construction in the proof of Lemma 2.5 also shows that the functions*

$$p \mapsto \mathbb{E}_p[Y_{1,1}^2], \quad p \mapsto \mathbb{E}_p[\tau_1] \quad \text{and in particular } p \mapsto \sigma^2(p) \tag{2.20}$$

are continuous on $(p_c, 1]$.

Proof. For fixed $z \in \mathbb{Z}, n \in \mathbb{N}$ note that by construction $\{Y_{1,1} = z, \tau_1 = n\} = D_{z,n} \cap \{(z, n) \rightarrow \infty\}$ where $D_{z,n} \in \sigma(\omega(x, m), \tilde{\omega}(x, m), (x, m) \in B_n) \subset \mathcal{G}_0^n$ (with $B_n := \{(x, m) : \|x\| \leq n, 0 \leq m < n\}$) can be expressed as a finite union

$$D_{z,n} = \bigcup_{\bar{\omega} \in C(z,n)} \left(\left\{ \omega(x, m) = \bar{\omega}(x, m) \text{ for } \|x\| \leq n, m < n \right\} \cap \left\{ (\tilde{\omega}(x, m) : \|x\| \leq n, m < n) \in \tilde{C}(z, n, \bar{\omega}) \right\} \right)$$

for certain $C(z, n) \subset \{0, 1\}^{B_n}$ and $\tilde{C}(z, n, \bar{\omega}) \subset \{\text{permutations of } 1, \dots, n\}^{B_n}$. Thus,

$$p \mapsto \mathbb{P}_p(Y_{1,1} = z, \tau_1 = n) = \mathbb{P}_p(D_{z,n}) \mathbb{P}_p(\{(z, n) \rightarrow \infty\}) = \mathbb{P}_p(D_{z,n}) \mathbb{P}_p(\{(0, 0) \rightarrow \infty\})$$

is continuous on $(p_c, 1]$ for any $(z, n) \in \mathbb{Z} \times \mathbb{N}$: Since $D_{z,n}$ depends only on finitely many coordinates of ω and $\tilde{\omega}$, continuity of $p \mapsto \mathbb{P}_p(D_{z,n})$ is obvious e.g. from a simple coupling argument; continuity of $p \mapsto \mathbb{P}_p(\{(0, 0) \rightarrow \infty\})$ is guaranteed by Theorem 2 from [10]. Combining this with exponential tail bounds for $|Y_{1,1}| \leq \tau_1$ that can be chosen uniform in $p \in [p_c + \delta, 1]$ for any $\delta > 0$ (cf. Lemma A.1 and Remark A.3 below) proves (2.20). \square

3 Joint dynamics of two walks on the same realisation of the cluster and the quenched CLT

In this section we study the joint dynamics of two walks on the same realisation of the cluster \mathcal{C} , in order to show the quenched CLT, Theorem 1.3.

For $x, x' \in \mathbb{Z}^d$, let $B_{x,x'}$ be the event $\{\xi_0(x) = \xi_0(x') = 1\}$. Conditioned on ω and $B_{x,x'}$, let $X := (X_n)_n$ and $X' := (X'_n)_n$ be two independent random walks with transition probabilities (1.4) started on $(x, 0)$ and $(x', 0)$ respectively. Observe that X and X' take their steps independently but in the same environment. Note also that, unlike true ancestral lines, the random walks X, X' can meet and then separate again.

We now extend the local construction of the random walk from Subsection 2.1 to the two-walk case. Assume that in addition, a collection of independent random permutations $\tilde{\omega}' = (\tilde{\omega}'(x, n))_{(x,n) \in \mathbb{Z}^d \times \mathbb{Z}}$ with the same distribution as $\tilde{\omega}$ on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is given, and define paths $\gamma_k'^{(x,n)}$ analogously as $\gamma_k^{(x,n)}$ using $\tilde{\omega}'$ instead of $\tilde{\omega}$ but the same ω . Note that for given n and k , the construction of $\gamma_k^{(x,n)}$ and $\gamma_k'^{(x,n)}$ is measurable with respect to

$$\hat{\mathcal{G}}_n^{n+k} := \sigma(\omega(y, i), \tilde{\omega}(y, i), \tilde{\omega}'(y, i) : y \in \mathbb{Z}^d, n \leq i < n + k). \tag{3.1}$$

On $B_{x,x'}$, using the same reasoning as in the previous section, we may couple the random walks X, X' started from x, x' with $\omega, \tilde{\omega}$ and $\tilde{\omega}'$ by

$$(X_k, k) = \lim_{n \rightarrow \infty} \gamma_n^{(x,0)}(k), \quad (X'_k, k) = \lim_{n \rightarrow \infty} \gamma_n'^{(x',0)}(k). \tag{3.2}$$

3.1 Joint regeneration structure of two random walks

The individual regeneration sequences are defined as in Section 2. We now define a joint regeneration sequence for the pair X, X' . We set $T_0 := 0, T'_0 := 0$ and for $r \in \mathbb{Z}_+$ put

$$T_{r+1} := \inf\{n > T_r : \xi(\gamma_n^{(x,0)}(n)) = 1\}, \tag{3.3}$$

$$T'_{r+1} := \inf\{n > T'_r : \xi(\gamma_n'^{(x',0)}(n)) = 1\}. \tag{3.4}$$

Note that if $(X_0, X'_0) = (x, x)$, then under $\mathbb{P}(\cdot | B_{x,x})$, $(X_{T_{r+1}} - X_{T_r}, T_{r+1} - T_r)_r$ is an i.i.d. sequence and $(X'_{T'_{r+1}} - X'_{T'_r}, T'_{r+1} - T'_r)_r$ has the same law but the two objects are of course not independent because both build on the same cluster given by the same ξ .

Now we define the sequence of simultaneous regeneration times. We set $J_0 := 0, J'_0 := 0$ and for $m \in \mathbb{Z}_+$ let

$$J_{m+1} := \min\{j > J_m : T_j = T'_{j'} \text{ for some } j' > J'_m\}, \tag{3.5}$$

$$J'_{m+1} := \min\{j' > J'_m : T'_{j'} = T_j \text{ for some } j > J_m\}, \tag{3.6}$$

then

$$T_m^{\text{sim}} := T_{J_m} = T'_{J'_m}, \quad m = 0, 1, 2, \dots \tag{3.7}$$

is the sequence of simultaneous regeneration times. Note that

$$T_m^{\text{sim}} = \min\left(\{T_j : T_j > T_{m-1}^{\text{sim}}\} \cap \{T'_{j'} : T'_{j'} > T_{m-1}^{\text{sim}}\}\right). \tag{3.8}$$

As in the one walk case we write for the increments $Y_k := X_{T_k} - X_{T_{k-1}}, \tau_k := T_k - T_{k-1}, Y'_k := X'_{T'_k} - X'_{T'_{k-1}}, \tau'_k := T'_k - T'_{k-1}$ and furthermore we define

$$\tilde{X}_m := X_{T_m}, \quad \tilde{X}'_m := X'_{T'_m}, \quad m \in \mathbb{Z}_+, \tag{3.9}$$

$$\hat{X}_\ell := X_{T_\ell^{\text{sim}}} = \tilde{X}_{J_\ell} = X_{T_{J_\ell}}, \quad \hat{X}'_\ell := X'_{T_\ell^{\text{sim}}} = \tilde{X}'_{J'_\ell} = X'_{T'_{J'_\ell}}, \quad \ell \in \mathbb{Z}_+. \tag{3.10}$$

Note that $\tilde{X}_m, \tilde{X}'_m$ will typically refer to the two walks X and X' at different real-time instants.

Let us denote the “pieces between simultaneous regenerations” by

$$\Xi_m := \left((Y_k, \tau_k)_{k=J_{m-1}+1}^{J_m}, (Y'_k, \tau'_k)_{k=J'_{m-1}+1}^{J'_m}, \hat{X}_m, \hat{X}'_m \right), \quad m = 1, 2, \dots \tag{3.11}$$

Note that Ξ_m takes values in $\mathbb{F} \times \mathbb{F} \times \mathbb{Z}^d \times \mathbb{Z}^d$, where $\mathbb{F} := \cup_{n=1}^\infty (\mathbb{Z}^d \times \mathbb{N})^n$.

The following result is the “joint” version of bound (2.11) in Lemma 2.5. Heuristically, since the individual regeneration times have exponential tails and immediate joint regeneration has “positive” probability one can use a “restart”-argument to construct joint regenerations. Because of the dependence of the two walks the proof that we actually give is somewhat more complicated than these heuristics.

Lemma 3.1 (Exponential tail bounds for joint regeneration times). *There exist constants $C, c \in (0, \infty)$ such that*

$$\mathbb{P}(T_1^{\text{sim}} \geq k \mid X_0 = x, X'_0 = x', B_{x,x'}) \leq C e^{-ck}, \quad \forall k \in \mathbb{N}, x, x' \in \mathbb{Z}^d. \tag{3.12}$$

Proof. Let $\gamma_k = \gamma_k^{(x,0)}$ and $\gamma'_k = \gamma'_k^{(x',0)}$. The proof is a variant of the proof of Lemma 2.5, but one should be a little bit more careful not to “discover too many sites where ξ is zero”. More precisely one must not check whether $\xi(\gamma_k(k)) = 1$ and $\xi(\gamma'_k(k)) = 1$ for all k . We proceed as follows. We first check whether $\xi(\gamma_1(1)) = 1$. If this is not the case, then we *do not check* $\xi(\gamma'_1(1))$, set $\sigma_1 = \ell(\gamma_1(1)) + 3$. When $\xi(\gamma_1(1)) = 1$, we check $\xi(\gamma'_1(1))$. When it is 1, then we are done and $T_1^{\text{sim}} = 1$. When it is 0, we set $\sigma_1 = \ell(\gamma'_1(1)) + 3$.

If we are not done, we proceed with the local construction of γ_k and γ'_k , but do not check any other value of ξ until reaching time σ_1 (as it is useless, we need first “discover locally” the fact that one of the ξ ’s we checked before was zero). At time σ_1 we have this information, so we check $\xi(\gamma_{\sigma_1}(\sigma_1))$ first. If it is zero, we set $\sigma_2 = \sigma_1 + \ell(\gamma_{\sigma_1}(\sigma_1)) + 2$. If it is one, we check also $\xi(\gamma'_{\sigma_1}(\sigma_1))$. When also this value is one, we are done, $T_1^{\text{sim}} = \sigma_1$. Otherwise we set $\sigma_2 = \sigma_1 + \ell(\gamma'_{\sigma_1}(\sigma_1)) + 2$. If we are not done, we continue the local construction up to the time σ_2 without checking any ξ ’s. At time σ_2 we check two end points of γ ’s as before, eventually defining σ_3 , etc.

Let, similarly as before, $\hat{\mathcal{F}}_k$ be the σ -algebra generated by $\hat{\mathcal{G}}_0^k$ and all the additional information about ξ ’s discovered by this algorithm (strictly) before time k . To estimate how many steps of the algorithm are necessary, we claim that when σ_k is defined

$$\mathbb{P} \left(\xi(\gamma_{\sigma_k}(\sigma_k)) = 1 = \xi(\gamma'_{\sigma_k}(\sigma_k)) \mid \hat{\mathcal{F}}_{\sigma_k}, B_{x,x'} \right) \geq \mathbb{P}(B_0)^2. \tag{3.13}$$

Indeed, this follows again by the FKG inequality, one should only observe that any negative information contained in the conditioning (that is the knowledge $\xi(z) = 0$ for some z) is contained in $\mathcal{G}_0^{\sigma_k}$ and can thus be removed from the conditioning, similarly to (2.17).

The number of steps of the algorithm is thus dominated by a geometric random variable. Moreover, as the random variables $\sigma_i - \sigma_{i-1}$ have exponential tails, the claim of the lemma follows as before. \square

We next show that $(\Xi_m)_{m \in \mathbb{Z}_+}$ defined in (3.11) form a (discrete) Markov chain.

Lemma 3.2. *Let $x, x' \in \mathbb{Z}^d, X_0 = x, X'_0 = x'$, put $\Xi_0 := (\alpha, \alpha', x, x')$ with arbitrary $\alpha, \alpha' \in \mathbb{F}$. Then under $\mathbb{P}(\cdot \mid B_{x,x'})$, $(\Xi_m)_{m \in \mathbb{Z}_+}$ is a (discrete) Markov chain, its transition probability function*

$$\Psi^{\text{joint}}((\alpha, \alpha', x, x'), (\beta, \beta', y, y')) =: \Psi^{\text{joint}}((x, x'), (\beta, \beta', y, y')) \tag{3.14}$$

depends in its first argument only on the last two coordinates (x, x') , not on (α, α') , and has a spatial homogeneity property:

$$\Psi^{\text{joint}}((x, x'), (\beta, \beta', y, y')) = \Psi^{\text{joint}}((x + z, x' + z), (\beta, \beta', y + z, y' + z)). \quad (3.15)$$

Proof. The proof is a straightforward adaptation of arguments given around (2.18)–(2.19) in the proof of Lemma 2.5. Therefore we do not repeat it here. \square

Remark 3.3. 1. In particular, $(\widehat{X}_\ell, \widehat{X}'_\ell)_\ell$ is in itself a Markov chain on $\mathbb{Z}^d \times \mathbb{Z}^d$ with transition probability

$$\widehat{\Psi}^{\text{joint}}((x, x'), (y, y')) := \Psi^{\text{joint}}((x, x'), \mathbb{F} \times \mathbb{F} \times \{(y, y')\}). \quad (3.16)$$

2. The full path $(\Xi_m)_m$ contains the same amount of information as the pair of sequences $((Y_i, \tau_i)_i, (Y'_i, \tau'_i)_i)$ since these can be reconstructed from a full Ξ -path.

We want to compare the joint distribution of (X, X') run in the same environment with the distribution of two walks run in two independent copies of the environment. To this end we consider the same construction as above, except that now we let X' use ξ' built on ω' , an independent copy of ω . That is, the sequences $(X_{T_{i+1}} - X_{T_i}, T_{i+1} - T_i)_i$ and $(X'_{T'_{i+1}} - X'_{T'_i}, T'_{i+1} - T'_i)_i$ are now independent copies since they use independent realisations of the medium. Obviously, then $(\Xi_m)_{m \in \mathbb{Z}_+}$ as defined in (3.11) is a Markov chain and we denote its transition probability function by

$$\Psi^{\text{ind}}((\alpha, \alpha', x, x'), (\beta, \beta', y, y')) =: \Psi^{\text{ind}}((x, x'), (\beta, \beta', y, y')). \quad (3.17)$$

It again only depends on the last two coordinates (x, x') , not on (α, α') , and has the same spatial homogeneity as Ψ^{joint} . Under Ψ^{ind} , the sequence $(T_\ell^{\text{sim}} - T_{\ell-1}^{\text{sim}})_\ell$ is i.i.d., the law of T_1^{sim} does not depend on the spatial separation (it is simply the law of the first joint renewal of two independent copies of a renewal process whose waiting time distribution is aperiodic and has exponential tails). Finally, similarly as in (3.16) we define

$$\widehat{\Psi}^{\text{ind}}((x, x'), (y, y')) := \Psi^{\text{ind}}((x, x'), \mathbb{F} \times \mathbb{F} \times \{(y, y')\}). \quad (3.18)$$

The next lemma allows us to compare Ψ^{joint} and Ψ^{ind} .

Lemma 3.4 (Total variation distance of Ψ^{joint} and Ψ^{ind}). *There exist constants $c, C > 0$ such that*

$$\|\Psi^{\text{joint}}((x, x'), \cdot) - \Psi^{\text{ind}}((x, x'), \cdot)\|_{\text{TV}} \leq C e^{-c\|x-x'\|}, \text{ for all } x, x' \in \mathbb{Z}^d, \quad (3.19)$$

where $\|\cdot\|_{\text{TV}}$ denotes the total variation norm.

Proof. The proof is an adaptation of the proof of Lemma 3.2(i) in [23]. We give it in the case $x \in \mathbb{Z}^d$ of the form $x = (x_1, 0, \dots, 0)$ for some positive x_1 and $x' = \mathbf{0}$. The general case has more complicated notation but the same arguments.

Let

$$\Omega_i = \{(\omega_i(z, n), \widetilde{\omega}_i(z, n)) : (z, n) \in \mathbb{Z}^d \times \mathbb{Z}_+\}, \quad i = 1, 2 \quad (3.20)$$

be two independent families of independent collections of random variables, where the random variables $\omega_i(z, n)$ are i.i.d. Bernoulli distributed with parameter $p > p_c$ (where p_c is the critical parameter for oriented percolation) and the random variables $\widetilde{\omega}_i(z, n)$ are i.i.d. random permutations of the sets $U(z, n)$ (defined in (1.3)).

Furthermore define $\Omega_3 = \{(\omega_3(z, n), \tilde{\omega}_3(z, n)) : (z, n) \in \mathbb{Z}^d \times \mathbb{Z}\}$ by setting for $z = (z_1, \dots, z_d)$

$$(\omega_3(z, n), \tilde{\omega}_3(z, n)) := \begin{cases} (\omega_1(z, n), \tilde{\omega}_1(z, n)) & : z_1 \leq x_1/2, \\ (\omega_2(z, n), \tilde{\omega}_2(z, n)) & : z_1 > x_1/2. \end{cases}$$

Then of course Ω_1, Ω_2 and Ω_3 have the same distribution and on each of the families we can define the random walks by the local construction of Section 2. To distinguish these walks throughout the proof we will need to denote several variables that we introduced earlier as functions of the Ω_i 's. In particular we write

$$\begin{aligned} &\xi_n(x; \Omega_i) \text{ for } \xi_n(x) \text{ constructed using } \Omega_i, \\ &\ell(x, n; \Omega_i) \text{ for } \ell(x, n) \text{ constructed using } \Omega_i, \\ &\gamma_k^{(x, n)}(m; \Omega_i) \text{ for the path } \gamma_k^{(x, n)}(m) \text{ obtained using } \Omega_i. \end{aligned}$$

Furthermore for $i, j \in \{1, 2, 3\}$ (we will consider the two cases $i = j = 3$ or $i = 1$ and $j = 2$) we set

$$\begin{aligned} B_{x, x'}(\Omega_i, \Omega_j) &:= \{\xi_0(x; \Omega_i) = 1 = \xi_0(x'; \Omega_j)\}, \\ T_{i, j}^{\text{sim}} &:= T^{\text{sim}}(\Omega_i, \Omega_j) := \inf\{n \geq 1 : \xi_n(\gamma_n^{(x, n)}(n; \Omega_i); \Omega_i) = \xi_n(\gamma_n^{(x', n)}(n; \Omega_j); \Omega_j) = 1\}. \end{aligned}$$

Note that on $B_{x, x'}(\Omega_3, \Omega_3)$ the regeneration time $T_{3,3}^{\text{sim}}$ is the simultaneous regeneration time T_1^{sim} defined as in (3.7) using Ω_3 and $T_{1,2}^{\text{sim}}$ is the first simultaneous regeneration time of two independent walks defined on Ω_1 and Ω_2 . In keeping with (3.5–3.6) we will write $J_1(\Omega_i, \Omega_j)$ and $J'_1(\Omega_i, \Omega_j)$ for the number of individual renewals until the first joint renewal of the first, respectively, the second walk when the first walk uses Ω_i and the second Ω_j .

Note also that there are constants $c, C \in (0, \infty)$ such that, $\forall i, j \in \{1, 2, 3\}$,

$$\mathbb{P}(T_{i, j}^{\text{sim}} > r | B_{x, x'}(\Omega_i, \Omega_j)) \leq C e^{-cr}. \tag{3.21}$$

For $i = j = 3$ this assertion was shown in Lemma 3.1. For $i = 1$ and $j = 2$ the inequality is true since the individual regeneration times of two independent random walks are aperiodic and by Lemma 2.5 have exponentially decaying tails.

Recall the definition of Ξ_m in (3.11) and define for $i, j \in \{1, 2, 3\}$

$$\Xi_1(\Omega_i, \Omega_j) := ((Y_k(\Omega_i), \tau_k(\Omega_i))_{k=1}^{J_1(\Omega_i, \Omega_j)}, (Y'_k(\Omega_j), \tau'_k(\Omega_j))_{k=1}^{J'_1(\Omega_i, \Omega_j)}, X_{T_{i, j}^{\text{sim}}}(\Omega_i), X'_{T_{i, j}^{\text{sim}}}(\Omega_j)).$$

Furthermore define, with some cemetery state Δ ,

$$\Xi_{x, x'}^{\text{joint}} := \begin{cases} \Xi_1(\Omega_3, \Omega_3), & \text{if } \xi_0(x; \Omega_3) = \xi_0(x'; \Omega_3) = 1, \\ \Delta, & \text{otherwise,} \end{cases}$$

and

$$\Xi_{x, x'}^{\text{ind}} := \begin{cases} \Xi_1(\Omega_1, \Omega_2), & \text{if } \xi_0(x; \Omega_1) = \xi_0(x'; \Omega_2) = 1, \\ \Delta, & \text{otherwise.} \end{cases}$$

Recall that we are considering the supercritical case $p > p_c$, hence the percolation probability $p_\infty := \mathbb{P}(\xi_0(\mathbf{0}) = 1)$ is strictly positive. Because of positive correlations, we have

$$\mathbb{P}(\Xi_{x, x'}^{\text{joint}} \neq \Delta), \mathbb{P}(\Xi_{x, x'}^{\text{ind}} \neq \Delta) \geq p_\infty^2 \tag{3.22}$$

uniformly in x, x' . Furthermore we have

$$\Psi^{\text{joint}}((x, x'), \cdot) = \mathbb{P} \left(\Xi_{x, x'}^{\text{joint}} = \cdot \mid \Xi_{x, x'}^{\text{joint}} \neq \Delta \right)$$

and

$$\Psi^{\text{ind}}((x, x'), \cdot) = \mathbb{P} \left(\Xi_{x, x'}^{\text{ind}} = \cdot \mid \Xi_{x, x'}^{\text{ind}} \neq \Delta \right).$$

Define $n^* = \lfloor x_1/2 \rfloor$ and set

$$L_1 = \{ \ell(x, 0; \Omega_1) \vee \ell(x', 0; \Omega_2) \leq n^* \}$$

$$L_2 = \{ \xi_0(x; \Omega_1) = \xi_0(x'; \Omega_2) = \xi_0(x; \Omega_3) = \xi_0(x'; \Omega_3) = 1, T_{3,3}^{\text{sim}} \leq n^*, T_{1,2}^{\text{sim}} \leq n^* \}.$$

By definition of Ω_3 and n^* we have

$$\gamma_n^{(x', 0)}(n; \Omega_1) = \gamma_n^{(x', 0)}(n; \Omega_3) \text{ and } \gamma_n^{(x, 0)}(n; \Omega_2) = \gamma_n^{(x, 0)}(n; \Omega_3) \text{ for all } n \in \{0, \dots, n^*\}. \tag{3.23}$$

Furthermore on $L_1 \cup L_2$ we have $\Xi_{x, x'}^{\text{joint}} = \Xi_{x, x'}^{\text{ind}}$. To see this note that on L_1 we have $\Xi_{x, x'}^{\text{joint}} = \Xi_{x, x'}^{\text{ind}} = \Delta$. On L_2 we have $T_{1,2}^{\text{sim}} = T_{3,3}^{\text{sim}}$ and since this is smaller than n^* we obtain by (3.23) that $\Xi_{x, x'}^{\text{joint}} = \Xi_{x, x'}^{\text{ind}}$.

The complement of $L_1 \cup L_2$ is contained in the union of the events

$$\begin{aligned} & \{n^* < \ell(x, 0; \Omega_1) < \infty\}, \{n^* < \ell(x, 0; \Omega_3) < \infty\}, \\ & \{n^* < \ell(x', 0; \Omega_2) < \infty\}, \{n^* < \ell(x', 0; \Omega_3) < \infty\}, \\ & \{ \xi_0(x; \Omega_1) = \xi_0(x'; \Omega_2) = 1, T_{1,2}^{\text{sim}} > n^* \}, \\ & \{ \xi_0(x; \Omega_3) = \xi_0(x'; \Omega_3) = 1, T_{3,3}^{\text{sim}} > n^* \}, \end{aligned}$$

each of which is exponentially decreasing in $\|x - x'\| = x_1$. For the events in the first two lines this follows by Lemma A.1, whereas for the events in the last two lines this is a consequence of (3.21). Thus, there are $c, C \in (0, \infty)$ such that

$$\sum_{w \in \mathcal{W} \cup \{\Delta\}} \left| \mathbb{P} \left(\Xi_{x, x'}^{\text{joint}} = w \right) - \mathbb{P} \left(\Xi_{x, x'}^{\text{ind}} = w \right) \right| = \mathbb{P} \left(\Xi_{x, x'}^{\text{joint}} \neq \Xi_{x, x'}^{\text{ind}} \right) \leq C e^{-cx_1},$$

where $\mathcal{W} := \mathbb{F} \times \mathbb{F} \times \mathbb{Z}^d \times \mathbb{Z}^d$. Now, as in [23] in the display after (3.9) on p. 2236, it follows that

$$\begin{aligned} & \|\Psi^{\text{joint}}((x, x'), \cdot) - \Psi^{\text{ind}}((x, x'), \cdot)\|_{\text{TV}} \\ &= \frac{1}{2} \sum_{w \in \mathcal{W}} \left| \frac{\mathbb{P} \left(\Xi_{x, x'}^{\text{joint}} = w \right)}{\mathbb{P} \left(\Xi_{x, x'}^{\text{joint}} \neq \Delta \right)} - \frac{\mathbb{P} \left(\Xi_{x, x'}^{\text{ind}} = w \right)}{\mathbb{P} \left(\Xi_{x, x'}^{\text{ind}} \neq \Delta \right)} \right| \\ &\leq \frac{1}{2\mathbb{P} \left(\Xi_{x, x'}^{\text{joint}} \neq \Delta \right)} \sum_{w \in \mathcal{W}} \left| \mathbb{P} \left(\Xi_{x, x'}^{\text{joint}} = w \right) - \mathbb{P} \left(\Xi_{x, x'}^{\text{ind}} = w \right) \right| \\ &\quad + \frac{1}{2} \left| \frac{1}{\mathbb{P} \left(\Xi_{x, x'}^{\text{joint}} \neq \Delta \right)} - \frac{1}{\mathbb{P} \left(\Xi_{x, x'}^{\text{ind}} \neq \Delta \right)} \right| \sum_{w \in \mathcal{W}} \mathbb{P} \left(\Xi_{x, x'}^{\text{ind}} = w \right) \\ &\leq \frac{C e^{-c\|x-x'\|}}{\mathbb{P} \left(\Xi_{x, x'}^{\text{joint}} \neq \Delta \right)} + \frac{1}{2} \frac{\left| \mathbb{P} \left(\Xi_{x, x'}^{\text{ind}} \neq \Delta \right) - \mathbb{P} \left(\Xi_{x, x'}^{\text{joint}} \neq \Delta \right) \right|}{\mathbb{P} \left(\Xi_{x, x'}^{\text{joint}} \neq \Delta \right) \mathbb{P} \left(\Xi_{x, x'}^{\text{ind}} \neq \Delta \right)} \\ &\leq \frac{C e^{-c\|x-x'\|}}{\mathbb{P} \left(\Xi_{x, x'}^{\text{joint}} \neq \Delta \right)} + \frac{C e^{-c\|x-x'\|}}{\mathbb{P} \left(\Xi_{x, x'}^{\text{joint}} \neq \Delta \right) \mathbb{P} \left(\Xi_{x, x'}^{\text{ind}} \neq \Delta \right)}. \end{aligned}$$

By (3.22) both denominators in the last line of the above display are bounded away from zero uniformly in x, x' . Thus, for suitably chosen C, c the assertion of the lemma follows in the case $x = (x_1, 0, \dots, 0)$. \square

3.2 Coupling of Ψ^{joint} and Ψ^{ind}

The following lemma is our “target lemma”, which forms the core of the proof of Theorem 1.3.

Lemma 3.5. *There exists $b > 0$ and a non-trivial centred d -dimensional normal law $\tilde{\Phi}$ such that for $f : \mathbb{R}^d \rightarrow \mathbb{R}$ bounded and Lipschitz we have*

$$\mathbb{E} \left[\left(E_\omega [f(\tilde{X}_m/\sqrt{m})] - \tilde{\Phi}(f) \right)^2 \right] \leq C_f m^{-b}. \tag{3.24}$$

We will show this lemma by coupling two Markov chains with transition matrices Ψ^{joint} and Ψ^{ind} , using Lemma 3.4. We need a few technical lemmata beforehand. The first one gives standard estimates on exit distribution from an annulus.

Lemma 3.6. *Write for $r > 0$*

$$\begin{aligned} h(r) &:= \inf \{ k \in \mathbb{Z}_+ : \|\hat{X}_k - \hat{X}'_k\|_2 \leq r \}, \\ H(r) &:= \inf \{ k \in \mathbb{Z}_+ : \|\hat{X}_k - \hat{X}'_k\|_2 \geq r \}, \end{aligned} \tag{3.25}$$

where $\|\cdot\|_2$ denotes the Euclidean norm on \mathbb{Z}^d , and set for $r_1 < r < r_2$

$$f_d(r; r_1, r_2) = \begin{cases} \frac{r_1^{2-d} - r^{2-d}}{r_1^{2-d} - r_2^{2-d}}, & \text{when } d \geq 3, \\ \frac{\log r - \log r_1}{\log r_2 - \log r_1}, & \text{when } d = 2, \\ \frac{r - r_1}{r_2 - r_1}, & \text{when } d = 1. \end{cases} \tag{3.26}$$

For every $\varepsilon > 0$ there are (large) R and \tilde{R} such that for all $r_2 > r_1 > R$ with $r_2 - r_1 > \tilde{R}$ and $x, y \in \mathbb{Z}^d$ satisfying $r_1 < r = \|x - y\|_2 < r_2$

$$(1 - \varepsilon) f_d(r; r_1, r_2) \leq \mathbb{P}_{x,y}^{\text{ind}}(H(r_2) < h(r_1)) \leq (1 + \varepsilon) f_d(r; r_1, r_2). \tag{3.27}$$

Remark 3.7. *We use for simplicity the sup-norm $\|x\| = \max_{1 \leq i \leq d} |x_i|$ instead of the Euclidean norm $\|x\|_2$ in the rest of the paper. Since all norms on \mathbb{R}^d are equivalent, we can and shall translate between them by an appropriate adjustments of constants. We will assume this implicitly below when applying Lemma 3.6.*

Proof of Lemma 3.6. From the construction of the transition probability Ψ^{ind} it follows that under \mathbb{P}^{ind} the Markov chain $(\hat{X}_n - \hat{X}'_n)_n$ is a random walk on \mathbb{Z}^d with i.i.d. increments whose distribution is symmetric (and thus centred) and has a finite variance. The claim is then a direct consequence of the usual invariance principle and exit probabilities from an annulus by a d -dimensional Brownian motion, see e.g. Theorem 3.18 in [16]. \square

We use $\mathbb{P}_{x,y}^{\text{joint}}$ to denote the distribution of the canonical Markov chain Ξ with transition probabilities Ψ^{joint} started from $\Xi_0 = (\alpha, \alpha', x, y)$. Note that by (3.14) this distribution does not depend on α, α' . Similarly, $\mathbb{P}_{x,y}^{\text{ind}}$ denotes the law of the chain with the transition matrix Ψ^{ind} . In both cases, with a slight abuse of notation, $\tilde{X}, \tilde{X}', \hat{X}, \hat{X}'$ denote the corresponding underlying chains which can be read from Ξ , see Remark 3.3.

In particular, under \mathbb{P}^{ind} , $(\widehat{X}_n, \widehat{X}'_n)_n$ is a Markov chain with transition probability $\widehat{\Psi}^{\text{ind}}$, given in (3.18).

From now on, we assume in this section that $d \geq 2$ and complete the proof of Lemma 3.5 under this assumption. The case $d = 1$ requires different arguments that are postponed to Section 3.4.

As we do not have a good control on the difference between $\Psi^{\text{ind}}((x, x'), \cdot)$ and $\Psi^{\text{joint}}((x, x'), \cdot)$ when $\|x - x'\|$ is small, we need to ensure that \widehat{X} and \widehat{X}' separate sufficiently quickly under $\mathbb{P}^{\text{joint}}$. This is shown in the next lemma:

Lemma 3.8 (Separation lemma). *Let $d \geq 2$ and let $H(r)$ be as in (3.25). There are $b_1, b_2 \in (0, 1/2)$, $b_3 > 0$, $b_4 \in (0, 1)$ such that for n large enough,*

$$\mathbb{P}_{\mathbf{0}, \mathbf{0}}^{\text{joint}}(H(n^{b_1}) \geq n^{b_2}) \leq e^{-b_3 n^{b_4}}. \tag{3.28}$$

Proof. We split the proof in six steps.

Step 1. We first observe that there exists a (small) $\varepsilon_1 > 0$ and $b_4 \in (0, 1/2)$, $b_5 > 0$ such that

$$\mathbb{P}_{x, y}^{\text{joint}}(H(\varepsilon_1 \log n) > n^{b_4}) \leq n^{-b_5} \quad \text{for } n \text{ large enough,} \tag{3.29}$$

uniformly in $x, y \in \mathbb{Z}^d$. To see this we use the definition of $(\widehat{X}_n, \widehat{X}'_n)$ via the joint law of the cluster and two walkers on it to construct suitable “corridors” in opposite directions in the random environment, and force the two walks to walk along these corridors. Namely, denoting $r = \lfloor \varepsilon_1 \log n \rfloor$, and assuming without loss of generality that $x \cdot e_1 \leq y \cdot e_1$, we require that $\omega(x - ke_1, k) = \omega(y + ke_1, k) = 1$ for all $k = 1, \dots, r$, that $\xi(x - re_1, r) = \xi(y + re_1, r) = 1$, and that the permutations $\tilde{\omega}, \tilde{\omega}'$ are such $\tilde{\omega}(x - ke_1, k)[1] = (x - (k + 1)e_1, k + 1)$, $\tilde{\omega}'(y + ke_1, k)[1] = (y + (k + 1)e_1, k + 1)$, for all $k = 0, \dots, r - 1$. The probability that these requirements are satisfied can be easily bounded from below by δ_1^r for some $\delta_1 = \delta_1(p, U) \in (0, 1)$. If the requirements are satisfied, then $T_k^{\text{sim}} = k$ and thus $\widehat{X}_k = x - ke_1$, $\widehat{X}'_k = y + ke_1$ for all $k = 1, \dots, r$. Therefore, we see that uniformly over $x, y \in \mathbb{Z}^d$

$$\mathbb{P}_{x, y}^{\text{joint}}(\|\widehat{X}_j - \widehat{X}'_j\| \geq j) \geq \delta_1^j. \tag{3.30}$$

Thus, the probability that (\widehat{X}_n) and (\widehat{X}'_n) have distance $\varepsilon_1 \log n$ after the first $\varepsilon_1 \log n$ steps is at least $n^{-\varepsilon_1 \log(1/\delta_1)}$. If this happens, we are done, otherwise, we can try again by the Markov property. By the uniformity of the bound in (3.30), we have

$$\mathbb{P}_{x, y}^{\text{joint}}(H(\varepsilon_1 \log n) > m\varepsilon_1 \log n) \leq (1 - n^{-\varepsilon_1 \log(1/\delta_1)})^m \leq \exp(-mn^{-\varepsilon_1 \log(1/\delta_1)}). \tag{3.31}$$

Now let ε_1 be so small that $-\varepsilon_1 \log \delta_1 \in (0, 1/2)$, pick $b_4 \in (-\varepsilon_1 \log \delta_1, 1/2)$, $b_5 > 0$ and set $m = b_5 n^{\varepsilon_1 \log(1/\delta_1)} \log n$ in (3.31).

Step 2. Next we claim that for any $K_2 > 0$ we can pick a $\delta_2 \in (0, 1)$ such that for all $x, y \in \mathbb{Z}^d$ with $\varepsilon_1 \log n \leq \|x - y\| < K_2 \log n$ and n large enough

$$\mathbb{P}_{x, y}^{\text{joint}}(H(K_2 \log n) < h(\frac{1}{2}\varepsilon_1 \log n) \wedge (K_2 \log n)^3) \geq \delta_2, \tag{3.32}$$

where h is the stopping time defined in (3.25). To this end we couple $\mathbb{P}_{x, y}^{\text{joint}}$ with $\mathbb{P}_{x, y}^{\text{ind}}$ using Lemma 3.4, in a standard way. This coupling implies that the left-hand side of (3.32) is bounded from below by

$$\mathbb{P}_{x, y}^{\text{ind}}(H(K_2 \log n) < h(\frac{1}{2}\varepsilon_1 \log n) \wedge (K_2 \log n)^3) - C(K_2 \log n)^3 n^{-c\varepsilon_1/2}, \tag{3.33}$$

where the second term is an upper bound on the probability that the coupling fails before the time $\min\{H(K_2 \log n), h(\frac{1}{2}\varepsilon_1 \log n), (K_2 \log n)^3\}$. This bound follows from the

fact that before this time the distance between \widehat{X} , \widehat{X}' is at least $\frac{1}{2}\varepsilon_1 \log n$ and thus the probability that the coupling fails in one step is at most $\exp\{-\frac{\varepsilon}{2}\varepsilon_1 \log n\} = n^{-c\varepsilon_1/2}$.

The first term in (3.33) is bounded from below by a small constant δ' , uniformly in n , as follows from Lemma 3.6 (with $r_1 = \frac{1}{2}\varepsilon_1 \log n$, $r_2 = K_2 \log n$) and the fact that $\mathbb{P}_{x,y}^{\text{ind}}(\inf\{k : \|\widehat{X}_k - \widehat{X}'_k\| \geq K_2 \log n\} > (K_2 \log n)^3) \rightarrow 0$ as $n \rightarrow \infty$. The latter assertion holds because by the invariance principle we have for a d -dimensional Brownian motion

$$\begin{aligned} & \mathbb{P}_{x,y}^{\text{ind}} \left(\inf \left\{ k : \|\widehat{X}_k - \widehat{X}'_k\| \geq K_2 \log n \right\} > (K_2 \log n)^3 \right) \\ & \approx \mathbb{P} \left(\inf \left\{ t : \|B_t\| \geq \frac{K_2 \log n}{\sqrt{2n}} \right\} > \frac{(K_2 \log n)^3}{n} \right) \leq \frac{n}{(K_2 \log n)^3} \frac{(K_2 \log n)^2}{\sqrt{2dn}}, \end{aligned}$$

where the last inequality follows from the Markov inequality and the fact that the expected exit time of a d -dimensional Brownian from a ball of radius r is bounded by r^2/d . As the second term in (3.33) converges to 0 as $n \rightarrow \infty$, the proof of (3.32) is completed.

Step 3. By repeating the argument from Step 1 and using (3.32) from Step 2, we see that we can choose a (large) K_3 , $b_6 \in (b_4, 1/2)$ such that uniformly in $x, y \in \mathbb{Z}^d$

$$\mathbb{P}_{x,y}^{\text{joint}} \left(H(K_3 \log n) \leq n^{b_6} \right) \geq \delta_3 > 0 \quad \text{for } n \text{ large enough.} \tag{3.34}$$

The previous steps work for all $d \geq 2$. In the next steps we shall treat separately the cases $d = 2$ and $d \geq 3$. We start with the case $d \geq 3$.

Step 4 ($d \geq 3$). Arguing as in Step 2, we can find $b_1 \in (0, 1/6)$ such that for all $x, y \in \mathbb{Z}^d$ with $K_3 \log n \leq \|x - y\| < n^{b_1}$ and n large enough

$$\mathbb{P}_{x,y}^{\text{joint}} \left(H(n^{b_1}) < h\left(\frac{1}{2}K_3 \log n\right) \wedge n^{3b_1} \right) \geq \delta_4 > 0. \tag{3.35}$$

Step 5 ($d \geq 3$). Now we recycle the argument from Step 1 as follows: Wait until (\widehat{X}_n) and (\widehat{X}'_n) have reached distance at least $K_3 \log n$ or stop if the waiting time exceeds n^{b_6} . Then, let $(\widehat{X}_n, \widehat{X}'_n)$ run until they have either reached distance n^{b_1} or have taken (another) n^{3b_1} steps without reaching that distance. Note that by construction, such an attempt takes at most $n^{b_6} + n^{3b_1}$ time steps, and by (3.34) and (3.35), with probability at least $\delta_3\delta_4$ leads to a separation of n^{b_1} , as required. If it does not, we start afresh, using the Markov property and the uniformity of (3.34, 3.35) in their respective initial conditions.

Pick b_2, b_7 such that $b_6 \vee 3b_1 < b_7 < b_2 < 1/2$ (in particular, $n^{b_7} \geq n^{b_6} + n^{3b_1}$ for n large enough), put $b_4 := b_2 - b_7$, $b_3 := -\log(1 - \delta_3\delta_4)$. The probability that the first n^{b_4} attempts all fail is bounded above by $(1 - \delta_3\delta_4)^{n^{b_4}} = \exp(-b_3 n^{b_4})$, and by construction these first n^{b_4} attempts take at most $n^{b_4}(n^{b_6} + n^{3b_1}) \leq n^{b_4+b_7} = n^{b_2}$ time steps, which proves the lemma for $d \geq 3$.

Step 6 ($d = 2$). In $d = 2$, the relation (3.35) should be replaced by

$$\mathbb{P}_{x,y}^{\text{joint}} \left(H(n^{b_1}) < h\left(\frac{1}{2}K_3 \log n\right) \wedge n^{3b_1} \right) \geq \frac{\log 2}{b_1 \log n}, \tag{3.36}$$

which can be shown using the same argument as in Step 2, together with Lemma 3.6. The argument in Step 5 is then analogous: The probability that the walks separate by distance n^{b_1} in the first $n^{b_6} + n^{3b_1}$ steps is bounded from below by $\delta_3 \log 2 / (b_1 \log n)$. Fixing $b_4 = (b_2 - b_7)/2$, we see that the probability that the first n^{2b_4} attempts fail is smaller than $(1 - \delta_3 \log 2 / (b_1 \log n))^{n^{2b_4}} \leq \exp(-b_3 n^{b_4})$, for some small b_3 and n large enough. These first n^{2b_4} attempts take at most $n^{2b_4}(n^{b_4} + n^{3b_1}) \leq n^{b_2}$ time steps, as required. \square

Using the above separation lemma, we now construct a coupling of $\mathbb{P}_{0,0}^{\text{joint}}$ and $\mathbb{P}_{0,0}^{\text{ind}}$ so that the increments differ only in few steps.

Lemma 3.9 (Coupling of dependent and independent Ξ -chains). *For any $d \geq 2$ there is a Markov chain $(\Xi_n^{\text{joint}}, \Xi_n^{\text{ind}})_n$ with state space $(\mathbb{F} \times \mathbb{F} \times \mathbb{Z}^d \times \mathbb{Z}^d)^2$ such that Ξ_n^{joint} is $\mathbb{P}_{0,0}^{\text{joint}}$ - and Ξ_n^{ind} is $\mathbb{P}_{0,0}^{\text{ind}}$ -distributed. Furthermore, writing $\Xi_n^{\text{joint}} = (\alpha_n^{\text{joint}}, \alpha_n^{\prime\text{joint}}, \widehat{X}_n^{\text{joint}}, \widehat{X}_n^{\prime\text{joint}})$ and $\Xi_n^{\text{ind}} = (\alpha_n^{\text{ind}}, \alpha_n^{\prime\text{ind}}, \widehat{X}_n^{\text{ind}}, \widehat{X}_n^{\prime\text{ind}})$, there exist $b_6 > 0$ and $b_5 \in (0, 1/2)$ so that for all N large enough,*

$$\mathbb{P}(\#\{k \leq N : (\alpha_k^{\text{joint}}, \alpha_k^{\prime\text{joint}}) \neq (\alpha_k^{\text{ind}}, \alpha_k^{\prime\text{ind}})\} \geq N^{b_5}) \leq N^{-b_6}. \tag{3.37}$$

Proof. First observe that under $\mathbb{P}_{0,0}^{\text{ind}}$ the sequences α^{ind} and $\alpha^{\prime\text{ind}}$ are independent and i.i.d., as the law of the increments up to the next regeneration do not depend on the positions $\widehat{X}^{\text{ind}}, \widehat{X}^{\prime\text{ind}}$.

Using this observation, the construction of the coupling in $d \geq 3$ is easy. We first run the Markov chains $\Xi^{\text{ind}}, \Xi^{\text{joint}}$ independently according to their prescribed laws up to the first time T when $\|\widehat{X}_T^{\text{joint}} - \widehat{X}_T^{\prime\text{joint}}\| \geq N^{b_1}$, for b_1 as in Lemma 3.8. Let A_1 be the event $A_1 = \{T \leq N^{b_2}\}$. According to Lemma 3.8, $\mathbb{P}(A_1) \geq 1 - e^{-b_3 N^4}$.

Let, for some large K , A_2 be the event

$$A_2 := \{\|\widehat{X}_k^{\text{joint}} - \widehat{X}_k^{\prime\text{joint}}\| \geq K \log N \text{ for all } T \leq k \leq N\}. \tag{3.38}$$

By comparing $\widehat{X}^{\text{joint}}$ with \widehat{X}^{ind} , as in the proof of Lemma 3.8, and using elementary properties of the random walk and the fact $\|\widehat{X}_T^{\text{joint}} - \widehat{X}_T^{\prime\text{joint}}\| \geq N^{b_1}$, it is easy to see that $\mathbb{P}[A_2^c] \leq N^{-c}$ for some $c > 0$

On $A_1 \cap A_2$, couple Ξ^{ind} and Ξ^{joint} so that $(\alpha_k^{\text{joint}}, \alpha_k^{\prime\text{joint}}) = (\alpha_k^{\text{ind}}, \alpha_k^{\prime\text{ind}})$ when possible for all $k \in [T, N]$. Using Lemma 3.4 and the observation in the first paragraph of this proof, the probability that this does not occur is at most $N e^{-cK \log N}$. On $A_1^c \cup A_2^c$, run $\Xi^{\text{ind}}, \Xi^{\text{joint}}$ independently up to time N . Obviously, this coupling satisfies (3.37).

In dimension $d = 2$ the situation is slightly more complicated, as the event A_2 has a small probability. We need to decompose the trajectory of Ξ^{joint} into excursions. For large constants K, K' , we define stopping times $\mathcal{R}_i, \mathcal{D}_i, \mathcal{U}$ by $\mathcal{R}_0 = 0$, and for $i \geq 1$

$$\begin{aligned} \mathcal{D}_i &= \inf\{k \geq \mathcal{R}_{i-1} : \|\widehat{X}_k^{\text{joint}} - \widehat{X}_k^{\prime\text{joint}}\| \geq N^{b_1}\}, \\ \mathcal{R}_i &= \inf\{k \geq \mathcal{D}_i : \|\widehat{X}_k^{\text{joint}} - \widehat{X}_k^{\prime\text{joint}}\| \leq K \log N\}, \\ \mathcal{U} &= \inf\{k \geq 0 : \|\widehat{X}_k^{\text{joint}} - \widehat{X}_k^{\prime\text{joint}}\| \geq K'N\}. \end{aligned} \tag{3.39}$$

We set J to be the unique integer such that $\mathcal{D}_J \leq \mathcal{U} \leq \mathcal{R}_J$. The random variable J has a geometric distribution with parameter converging to b_1 as $N \rightarrow \infty$, as can be easily shown by comparing Ξ^{joint} with Ξ^{ind} as in the proof of Lemma 3.8 and applying Lemma 3.6. Therefore, $\mathbb{P}(J \geq \log N) \leq N^{-c}$ for a $c > 0$. Applying Lemma 3.8, we get

$$\mathbb{P}(\mathcal{D}_i - \mathcal{R}_{i-1} \geq N^{b_2}) \leq e^{-b_3 N^{b_4}}. \tag{3.40}$$

Combining these two facts we obtain

$$\mathbb{P}\left(\sum_{i=1}^J \mathcal{D}_i - \mathcal{R}_{i-1} \geq N^{b_2} \log N\right) \leq N^{-c}. \tag{3.41}$$

On the other hand, comparing again Ξ^{joint} with Ξ^{ind} , and using simple random walk large deviation estimates, for K' large enough,

$$\mathbb{P}(\mathcal{U} \leq N) \leq e^{-cN} \leq N^{-c}. \tag{3.42}$$

Inequalities (3.41) and (3.42) yield

$$\begin{aligned} & \mathbb{P} \left(\#\{k \leq N : \|\widehat{X}_k^{\text{joint}} - \widehat{X}'_k^{\text{joint}}\| \leq K \log N\} \geq N^{b_2} \log N \right) \\ & \leq \mathbb{P} \left(\sum_{i: \mathcal{R}_{i-1} \leq N} \mathcal{D}_i - \mathcal{R}_{i-1} \geq N^{b_2} \log N \right) \leq N^{-c}. \end{aligned} \tag{3.43}$$

If the event on the left-hand side of the last display does not occur, we can with probability at least $1 - N^{-c}$ couple Ξ^{joint} and Ξ^{ind} so that $(\alpha_k^{\text{joint}}, \alpha'_k{}^{\text{joint}}) = (\alpha_k^{\text{ind}}, \alpha'_k{}^{\text{ind}})$ for all k satisfying $\mathcal{D}_i \leq k \leq \mathcal{R}_i$ for some i , using Lemma 3.4 (and noting that under Ψ^{ind} , the law of $(\alpha_1^{\text{ind}}, \alpha'_1{}^{\text{ind}})$ does not depend on the starting point). Taking b_5 satisfying $b_2 < b_5 < 1/2$, (3.37) is proved for $d = 2$. \square

Lemma 3.10. *Let $d \geq 2$. Recall that $\widetilde{X}, \widetilde{X}'$ are read from Ξ as in Remark 3.3. Then, there exist $b, C > 0$ such that for every pair of bounded Lipschitz functions $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$*

$$\begin{aligned} & \left| \mathbb{E}_{\mathbf{0}, \mathbf{0}}^{\text{joint}} [f(\widetilde{X}_n/\sqrt{n})g(\widetilde{X}'_n/\sqrt{n})] - \mathbb{E}_{\mathbf{0}, \mathbf{0}}^{\text{ind}} [f(\widetilde{X}_n/\sqrt{n})g(\widetilde{X}'_n/\sqrt{n})] \right| \\ & \leq C(1 + \|f\|_\infty + L_f)(1 + \|g\|_\infty + L_g)n^{-b}, \end{aligned} \tag{3.44}$$

where $L_f := \sup_{x \neq y} |f(y) - f(x)|/\|y - x\|$ and L_g are the Lipschitz constants of f and g .

Proof. We use the coupling constructed in the last lemma. Let \mathcal{I} be the complement of the set of indices appearing in (3.37),

$$\mathcal{I} = \{k \leq n : (\alpha_k^{\text{joint}}, \alpha'_k{}^{\text{joint}}) = (\alpha_k^{\text{ind}}, \alpha'_k{}^{\text{ind}})\}, \tag{3.45}$$

and set $\mathcal{I}^c = \{0, \dots, n\} \setminus \mathcal{I}$. By the last lemma,

$$\mathbb{P}(|\mathcal{I}^c| \geq n^{b_5}) \leq n^{-b_6}. \tag{3.46}$$

We now read the processes $\widetilde{X}^{\text{joint}}, \widetilde{X}'^{\text{joint}}, \widetilde{X}^{\text{ind}}, \widetilde{X}'^{\text{ind}}$ out of $\Xi^{\text{joint}}, \Xi^{\text{ind}}$. Recall notation (3.3)–(3.10). We use the additional superscript \cdot^{ind} or \cdot^{joint} in this notation in the obvious way, and write $T^{\text{joint}}, T^{\text{ind}}$ for T^{sim} corresponding to those processes.

By Lemma 3.1, and standard large deviation estimates, there is a large constant K so that

$$\mathbb{P}(T_n^{\text{joint}} \geq Kn) \leq e^{-cn} \tag{3.47}$$

and, using (3.46) as well,

$$\mathbb{P} \left(\sum_{i \in \mathcal{I}^c} (T_i^{\text{joint}} - T_{i-1}^{\text{joint}}) \geq Kn^{b_5} \right) \leq n^{-b_6}, \tag{3.48}$$

and similarly for T^{ind} .

We now consider the process $\widetilde{X}^{\text{ind}}$ and define two sets

$$\begin{aligned} \mathcal{G}^{\text{ind}} & := \{1 \leq k \leq J_n^{\text{ind}} : [T_{k-1}, T_k] \subset [T_{j-1}^{\text{ind}}, T_j^{\text{ind}}] \text{ for some } j \in \mathcal{I}\} \\ \mathcal{B}^{\text{ind}} & := \{1, \dots, J_n^{\text{ind}}\} \setminus \mathcal{G}^{\text{ind}}. \end{aligned} \tag{3.49}$$

We define $\mathcal{G}'^{\text{ind}}, \mathcal{G}'^{\text{joint}}, \dots$ analogously. Note that \mathcal{G} 's are the sets of indices of those increments that ‘occur during coupled periods of Ξ 's’. More precisely, by the coupling construction, there is an order-preserving bijection ϕ of \mathcal{G}^{ind} and $\mathcal{G}^{\text{joint}}$ so that for every $j \in \mathcal{G}^{\text{ind}}$ (using the notation introduced before (3.9)),

$$Y_j^{\text{ind}} = \widetilde{X}_j^{\text{ind}} - \widetilde{X}_{j-1}^{\text{ind}} = \widetilde{X}_{\phi(j)}^{\text{joint}} - \widetilde{X}_{\phi(j)-1}^{\text{joint}} = Y_{\phi(j)}^{\text{joint}}. \tag{3.50}$$

Therefore, setting $[n] := \{1, \dots, n\}$, and $\mathcal{R}_{[n]} := \mathcal{R} \cap [n]$ for any set \mathcal{R} , we can write

$$\begin{aligned} \tilde{X}_n^{\text{ind}} &= \sum_{i=1}^n Y_i^{\text{ind}} = \sum_{i \in \mathcal{G}_{[n]}^{\text{ind}}} Y_i^{\text{ind}} + \sum_{i \in \mathcal{B}_{[n]}^{\text{ind}}} Y_i^{\text{ind}} \\ &= \sum_{j \in \phi(\mathcal{G}_{[n]}^{\text{ind}})} Y_j^{\text{joint}} + \sum_{i \in \mathcal{B}_{[n]}^{\text{ind}}} Y_i^{\text{ind}} \\ &= \tilde{X}_n^{\text{joint}} + \sum_{j \in \phi(\mathcal{G}_{[n]}^{\text{ind}}) \setminus [n]} Y_j^{\text{joint}} - \sum_{j \in \mathcal{G}_{[n]}^{\text{joint}} \setminus \phi(\mathcal{G}_{[n]}^{\text{ind}})} Y_j^{\text{joint}} + \sum_{i \in \mathcal{B}_{[n]}^{\text{ind}}} Y_i^{\text{ind}} - \sum_{i \in \mathcal{B}_{[n]}^{\text{joint}}} Y_i^{\text{joint}}. \end{aligned} \tag{3.51}$$

Similar claims hold for the processes with primes.

Inequality (3.48) implies that

$$\mathbb{P}(|\mathcal{B}^{\text{ind}}| \vee |\mathcal{B}'^{\text{ind}}| \vee |\mathcal{B}^{\text{joint}}| \vee |\mathcal{B}'^{\text{joint}}| \geq Kn^{b_5}) \leq n^{-b_6}. \tag{3.52}$$

When the complement of event in (3.52) holds, then all four sums on the right-hand side of (3.51) have length at most $2Kn^{b_5}$. Since all relevant increments have (uniformly bounded) exponential tails, by standard large deviation estimates we can choose K' large so that the probability that the absolute value of these four sums exceed $K'n^{b_5}$ is at most $e^{-cn^{b_5}}$, for all n large enough.

Putting all these claims together we see that there is an event A satisfying $\mathbb{P}(A^c) \leq n^{-b'}$ with $b' > 0$, so that on A

$$|\tilde{X}_n^{\text{ind}} - \tilde{X}_n^{\text{joint}}| \leq K'n^{b_5} \quad \text{and} \quad |\tilde{X}'_n{}^{\text{ind}} - \tilde{X}'_n{}^{\text{joint}}| \leq K'n^{b_5}. \tag{3.53}$$

Therefore,

$$\begin{aligned} & \left| \mathbb{E}_{\mathbf{0}, \mathbf{0}}^{\text{joint}} [f(\tilde{X}_n/\sqrt{n})g(\tilde{X}'_n/\sqrt{n})] - \mathbb{E}_{\mathbf{0}, \mathbf{0}}^{\text{ind}} [f(\tilde{X}_n/\sqrt{n})g(\tilde{X}'_n/\sqrt{n})] \right| \\ &= \left| \mathbb{E} [f(\tilde{X}_n^{\text{joint}}/\sqrt{n})g(\tilde{X}'_n{}^{\text{joint}}/\sqrt{n}) - f(\tilde{X}_n^{\text{ind}}/\sqrt{n})g(\tilde{X}'_n{}^{\text{ind}}/\sqrt{n})] \right| \\ &\leq 2\mathbb{P}(A^c)\|f\|_\infty\|g\|_\infty \\ &\quad + \mathbb{E} \left[\mathbb{1}_A |f(\tilde{X}_n^{\text{joint}}/\sqrt{n})g(\tilde{X}'_n{}^{\text{joint}}/\sqrt{n}) - f(\tilde{X}_n^{\text{ind}}/\sqrt{n})g(\tilde{X}'_n{}^{\text{ind}}/\sqrt{n})| \right]. \end{aligned} \tag{3.54}$$

Observing that

$$|f(x)g(y) - f(x')g(y')| \leq \|g\|_\infty L_f \|x - x'\| + \|f\|_\infty L_g \|y - y'\|, \tag{3.55}$$

and $b_5 < 1/2$, using (3.53), this implies the claim with $b := b' \wedge (1/2 - b_5)$. \square

Remark 3.11 (Functional limit theorem). *For the quenched functional limit theorem in the case $d \geq 2$ we need also a functional version of the above theorem. The proof above can be adapted to show the analogue of (3.53), i.e. that on some event \tilde{A} , satisfying $\mathbb{P}(\tilde{A}^c) \leq n^{-b'}$ with $b' > 0$, we have*

$$\sup_{t \in [0,1]} |\tilde{X}_{[nt]}^{\text{ind}} - \tilde{X}_{[nt]}^{\text{joint}}| \leq \tilde{K}n^{b_5} \quad \text{and} \quad \sup_{t \in [0,1]} |\tilde{X}'_{[nt]}{}^{\text{ind}} - \tilde{X}'_{[nt]}{}^{\text{joint}}| \leq \tilde{K}n^{b_5}.$$

Proof of Lemma 3.5, case $d \geq 2$. Lemma 3.5 follows now easily from Lemma 3.10 together with standard Berry-Esseen type estimates for the term $\mathbb{E}[f(\tilde{X}_n/\sqrt{n})g(\tilde{X}'_n/\sqrt{n})]$ appearing in (3.44) there.

The case $d = 1$ requires a somewhat different approach and is given in Section 3.4. \square

Lemma 3.12. Assume that for some $b > 1$, and any bounded Lipschitz function $f : \mathbb{R}^d \rightarrow \mathbb{R}$

$$E_\omega [f(\tilde{X}_{k^b}/k^{b/2})] \xrightarrow[k \rightarrow \infty]{} \tilde{\Phi}(f) \quad \text{for } \mathbb{P}(\cdot | B_0)\text{-a.a. } \omega, \tag{3.56}$$

where $\tilde{\Phi}$ is some non-trivial centred d -dimensional normal law. Then we have for any bounded Lipschitz function f

$$E_\omega [f(\tilde{X}_m/m^{1/2})] \xrightarrow[m \rightarrow \infty]{} \tilde{\Phi}(f) \quad \text{for } \mathbb{P}(\cdot | B_0)\text{-a.a. } \omega. \tag{3.57}$$

Proof. Note that under $\mathbb{P}(\cdot | B_0)$, the increments $Y_i = \tilde{X}_i - \tilde{X}_{i-1}$ are i.i.d., centred and satisfy $\mathbb{E}[\exp(\lambda \cdot Y_1) | B_0] < \infty$ for all λ in some neighbourhood of the origin. Let $(2-b) \vee 0 < \varepsilon < 1$. By a moderate deviation principle (e.g. Theorem 3.7.1 in [3] where we set $n = bk^{b-1}$, $a_n = bk^{-1+\varepsilon}$, then $a_n \rightarrow 0$, $na_n \rightarrow \infty$, $\sqrt{a_n/n} = k^{-(b-\varepsilon)/2}$), we have for any $\delta > 0$

$$\mathbb{P}\left(\max_{1 \leq i \leq bk^{b-1}} \frac{|\tilde{X}_i|}{k^{(b-\varepsilon)/2}} \geq \delta \mid B_0\right) \leq Ck^{b-1} \exp(-ck^\eta) \tag{3.58}$$

for some $C, c, \eta > 0$ (η can be chosen in $(0, 1-\varepsilon)$) and all $k \in \mathbb{N}$. Since the right-hand side of (3.58) is summable in k , noting that $k^b - (k-1)^b \leq bk^{b-1}$, we obtain by Borel-Cantelli

$$\limsup_{k \rightarrow \infty} \max_{(k-1)^b \leq \ell \leq k^b} \frac{|\tilde{X}_\ell - \tilde{X}_{(k-1)^b}|}{k^{b/2}} \leq \limsup_{k \rightarrow \infty} \max_{(k-1)^b \leq \ell \leq k^b} \frac{|\tilde{X}_\ell - \tilde{X}_{(k-1)^b}|}{k^{(b-\varepsilon)/2}} = 0 \tag{3.59}$$

for $\mathbb{P}(\cdot | B_0)$ -a.a. ω . □

3.3 Proof of Theorem 1.3

The idea is to use the control of variance provided by Lemma 3.10 to obtain the quenched CLT. This approach seems to have appeared in the literature on random walk in random environments for the first time in [2]. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be bounded and Lipschitz, $b' > 1/b \vee 1$ with b from Lemma 3.5. By (3.24) and Markov's inequality,

$$\mathbb{P}\left(\left|E_\omega[f(\tilde{X}_{[n^{b'}]}/\sqrt{[n^{b'}]})] - \tilde{\Phi}(f)\right| > \varepsilon\right) \leq C_f \varepsilon^{-2} n^{-b'b}, \tag{3.60}$$

which is summable, hence

$$E_\omega [f(\tilde{X}_{[n^{b'}]}/\sqrt{[n^{b'}]})] \rightarrow \tilde{\Phi}(f) \quad \text{a.s. as } n \rightarrow \infty \tag{3.61}$$

by Borel-Cantelli. Now Lemma 3.12 yields

$$E_\omega [f(\tilde{X}_m/m^{1/2})] \xrightarrow[m \rightarrow \infty]{} \tilde{\Phi}(f) \quad \text{for } \mathbb{P}(\cdot | B_0)\text{-a.a. } \omega. \tag{3.62}$$

Put

$$V_n := \max\{m \in \mathbb{Z}_+ : T_m \leq n\}. \tag{3.63}$$

We have $V_n/n \rightarrow 1/\mathbb{E}[\tau_1]$ a.s. as $n \rightarrow \infty$ by classical renewal theory, in fact

$$\limsup_{n \rightarrow \infty} \frac{|V_n - n/\mathbb{E}[\tau_1]|}{\sqrt{n \log \log n}} < \infty \quad \text{a.s.} \tag{3.64}$$

(see, e.g. Theorem III.11.1 in [11]). Since the differences of the T_n are i.i.d. with exponential tail bounds,

$$\limsup_{n \rightarrow \infty} \frac{\max_{j \leq n} \{j - T_{V_j}\}}{\log n} < \infty \quad \text{a.s.} \tag{3.65}$$

Recall that $X_{T_{V_n}} = \tilde{X}_{V_n}$. Now

$$P_\omega(\|X_n - \tilde{X}_{V_n}\| \geq \log^2 n) \rightarrow 0 \quad \text{a.s.} \tag{3.66}$$

by (3.65) and the fact that X has bounded increments. Furthermore, for any $\varepsilon > 0$

$$P_\omega(|V_n - n/\mathbb{E}[\tau_1]| \geq n^{1/2+\varepsilon}) \rightarrow 0 \quad \text{a.s.} \tag{3.67}$$

by (3.64).

Moreover, there exist $\beta \in (1/2, 1)$ and $\gamma \in (\beta/2, 1/2)$ such that for any $\theta \geq 0$,

$$\limsup_{n \rightarrow \infty} \sup_{|k-\theta n| \leq n^\beta} \frac{|\tilde{X}_k - \tilde{X}_{[\theta n]}|}{n^\gamma} \rightarrow 0 \quad \text{a.s.} \tag{3.68}$$

To prove (3.68) note that we can take w.l.o.g. $\theta = 0$. By Doob's L^6 -inequality, for any $\varepsilon > 0$

$$\mathbb{P}\left(\sup_{k \leq n^\beta} |\tilde{X}_k - \tilde{X}_0| > \varepsilon n^\gamma\right) \leq \varepsilon^{-6} n^{-6\gamma} \mathbb{E}\left[\|\tilde{X}_{n^\beta} - \tilde{X}_0\|^6\right] \leq \text{Const.} \varepsilon^{-6} n^{3\beta-6\gamma} \tag{3.69}$$

(note that $\mathbb{E}_0[\|S_k\|^6] \leq Ck^3$ for a random walk (S_k) whose increments are centred and have bounded 6th moments), thus we can choose $\beta > 1/2 > \gamma$ and both sufficiently close to $1/2$ that $3\beta - 6\gamma < -1$ and the right-hand side of (3.69) becomes summable in n . The usual Borel-Cantelli argument allows to conclude (3.68).

Now write

$$\frac{X_n}{\sqrt{n}} = \frac{X_n - \tilde{X}_{V_n}}{\sqrt{n}} + \frac{\tilde{X}_{V_n} - \tilde{X}_{[n/\mathbb{E}[\tau_1]]}}{\sqrt{n}} + \frac{\tilde{X}_{[n/\mathbb{E}[\tau_1]]}}{\sqrt{n/\mathbb{E}[\tau_1]}} \sqrt{1/\mathbb{E}[\tau_1]} \tag{3.70}$$

and let Φ be defined by $\Phi(f) := \tilde{\Phi}(f((\mathbb{E}[\tau_1])^{-1/2} \cdot))$, i.e. Φ is the image measure of $\tilde{\Phi}$ under $x \mapsto x/\sqrt{\mathbb{E}[\tau_1]}$. Then

$$\begin{aligned} & \left| E_\omega[f(X_n/n^{1/2})] - \Phi(f) \right| \\ & \leq 2\|f\|_\infty \left(P_\omega(\|X_n - \tilde{X}_{V_n}\| \geq \log^2 n) + P_\omega(|V_n - n/\mathbb{E}[\tau_1]| \geq n^\beta) \right. \\ & \quad \left. + P_\omega\left(\sup_{|k-n/\mathbb{E}[\tau_1]| \leq n^\beta} |\tilde{X}_k - \tilde{X}_{[n/\mathbb{E}[\tau_1]]}| > n^\gamma\right) \right) \\ & \quad + L_f(\log^2 n/\sqrt{n} + n^{\gamma-1/2}) \\ & \quad + \left| E_\omega[f((\mathbb{E}[\tau_1])^{-1/2} \times \tilde{X}_{[n/\mathbb{E}[\tau_1]]}/\sqrt{n/\mathbb{E}[\tau_1]})] - \tilde{\Phi}(f((\mathbb{E}[\tau_1])^{-1/2} \cdot)) \right| \\ & \rightarrow 0 \quad \text{a.s. as } n \rightarrow \infty. \end{aligned}$$

This proves (1.8) for bounded Lipschitz functions f , which in particular implies that the laws of (X_n/\sqrt{n}) under P_ω are tight, for almost all ω . Finally, note that a general continuous bounded f can be approximated by bounded Lipschitz functions in a locally uniform way. \square

3.4 Proof of Lemma 3.5, case $d = 1$

Here we prove Lemma 3.5 in the case $d = 1$. In the cases $d \geq 2$ our proof made substantial use of the fact that two d -dimensional random walks typically do not spend much time near each other: The “approximate collision time” during the first n steps is typically $O(\log n)$ in $d = 2$ and $O(1)$ in $d \geq 3$. Thus, we could couple two walks on the same cluster with two walks on independent copies of the cluster and the error incurred becomes negligible when dividing by \sqrt{n} in the CLT (see the proofs of Lemmas 3.9 and 3.10). An analogous strategy can not be “naïvely” implemented in $d = 1$ because now we expect typically $O(\sqrt{n})$ hits of the two walks, so using simple “worst case bounds” in the region where the two walkers are close would yield an (overly pessimistic) error term that does not vanish upon dividing by \sqrt{n} . Instead, we now use a martingale decomposition for the dynamics of $(\widehat{X}_n, \widehat{X}'_n)_n$ under $\widehat{\Psi}^{\text{joint}}$, combined with a quantitative martingale CLT from [18] to estimate the Kantorovich distance from the bivariate normal.

Consider as a toy example a Markov chain on \mathbb{Z} that behaves as a symmetric simple random walk as long as $X_n \neq 0$. Upon hitting 0 the chain stays there for some random time and leaves this state symmetrically. If the distribution of the time the chain spends in 0 is suitably controlled, one can prove a central limit theorem with non-trivial limit by using a martingale central limit theorem. A similar argument works for our two random walks (observed along joint renewal times) where in this case “being at zero” corresponds to the event that the two walks are closer together than $K \log n$ for some appropriate constant K . This is the “black box” region in which we cannot couple the two walks to independent copies. If they are more than n^b for small b away from each other then we can couple with very good control of the error (cf. Lemma 3.4). Then we make use of the symmetries of the model and the fact that in $d = 1$, the walks \widehat{X} and \widehat{X}' have many overcrossings to verify that the error terms stemming from times inside the black box up to time n are in fact $o(\sqrt{n})$ (in a suitably quantitative sense).

Let $(\widehat{X}_n, \widehat{X}'_n)_n$ be a pair of walks in $d = 1$ on the same cluster observed along joint renewal times as in (3.10), which is in itself a Markov chain when averaging over the cluster, and let $\widehat{\Psi}^{\text{joint}}((x, x'), (y, y'))$ be its transition probability, as defined in (3.16) in Remark 3.3. We write $\widehat{\mathcal{F}}_n := \sigma(\widehat{X}_i, \widehat{X}'_i, 0 \leq i \leq n)$ for the canonical filtration of this chain.

Furthermore set

$$\begin{aligned} \phi_1(x, x') &:= \sum_{y, y'} (y - x) \widehat{\Psi}^{\text{joint}}((x, x'), (y, y')), \\ \phi_2(x, x') &:= \sum_{y, y'} (y' - x') \widehat{\Psi}^{\text{joint}}((x, x'), (y, y')), \\ \phi_{11}(x, x') &:= \sum_{y, y'} (y - x - \phi_1(x, x'))^2 \widehat{\Psi}^{\text{joint}}((x, x'), (y, y')), \\ \phi_{22}(x, x') &:= \sum_{y, y'} (y' - x' - \phi_2(x, x'))^2 \widehat{\Psi}^{\text{joint}}((x, x'), (y, y')), \\ \phi_{12}(x, x') &:= \sum_{y, y'} (y - x - \phi_1(x, x'))(y' - x' - \phi_2(x, x')) \widehat{\Psi}^{\text{joint}}((x, x'), (y, y')). \end{aligned}$$

Note that by Lemma 3.1 these are uniformly bounded, i.e.

$$C_\phi := \|\phi_1\|_\infty \vee \|\phi_2\|_\infty \vee \|\phi_{11}\|_\infty \vee \|\phi_{12}\|_\infty \vee \|\phi_{22}\|_\infty < \infty. \tag{3.71}$$

Define

$$\begin{aligned}
 A_n^{(1)} &:= \sum_{j=0}^{n-1} \phi_1(\widehat{X}_j, \widehat{X}'_j), & A_n^{(2)} &:= \sum_{j=0}^{n-1} \phi_2(\widehat{X}_j, \widehat{X}'_j), \\
 A_n^{(11)} &:= \sum_{j=0}^{n-1} \phi_{11}(\widehat{X}_j, \widehat{X}'_j), & A_n^{(22)} &:= \sum_{j=0}^{n-1} \phi_{22}(\widehat{X}_j, \widehat{X}'_j), & A_n^{(12)} &:= \sum_{j=0}^{n-1} \phi_{12}(\widehat{X}_j, \widehat{X}'_j), \\
 M_n &:= \widehat{X}_n - A_n^{(1)}, & M'_n &:= \widehat{X}'_n - A_n^{(2)}.
 \end{aligned} \tag{3.72}$$

Then, (M_n) , (M'_n) , $(M_n^2 - A_n^{(11)})$, $(M_n'^2 - A_n^{(22)})$ and $(M_n M'_n - A_n^{(12)})$ are martingales whose increments have exponential tails (by Lemma 3.1, which in particular shows that exponential tail bounds can be chosen uniformly in n).

Write $\widehat{\sigma}^2 := \sum_{y,y'} y^2 \widehat{\Psi}^{\text{ind}}((0,0), (y,y'))$ for the variance of a single increment under $\widehat{\Psi}^{\text{ind}}$ (recall (3.18)).

Lemma 3.13. *There exist $C > 0$, $b \in (0, 1/4)$ such that for all bounded Lipschitz continuous $f : \mathbb{R}^2 \rightarrow \mathbb{R}$*

$$\left| \mathbb{E} \left[f \left(\frac{\widehat{X}_n}{\widehat{\sigma}\sqrt{n}}, \frac{\widehat{X}'_n}{\widehat{\sigma}\sqrt{n}} \right) \right] - \mathbb{E} [f(Z)] \right| \leq L_f \frac{C}{n^b} \tag{3.73}$$

where Z is two-dimensional standard normal and L_f the Lipschitz constant of f .

By Lemma 3.4, there exist $C_1, K > 0$ such that for $x, x' \in \mathbb{Z}$ with $|x - x'| \geq K \log n$

$$|\phi_1(x, x')|, |\phi_2(x, x')|, |\phi_{12}(x, x')| \leq \frac{C_1}{n^2}, \tag{3.74}$$

$$|\phi_{11}(x, x') - \widehat{\sigma}^2|, |\phi_{22}(x, x') - \widehat{\sigma}^2| \leq \frac{C_1}{n^2}. \tag{3.75}$$

Put

$$R_n := \#\{0 \leq j \leq n : |\widehat{X}_j - \widehat{X}'_j| \leq K \log n\}. \tag{3.76}$$

We expect that $R_n = o(n)$ in probability (see below for a proof), which combined with (3.74), (3.75) would yield

$$\frac{A_n^{(11)}}{n} \rightarrow \widehat{\sigma}^2, \quad \frac{A_n^{(22)}}{n} \rightarrow \widehat{\sigma}^2, \quad \frac{A_n^{(12)}}{n} \rightarrow 0 \quad \text{in } \mathbb{P}(\cdot|B_0)\text{-probability as } n \rightarrow \infty. \tag{3.77}$$

Note that (3.77) together with our (exponential) tail bounds for the differences would already imply a two-dimensional CLT for $(M_n/\sqrt{n}, M'_n/\sqrt{n})$ by standard martingale CLT results. Since we require quantitative control in the martingale CLT, we have to estimate a little more carefully.

For $n \in \mathbb{N}$, let

$$Q_n := \begin{pmatrix} \phi_{11}(\widehat{X}_{n-1}, \widehat{X}'_{n-1}) & \phi_{12}(\widehat{X}_{n-1}, \widehat{X}'_{n-1}) \\ \phi_{12}(\widehat{X}_{n-1}, \widehat{X}'_{n-1}) & \phi_{22}(\widehat{X}_{n-1}, \widehat{X}'_{n-1}) \end{pmatrix} \tag{3.78}$$

be the conditional covariance matrix given \widehat{F}_{n-1} of the \mathbb{R}^2 -valued random variable $(M_n - M_{n-1}, M'_n - M'_{n-1})$, let $\lambda_{n,1} \geq \lambda_{n,2} \geq 0$ be its eigenvalues. We obtain from (3.74), (3.75) and (3.71) together with well-known stability properties for the eigenvalues of symmetric matrices that

$$|\lambda_{j+1,1} - \widehat{\sigma}^2| + |\lambda_{j+1,2} - \widehat{\sigma}^2| \leq C_2 \mathbb{1}_{\{|\widehat{X}_j - \widehat{X}'_j| \leq K \log n\}} + \frac{C_2}{n^2} \mathbb{1}_{\{|\widehat{X}_j - \widehat{X}'_j| > K \log n\}} \tag{3.79}$$

for some $C_2 < \infty$, see, e.g., §41 in [9] or Chapter IV, Theorem 3.1 in [21]. In particular,

$$\sum_{i=1}^2 \left| n\widehat{\sigma}^2 - \sum_{j=1}^n \lambda_{j,i} \right| \leq \frac{C_2}{n} + C_2 R_n. \tag{3.80}$$

Lemma 3.14. 1. *There exist $0 \leq \delta_R < 1/2$, $c_R < \infty$ such that*

$$\mathbb{E}[R_n^{3/2}] \leq c_R n^{1+\delta_R} \quad \text{for all } n. \tag{3.81}$$

2. *There exist $\delta_C > 0$, $c_C < \infty$ such that*

$$\mathbb{E} \left[\frac{|A_n^{(1)}|}{\sqrt{n}} \right], \mathbb{E} \left[\frac{|A_n^{(2)}|}{\sqrt{n}} \right] \leq \frac{c_C}{n^{\delta_C}} \quad \text{for all } n. \tag{3.82}$$

Now we have all ingredients for the

Proof of Lemma 3.13. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be bounded Lipschitz continuous with Lipschitz constant L_f and Z two-dimensional standard normal. We obtain from (3.80), (3.81) and Corollary 1.3 in [18] that

$$\left| \mathbb{E} \left[f \left(\frac{M_n}{\widehat{\sigma}\sqrt{n}}, \frac{M'_n}{\widehat{\sigma}\sqrt{n}} \right) \right] - \mathbb{E}[f(Z)] \right| \leq L_f \frac{C}{n^{b'}} \quad \text{for all } n \tag{3.83}$$

for some $C < \infty$ and $b' = (\frac{1}{2} - \delta_R)/3$. (Read $X_k = ((M_k - M_{k-1})/\sqrt{\widehat{\sigma}^2 n}, (M'_k - M'_{k-1})/\sqrt{\widehat{\sigma}^2 n})$ in Corollary 1.3 in [18], note that $\sup_k \mathbb{E}[\|(M_k - M_{k-1}, M'_k - M'_{k-1})\|^3] < \infty$ by the uniform exponential tail bounds from Lemma 3.1.)

Combining (3.83) and (3.82) yields

$$\left| \mathbb{E} \left[f \left(\frac{\widehat{X}_n}{\widehat{\sigma}\sqrt{n}}, \frac{\widehat{X}'_n}{\widehat{\sigma}\sqrt{n}} \right) \right] - \mathbb{E}[f(Z)] \right| \leq L_f \frac{C}{n^{b'}} + L_f \frac{c_C}{\widehat{\sigma} n^{\delta_C}}. \tag{3.84}$$

□

To prepare the proof of Lemma 3.14 we need some further notation: Put (with suitable $b \in (0, 1/2)$ and $K \gg 1$, see below) for $n \in \mathbb{N}$ $\mathcal{R}_{n,0} := 0$ and for $i \in \mathbb{N}$

$$\begin{aligned} \mathcal{D}_{n,i} &= \min\{m > \mathcal{R}_{n,i-1} : |\widehat{X}_m - \widehat{X}'_m| \geq n^b\}, \\ \mathcal{R}_{n,i} &= \min\{m > \mathcal{D}_{n,i} : |\widehat{X}_m - \widehat{X}'_m| \leq K \log n\}, \end{aligned}$$

then $[\mathcal{R}_{n,i-1}, \mathcal{D}_{n,i})$ is the i th “black box interval” on “coarse-graining level” n (note that for $m \in \cup_i [\mathcal{R}_{n,i-1}, \mathcal{D}_{n,i})$ the coupling result, Lemma 3.4, does not help; whereas for $m \in \cup_i [\mathcal{D}_{n,i}, \mathcal{R}_{n,i})$, we can couple $(\widehat{X}_m, \widehat{X}'_m)$ with a pair of walks on independent copies of the cluster up to a small error term).

We distinguish four possible types of such black box intervals, depending on the ordering of \widehat{X} and \widehat{X}' at the beginning and end of the interval: Set

$$W_{n,i} := \begin{cases} 1 & \text{if } \widehat{X}_{\mathcal{R}_{n,i-1}} > \widehat{X}'_{\mathcal{R}_{n,i-1}}, \widehat{X}_{\mathcal{D}_{n,i}} < \widehat{X}'_{\mathcal{D}_{n,i}}, \\ 2 & \text{if } \widehat{X}_{\mathcal{R}_{n,i-1}} > \widehat{X}'_{\mathcal{R}_{n,i-1}}, \widehat{X}_{\mathcal{D}_{n,i}} > \widehat{X}'_{\mathcal{D}_{n,i}}, \\ 3 & \text{if } \widehat{X}_{\mathcal{R}_{n,i-1}} < \widehat{X}'_{\mathcal{R}_{n,i-1}}, \widehat{X}_{\mathcal{D}_{n,i}} > \widehat{X}'_{\mathcal{D}_{n,i}}, \\ 4 & \text{if } \widehat{X}_{\mathcal{R}_{n,i-1}} < \widehat{X}'_{\mathcal{R}_{n,i-1}}, \widehat{X}_{\mathcal{D}_{n,i}} < \widehat{X}'_{\mathcal{D}_{n,i}}. \end{cases} \tag{3.85}$$

By construction and the strong Markov property of $(\widehat{X}_m, \widehat{X}'_m)_m$ [with a grain of salt because at a time $\mathcal{R}_{n,i}$, the difference may be $[K \log n]$ or $[K \log n] - 1$, etc.] we have the following: For each $n \in \mathbb{N}$,

$$(\mathcal{R}_{n,i} - \mathcal{D}_{n,i})_{i=1,2,\dots} \text{ is an i.i.d. sequence, and} \tag{3.86}$$

$$(W_{n,i}, \mathcal{D}_{n,i} - \mathcal{R}_{n,i-1})_{i=2,3,\dots} \text{ is a Markov chain;} \tag{3.87}$$

furthermore, the two objects are independent, the transition probabilities of the chain $(W_{n,i}, \mathcal{D}_{n,i} - \mathcal{R}_{n,i-1})_i$ depend only on the first (the “type”) coordinate, and a bound of the form analogous to Lemma 3.8 holds.

Lemma 3.15 (Separation and overcrossing lemma for $d = 1$). *We can choose $0 < b_2 < 1/4$, $b_3, b_4 > 0$ such that*

$$\mathbb{P}(\mathcal{D}_{n,i} - \mathcal{R}_{n,i-1} \geq n^{b_2} \mid W_{n,i} = w) \leq e^{-b_3 n^{b_4}}, \quad w \in \{1, 2, 3, 4\}, \quad n \in \mathbb{N}. \quad (3.88)$$

Furthermore, there exists $\epsilon > 0$ such that uniformly in n

$$\mathbb{P}(W_{n,2} = a' \mid W_{n,1} = a) \geq \epsilon \quad (3.89)$$

for all pairs of types $(a, a') \in \{1, 2, 3, 4\}^2$ where a transition is “logically possible” (cf. (3.85)).

Proof sketch. The proof of (3.88) is analogous to that of Lemma 3.8, making use of the $d = 1$ -case of Lemma 3.6.

For (3.89) the crucial point is to show that when \widehat{X} and \widehat{X}' have come closer than $K \log n$ at time $m = \mathcal{R}_{n,i}$ with $\widehat{X}_m > \widehat{X}'_m$, say, there is a chance of at least $\delta > 0$ (uniformly in n) that they reverse their roles before reaching a distance of n^b , i.e. there is $j < \mathcal{D}_{n,i+1}$ such that $\widehat{X}_j \leq \widehat{X}'_j$.

To see this, write $D_j := \widehat{X}_j - \widehat{X}'_j$, pick $0 < \epsilon \ll K$ (to be tuned later). When the process D starts from $K \log n$, there is a chance $\geq \delta' > 0$ that it reaches $\epsilon \log n$ before $2K \log n$ within less than $\log^3 n$ steps (use the coupling from Lemma 3.4 and analogous results for simple random walk on \mathbb{Z}^1 , note that the probability that the coupling fails is at most $Ce^{-c\epsilon n} \log^3 n$); once $D_j \leq \epsilon \log n$ there is a chance of at least $\exp(-c'\epsilon \log n) = n^{-c'\epsilon}$ that D hits $(-\infty, 0]$ with the next $(\epsilon \log n)/2$ steps (construct suitable “corridors” as in Step 1 of the proof of Lemma 3.8). When D does not hit $(-\infty, 0]$ but instead reaches $2K \log n$ observe that

$$\mathbb{P}(D \text{ hits } K \log n \text{ before } n^b \mid D_0 = 2K \log n) \approx \frac{n^b - 2K \log n}{n^b - K \log n} = 1 - \frac{K \log n}{n^b - K \log n}.$$

By the Markov property, we will thus have a geometric number of excursions from $2K \log n$ that reach $K \log n$ but not n^b , and each of these has a chance $\geq \delta' n^{-c'\epsilon}$ to hit $(-\infty, 0]$. Thus, if ϵ is chosen so small that $c'\epsilon < b$ there is a substantial chance that one of them will be successful. \square

Note that (3.89) guarantees that the chain $(W_{n,i})_i$ is uniformly in n (exponentially) mixing. By symmetry of the construction,

$$\mathbb{P}(W_{n,j} = 1) = \mathbb{P}(W_{n,j} = 3) \quad \text{and} \quad \mathbb{P}(W_{n,j} = 2) = \mathbb{P}(W_{n,j} = 4) \quad \text{for all } j, n, \quad (3.90)$$

and the same holds for the stationary distribution $\pi_{W,n}$ of $(W_{n,j})_j$.

Proof of Lemma 3.14. Let Y have distribution

$$\mathbb{P}(Y \geq \ell) = \widehat{\Psi}^{\text{ind}}\left(\inf\{m \geq 0 : \widehat{X}_m < \widehat{X}'_m\} \geq \ell \mid (\widehat{X}_0, \widehat{X}'_0) = (1, 0)\right), \quad \ell \in \mathbb{N}$$

and let V be an independent, Bernoulli($1 - 1/n$)-distributed random variable. A simple coupling construction based on Lemma 3.4 shows that $\mathcal{R}_{n,1} - \mathcal{D}_{n,1}$ is stochastically larger than

$$((1 - V) + VY) \wedge n \quad (3.91)$$

(by choosing K appropriately, we can ensure that the probability that the relevant coupling between $\widehat{\Psi}^{\text{ind}}$ and $\widehat{\Psi}^{\text{joint}}$ fails during the first n steps is less than $1/n$). In particular, by well-known tail probability estimates for ladder times of one-dimensional random walks, there exist $c > 0$ and $c_Y > 0$ such that uniformly in n ,

$$\mathbb{E}[e^{-\lambda((1-V)+VY)}] \leq \exp(-c_Y\sqrt{\lambda}), \quad \lambda \geq 0 \quad \text{and} \quad (3.92)$$

$$\mathbb{P}(\mathcal{R}_{n,1} - \mathcal{D}_{n,1} \geq \ell) \geq \frac{c}{\sqrt{\ell}}, \quad \ell = 1, \dots, n. \quad (3.93)$$

Let $I_n := \max\{i : T_i \leq n\}$ be the number of “black boxes” that we see up to time n . We have $I_n = \mathcal{O}(\sqrt{n})$ in probability and in fact

$$\mathbb{E}[I_n^2] \leq Cn \quad (3.94)$$

as can be seen from (3.93) by comparison with a renewal process with inter-arrival law given by the return times of a (fixed) one-dimensional random walk.

More quantitatively, there is $c > 0$ such that for $1 \leq k \leq n$,

$$\mathbb{P}(I_n \geq k) \leq \exp(-ck^2/n), \quad (3.95)$$

so in particular

$$\mathbb{E}[I_n \mathbb{1}_{\{I_n \geq n^{3/4}\}}] = \sum_{k=\lceil n^{3/4} \rceil}^n \mathbb{P}(I_n \geq k) \leq ne^{-c\sqrt{n}}. \quad (3.96)$$

((3.95) is a standard result for lower deviations of a heavy-tailed renewal process. For completeness' sake (and lack of a point reference), here are some details: Let Y'_1, Y'_2, \dots be i.i.d. copies of $((1-V) + VY)$ from (3.91), then for $1 \leq k \leq n$

$$\begin{aligned} \mathbb{P}(I_n \geq k) &= \mathbb{P}((Y'_1 \wedge n) + \dots + (Y'_k \wedge n) \leq n) = \mathbb{P}(Y'_1 + \dots + Y'_k \leq n) \\ &\leq e^{\lambda n} \left(\mathbb{E}[\exp(-\lambda Y_1)] \right)^k \leq \exp(\lambda n - kc_Y\sqrt{\lambda}) \end{aligned} \quad (3.97)$$

for any $\lambda > 0$ by (3.92), now put $\lambda := (c_Y k/n)^2$.)

Note that

$$R_n \leq \sum_{j=1}^{I_n+1} (\mathcal{D}_{n,j} - \mathcal{R}_{n,j-1}), \quad (3.98)$$

thus, using (3.88), indeed $R_n = o(n)$ in probability, and (3.94) together with (3.98), (3.88) implies (3.81):

$$\mathbb{E}[R_n^2] \leq n^2 \mathbb{P}(\exists j \leq n : \mathcal{D}_{n,j+1} - \mathcal{R}_{n,j} \geq n^{b_2}) + n^{2b_2} \mathbb{E}[I_{n+1}^2] \leq Cn^{1+2b_2},$$

and $\mathbb{E}[R_n^{3/2}] \leq (\mathbb{E}[R_n^2])^{3/4}$.

For (3.82) we must make use of cancellations in the increments of $A_n^{(1)}, A_n^{(2)}$, making use of the fact that “opposite” types of crossings of \widehat{X} and \widehat{X}' appear asymptotically with the same frequency. Let

$$D_{n,m} := A_{\mathcal{D}_{n,m}}^{(1)} - A_{\mathcal{R}_{n,m-1}}^{(1)}, \quad D'_{n,m} := A_{\mathcal{D}_{n,m}}^{(2)} - A_{\mathcal{R}_{n,m-1}}^{(2)}.$$

By symmetry,

$$\begin{aligned} \mathbb{E}[D_{n,j}] &= 0, \quad \mathbb{E}[D_{n,j} \mid W_{n,j} = 1] = -\mathbb{E}[D_{n,j} \mid W_{n,j} = 3], \\ \mathbb{E}[D_{n,j} \mid W_{n,j} = 2] &= -\mathbb{E}[D_{n,j} \mid W_{n,j} = 4], \end{aligned} \quad (3.99)$$

(3.88) and (3.71) together show that for some $C < \infty$, $b > 0$ uniformly in $j, n \in \mathbb{N}$, $w \in \{1, 2, 3, 4\}$

$$\mathbb{E}[|D_{n,j}| \mid W_{n,j} = w] \leq Cn^b \tag{3.100}$$

and analogously for $D'_{n,j}$. Put $\mathcal{G}_j := \widehat{\mathcal{F}}_{\mathcal{D}_{n,j}}$ (the σ -field of the $\mathcal{D}_{n,j}$ -past) for $j \in \mathbb{N}$, for $j \leq 0$ let \mathcal{G}_j be the trivial σ -algebra. $D_{n,j}, D'_{n,j}$ are \mathcal{G}_j -adapted for $j \in \mathbb{N}$. Since for $k < m$

$$\mathbb{E}[D_{n,m} \mid \mathcal{G}_k] = \mathbb{E}[\mathbb{E}[D_{n,m} \mid W_{n,m}] \mid \mathcal{G}_k]$$

by construction and $(W_{n,j})_j$ is (uniformly in n) exponentially mixing we have, observing (3.99), (3.90), and (3.100)

$$\mathbb{E} \left[\left(\mathbb{E}[D_{n,m} \mid \mathcal{G}_{m-j}] \right)^2 \right] \leq Ce^{-cj}, \quad m, j \in \mathbb{N}, n \in \mathbb{N} \tag{3.101}$$

for some $C, c \in (0, \infty)$ and analogous bounds for $D'_{n,m}$. Let $S_{n,m} := \sum_{j=1}^m D_{n,j}$, $S'_{n,m} := \sum_{j=1}^m D'_{n,j}$ then for each $n \in \mathbb{N}$, $(S_{n,m})_m$ is a mixingale with uniformly (in n) controlled mixing rate (see [12], p. 19).

Thus, using McLeish's analogue of Doob's \mathcal{L}^2 -inequality for mixingales (e.g. [12], Lemma 2.1), we have

$$\mathbb{E} \left[\max_{m=1, \dots, n^{3/4}} S_{n,m}^2 \right] \leq Kn^{3/4}, \tag{3.102}$$

hence

$$\mathbb{E} \left[\frac{|S_{n,I_n}|}{\sqrt{n}} \mathbb{1}_{\{I_n \leq n^{3/4}\}} \right] \leq \frac{1}{\sqrt{n}} \left(\mathbb{E} \left[\max_{m=1, \dots, n^{3/4}} S_{n,m}^2 \right] \right)^{1/2} \leq \frac{K^{1/2}}{n^{1/8}}. \tag{3.103}$$

By (3.74) we have $A_n^{(1)} = \sum_{j=1}^{I_n} D_{n,j} + O(1)$, so

$$\begin{aligned} \mathbb{E} \left[\frac{|A_n^{(1)}|}{\sqrt{n}} \right] &\leq \frac{c}{\sqrt{n}} + \frac{1}{\sqrt{n}} \mathbb{E} \left[\frac{|S_{n,I_n}|}{\sqrt{n}} \mathbb{1}_{\{I_n \leq n^{3/4}\}} \right] + \frac{1}{\sqrt{n}} Cn^{1+b_2} \mathbb{E}[I_n \mathbb{1}_{\{I_n \geq n^{3/4}\}}] \\ &\quad + \frac{1}{\sqrt{n}} cn \mathbb{P}(\exists j \leq n : \mathcal{D}_{n,j} - \mathcal{R}_{n,j-1} \geq n^{b_2}). \end{aligned} \tag{3.104}$$

Using (3.103), (3.96) and (3.88), respectively on the last three terms on the right hand side (and analogous bounds for $A_n^{(2)}$) yields (3.82). \square

Remark 3.16. *The arguments used in the proof of Lemma 3.14 can be used to show that there exists $C < \infty$ such that for all $n \in \mathbb{N}$*

$$\sup_{\ell \in \mathbb{N}_0} \mathbb{E} \left[\sup_{0 \leq k \leq n^{2/3}} \left(\widehat{X}_{\ell+k} - \widehat{X}_\ell \right)^2 \right] \leq Cn^{2/3} \tag{3.105}$$

and analogously for \widehat{X}' .

Sketch of proof. Decompose $\widehat{X}_n = M_n + A_n^{(1)}$ (recall (3.72)). The analogue of (3.105) for the martingale (M_n) holds by Doob's \mathcal{L}^2 -inequality (note that by the uniform exponential tail bounds we have $\sup_{n \in \mathbb{N}} \mathbb{E}[(M_n - M_{n-1})^2] < \infty$).

To obtain the analogue of (3.105) for the process $(A_n^{(1)})$ note that by the Markov property of $(\widehat{X}, \widehat{X}')$ it suffices to verify that uniformly in $x_0, x'_0 \in \mathbb{Z}$

$$\mathbb{E} \left[\sup_{0 \leq k \leq n^{2/3}} \left(A_k^{(1)} - A_0^{(1)} \right)^2 \mid (\widehat{X}_0, \widehat{X}'_0) = (x_0, x'_0) \right] \leq Cn^{2/3} \tag{3.106}$$

for some $C < \infty$. This can be proved by expressing $A_k^{(1)}$ as a sum of mixingale increments as in the proof of Lemma 3.14 and suitably adapting the argument around (3.102)–(3.104). \square

3.4.1 Transferring from \widehat{X} to \widetilde{X} and completion of the proof of Lemma 3.5, for the case $d = 1$

Note that Lemma 3.13 gives almost the required result except that it speaks about $(\widehat{X}, \widehat{X}')$, a pair of walks observed along *joint* regeneration times, rather than two walks each observed along its individual sequence of regeneration times. Here, we indicate how to remedy this.

Let (with regeneration times T_m, T'_m as in (3.3–3.4) and T_m^{sim} from (3.7))

$$\begin{aligned} L_n &:= \max\{m \leq n : T_m \in \{T'_0, T'_1, \dots\}\}, \\ L'_n &:= \max\{m \leq n : T'_m \in \{T_0, T_1, \dots\}\}, \\ \widehat{L}_n &:= \max\{m \leq n : T_m^{\text{sim}} \leq T_n\}, \\ \widehat{L}'_n &:= \max\{m \leq n : T_m^{\text{sim}} \leq T'_n\}, \end{aligned}$$

i.e., L_n and L'_n are the indices of the last joint regeneration time before the n -th with respect to the walks X respectively X' and \widehat{L}_n respectively \widehat{L}'_n is the corresponding number of joint regeneration times, in particular

$$(\widetilde{X}_{L_n}, T_{L_n}) = (\widehat{X}_{\widehat{L}_n}, T_{\widehat{L}_n}^{\text{sim}}) \quad \text{and} \quad (\widetilde{X}'_{L'_n}, T'_{L'_n}) = (\widehat{X}'_{\widehat{L}'_n}, T_{\widehat{L}'_n}^{\text{sim}}). \tag{3.107}$$

Lemma 3.17. *There exist $C < \infty, q \in (0, 1]$ such that for all $n \in \mathbb{N}$*

$$\mathbb{E}_{\mathbf{0}, \mathbf{0}}^{\text{joint}} \left[(n - L_n)^2 \right] \leq C, \tag{3.108}$$

$$\mathbb{E}_{\mathbf{0}, \mathbf{0}}^{\text{joint}} \left[\left(\widehat{L}_n - nq \right)^2 \right] \leq Cn, \tag{3.109}$$

and analogously for L'_n and \widehat{L}'_n .

Sketch of proof. Note that under Ψ^{ind} , when the two walks use independent copies of the cluster, (T_n) and (T'_n) are two independent renewal processes (whose waiting times have exponential tail bounds), hence $T_n/n \rightarrow \mu, T_n^{\text{sim}}/n \rightarrow \widehat{\mu}$ a.s. and in \mathcal{L}^2 with some $0 < \mu < \widehat{\mu} < \infty$, and (3.108), (3.109) hold (with $q = \mu/\widehat{\mu}$).

By suitably “enriching” the coupling arguments used above, i.e. using Ψ^{joint} instead of $\widehat{\Psi}^{\text{joint}}$, and then reading off the number of individual renewals between joint renewals we see that (3.108), (3.109) also hold for two walks on the same cluster. \square

Now write

$$(\widetilde{X}_n, \widetilde{X}'_n) = (\widehat{X}_{[nq]}, \widehat{X}'_{[nq]}) + (\widehat{X}_{\widehat{L}_n} - \widehat{X}_{[nq]}, \widehat{X}'_{\widehat{L}'_n} - \widehat{X}'_{[nq]}) + (\widetilde{X}_n - \widetilde{X}_{L_n}, \widetilde{X}'_n - \widetilde{X}'_{L'_n}), \tag{3.110}$$

hence

$$\begin{aligned} &\mathbb{E} \left[|\widetilde{X}_n - \widehat{X}_{[nq]}| + |\widetilde{X}'_n - \widehat{X}'_{[nq]}| \right] \\ &\leq \mathbb{E} \left[|\widehat{X}_{\widehat{L}_n} - \widehat{X}_{[nq]}| + |\widehat{X}'_{\widehat{L}'_n} - \widehat{X}'_{[nq]}| \right] + \mathbb{E} \left[|\widetilde{X}_n - \widetilde{X}_{L_n}| + |\widetilde{X}'_n - \widetilde{X}'_{L'_n}| \right] \leq \frac{C}{n^b} \end{aligned} \tag{3.111}$$

for some $C < \infty, b > 0$, using Lemma 3.17 and the fact that the increments of $\widetilde{X}, \widetilde{X}', \widehat{X}$, and \widehat{X}' have uniformly exponentially bounded tails, and Chebyshev’s inequality. (3.111) together with Lemma 3.13 implies Lemma 3.5 for $d = 1$. \square

A An auxiliary result

Recall the definition of τ^0 in (and after) equation (1.1). The following result is “folklore” but we did not find a suitable reference (only for the corresponding contact process version of the result or for a special case; see Remark A.2 below).

Lemma A.1. *For $p > p_c$ there exist $C, \gamma \in (0, \infty)$ such that*

$$P_p(n \leq \tau^0 < \infty) \leq Ce^{-\gamma n}. \tag{A.1}$$

Remark A.2. *The above result is proven in [7] for the “conventional oriented percolation” on $\mathbb{Z} \times \mathbb{Z}_+$ with critical value $p_c^{(1)}$. Dominating the “conventional oriented percolation” it is easy to see that the above result is true for any $d \geq 1$ and $p > p_c^{(1)}$ where $p_c^{(1)}$ is the critical value for the “conventional oriented percolation” on $\mathbb{Z} \times \mathbb{Z}_+$ considered in [7]. It is also clear that for $d \geq 1$ we have $p_c \leq p_c^{(1)}$. Our task here is to extend the result to $p \in (p_c, p_c^{(1)})$.*

Proof. We will adapt arguments from p. 57-58 in [15], where this result was proven for the contact process.

According to [10] (see p. 7 in arXiv version) there exist r (large enough) such that

$$\mathbb{P}(\tau^{[-r,r]^d} < \infty) < \varepsilon.$$

Furthermore, by standard arguments, on a coarse grained grid one can construct a percolation structure with probability of open sites $p_{\text{coarse}} > p_c^{(1)}$, such that it is dominated by the process of suitably defined space-time blocks of the original percolation (see [10]), where the blocks are such that on the “bottom” of the block all sites in some space-time translate of $[-r, r]^d \times \{0\}$ are open.

Now the idea is to show that for large n on the event $\{\tau^A > n\}$ it is likely that the “domination” described above has started.

We set

$$\delta := \mathbb{P}(\eta_r^0 = [-r, r]^d).$$

Now we define a random variable N (measurable w.r.t. to $\sigma(\Omega_p)$) such that

$$\mathbb{P}(N = k) = \delta(1 - \delta)^k, \quad k \geq 0,$$

and

$$\text{either } \eta_{Nr}^0 = \emptyset, \text{ or } x + [-r, r]^d \subset \eta_{(N+1)r}^0 \text{ for some } x \in \mathbb{Z}^d.$$

Set $N = 0$ if $\eta_r^0 = [-r, r]^d$. If $\eta_r^0 \neq [-r, r]^d$, i.e. on $\{N > 0\}$ we have either $\eta_r^0 = \emptyset$ or $\eta_r^0 \neq \emptyset$. In the first case $\eta_n^0 = \emptyset$ for all $n \geq r$ and therefore $N \geq 1$. In the second case, restart (a subprocess) in some $x \in \eta_r^0$, and let $N = 1$ if

$${}_r\eta_{2r}^x \subset x + [-r, r]^d.$$

Here, for $m \leq n$ we use ${}_m\eta_n^x$ to denote the discrete time contact process at time n starting in $\{x\}$ at time m . Again on the complement $\{N = 1\}$ either ${}_r\eta_{2r}^x = \emptyset$, in which case $N \geq 2$, or ${}_r\eta_{2r}^x \neq \emptyset$, in which case we can proceed as before.

If $x + [-r, r]^d \subset \eta_{(N+1)r}^0$ for some $x \in \mathbb{Z}^d$ then we can start the coupling with percolation on a coarse grained grid described at the beginning of the proof.

Let $(N + 1)r + M$ be the extinction time of this block percolation process. Note that $(N + 1)r$ is the time at which the comparison starts and M is the extinction time of the discrete contact process with probability of open sites given by p_{coarse} .

As noted in Remark A.2 we have $\mathbb{P}(M > n | M < \infty) \leq Ce^{-\gamma n}$ for suitable $C, \gamma > 0$. If $M = \infty$ then $\tau^0 = \infty$. If $M < \infty$ then at time $M + (N + 1)r$ the configuration $\eta_{M+(N+1)r}$ is empty or not. If it is not empty then we repeat the procedure and obtain an i.i.d. sequence of independent random variables N_i with the same law as N and independent random variables M_i with the same law as M conditioned on $M < \infty$. Let L be such that at time

$$\sigma = \sum_{i=1}^L ((N_i + 1)r + M_i)$$

either $\eta_\sigma^0 = \emptyset$ or $\tau^0 = \infty$. Thus, $\sigma > \tau^0$ on $\{\tau^0 < \infty\}$ and we obtain

$$\mathbb{P}(n < \tau^0 < \infty) \leq \mathbb{P}(\sigma > n) \leq Ce^{-\gamma n}$$

for suitable $C, \gamma > 0$. □

Remark A.3. *Inspection of the proof of Lemma A.1 shows that the constants γ and C in (A.1) can be chosen to apply uniformly in $p \in [p_c + \delta, 1]$ for any $\delta > 0$.*

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