RANDOM MATRICES AND COMPLEXITY OF SPIN GLASSES

ANTONIO AUFFINGER, GÉRARD BEN AROUS, AND JIŘÍ ČERNÝ

ABSTRACT. We give an asymptotic evaluation of the complexity of spherical p-spin spin-glass models via random matrix theory. This study enables us to obtain detailed information about the bottom of the energy landscape, including the absolute minimum (the ground state), the other local minima, and describe an interesting layered structure of the low critical values for the Hamiltonians of these models. We also show that our approach allows us to compute the related TAP-complexity and extend the results known in the physics literature. As an independent tool, we prove a LDP for the k-th largest eigenvalue of the GOE, extending the results of [BDG01].

1. INTRODUCTION

How many critical values of given index and below a given level does a typical random Morse function have on a high dimensional manifold? Our work addresses this question in a very special case. We look at certain natural random Gaussian functions on the N-dimensional sphere known as p-spin spherical spin glass models. We cannot yet answer the question above about the typical number, but we can study thoroughly the mean number, which we show is exponentially large in N. We introduce a new identity, based on the classical Kac-Rice formula, relating random matrix theory and the problem of counting these critical values. Using this identity and tools from random matrix theory, we give an asymptotic evaluation of the complexity of these spherical spin-glass models.

The complexity mentioned here is defined as the mean number of critical points of given index whose value is below (or above) a given level. This includes the important question of counting the mean number of local minima below a given level, and in particular the question of finding the ground state energy (the minimal value of the Hamiltonian). We show that this question is directly related to the study of the edge of the spectrum of the Gaussian Orthogonal Ensemble (GOE).

The question of computing the complexity of mean-field spin glass models has recently been thoroughly studied in the physics literature (see for example [CLR03] and the references therein), mainly for a different measure of the complexity, i.e. the mean number of solutions to the Thouless-Anderson-Palmer equations, or TAP-complexity. Our approach to the complexity enables us to recover known results in the physics literature about TAP-complexity, to compute the ground state energy (when p is even), and to describe an interesting layered structure of the low energy levels of the Hamiltonians of these models, which might prove useful for the study of the metastability of Langevin dynamics for these models (in longer time scales than those studied in [BDG01]).

The paper is organised as follows. In Section 2, we give our main results. In Section 3, we prove two main formulas (Theorem 2.1 and 2.2), relating random matrix theory (specifically the GOE) and spherical spin glasses. These formulas are direct consequences of the Kac-Rice formula (we learned the version needed here in the book [AT07], for another modern account see [AW09]). The main new ingredient is the fact that, for spherical spin-glass models, the Hessian of the Hamiltonian at a critical point, conditioned on the value of the Hamiltonian, is a symmetric Gaussian random matrix with independent entries (up to symmetry) plus a diagonal matrix. This implies, in particular, that it is possible to

relate statistics of critical points of index k to statistics of the $k+1$-th smallest eigenvalue of a matrix sampled from the GOE.

In Section 4, we compute precise logarithmic estimates of the complexity using the known large deviation principle (LDP) for the empirical spectral measure [BG97] and for the largest eigenvalue of the GOE [BDG01]. In fact we need a simple extension of the last LDP, i.e. an LDP for the law of the k-th largest eigenvalue, which is of independent interest and proven in Appendix A.

In Section 5, we show how these logarithmic results can be used to extract information about the lowest lying critical values. We first prove that the lowest lying critical points have an interesting layered structure, Theorem 2.15. We then show how our logarithmic results imply a lower bound on the ground state energy (the minimal value of the Hamiltonian). At this point it would be useful to have a concentration result for the number of local minima, for instance using a control of its second moment. Unfortunately, we cannot prove directly such a concentration result. Nevertheless we prove that our lower bound is tight (for p even), by proving the corresponding upper bound, using the Parisi formula for the free energy at positive temperature, as established by Talagrand [Tal06]. It is remarkable that the ground state is indeed correctly predicted by our very naive approach, i.e. by the vanishing of the "annealed" complexity of the number of local minima. We expect this to be true for all models where Parisi's one-step replica symmetry breaking holds at low temperature. In Section 6, we extend our results to the TAP-complexity and compare our results to the physics literature [CLR03], [CS95].

In Section 7, we show how one can go further and obtain sharper than logarithmic asymptotic results for the complexity, using classical tools from orthogonal polynomials theory, i.e. Plancherel-Rotach asymptotics for Hermite functions. This section is technically involved, so we restrict it to the study of the global complexity, i.e. the mean number of critical points below a given level, and we do not push it to include the mean number of critical points below a given level with a fixed index. However, we remark that at low energy levels, the total number of critical points coincides with the total number of local minima.

2. Notations and main results

We first introduce the p-spin spherical spin-glass model. We will fix p an integer larger or equal to 2 (the case $p = 2$ is rather trivial regarding our complexity questions, it will be discussed below only in Remark 2.3).

A configuration $\boldsymbol{\sigma}$ of the p-spin spherical spin-glass model is a vector of $\mathbb{R}^N$ satisfying the spherical constraint

$$\frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 = 1. \tag{2.1}$$

Thus the state space of the p-spin spherical spin-glass model is $S^{N-1}(\sqrt{N}) \subset \mathbb{R}^N$, the Euclidean sphere of radius $\sqrt{N}$.

The Hamiltonian of the model is the random function defined on $S^{N-1}(\sqrt{N})$ by

$$H_{N,p}(\boldsymbol{\sigma}) = \frac{1}{N^{(p-1)/2}} \sum_{i_1,\dots,i_p=1}^{N} J_{i_1,\dots,i_p}\sigma_{i_1}\dots\sigma_{i_p}, \qquad \boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_N) \in S^{N-1}(\sqrt{N}), \quad (2.2)$$

where $J_{i_1,\dots,i_p}$ are independent centered standard Gaussian random variables.

Equivalently, $H_{N,p}$ is the centered Gaussian process on the sphere $S^{N-1}(\sqrt{N})$ whose covariance is given by

$$\mathbb{E}\big[H_{N,p}(\boldsymbol{\sigma})H_{N,p}(\boldsymbol{\sigma}')\big] = N^{1-p}\Big(\sum_{i=1}^{N}\sigma_i\sigma_i'\Big)^p = NR(\boldsymbol{\sigma},\boldsymbol{\sigma}')^p, \qquad (2.3)$$

where the normalised inner product $R(\boldsymbol{\sigma},\boldsymbol{\sigma}') = \frac{1}{N}\langle\boldsymbol{\sigma},\boldsymbol{\sigma}'\rangle = \frac{1}{N}\sum_{i=1}^{N}\sigma_i\sigma_i'$ is usually called the overlap of the configurations $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$.

We now want to introduce the complexity of spherical spin glasses. For any Borel set $B \subset \mathbb{R}$ and integer $0 \le k < N$, we consider the (random) number $\mathrm{Crt}_{N,k}(B)$ of critical values of the Hamiltonian $H_{N,p}$ in the set $NB = \{Nx : x \in B\}$ with index equal to k,

$$\mathrm{Crt}_{N,k}(B) = \sum_{\boldsymbol{\sigma}:\nabla H_{N,p}(\boldsymbol{\sigma})=0} \mathbf{1}\{H_{N,p}(\boldsymbol{\sigma}) \in NB\}\mathbf{1}\{i(\nabla^2 H_{N,p}(\boldsymbol{\sigma})) = k\}. \qquad (2.4)$$

Here ∇, ∇^2 are the gradient and the Hessian restricted to $S^{N-1}(\sqrt{N})$, and $i(\nabla^2 H_{N,p}(\boldsymbol{\sigma}))$ is the index of $\nabla^2 H_{N,p}$ at $\boldsymbol{\sigma}$, that is the number of negative eigenvalues of the Hessian $\nabla^2 H_{N,p}$. We will also consider the (random) total number $\mathrm{Crt}_N(B)$ of critical values of the Hamiltonian $H_{N,p}$ in the set NB (whatever their index)

$$\mathrm{Crt}_N(B) = \sum_{\boldsymbol{\sigma}:\nabla H_{N,p}(\boldsymbol{\sigma})=0} \mathbf{1}\{H_{N,p}(\boldsymbol{\sigma}) \in NB\}. \qquad (2.5)$$

Our results will give exact formulas and asymptotic estimates for the mean values $\mathbb{E}(\mathrm{Crt}_{N,k}(B))$ and $\mathbb{E}(\mathrm{Crt}_N(B))$, when $N \to \infty$ and B, k and p are fixed. In particular, we will compute $\lim \frac{1}{N}\log\mathbb{E}\,\mathrm{Crt}_{N,k}(B)$ and $\lim \frac{1}{N}\log\mathbb{E}\,\mathrm{Crt}_N(B)$ as N tends to infinity.

Before giving the central identity relating the GOE to the complexity of spherical spin-glass models, we fix our notations for the GOE.

The GOE ensemble is a probability measure on the space of real symmetric matrices. Namely, it is the probability distribution of the $N \times N$ real symmetric random matrix M^N, whose entries $(M_{ij}, i \le j)$ are independent centered Gaussian random variables with variance

$$\mathbb{E}M_{ij}^2 = \frac{1+\delta_{ij}}{2N}. \qquad (2.6)$$

We will denote by $\mathbb{E}_{\mathrm{GOE}} = \mathbb{E}_{\mathrm{GOE}}^N$ the expectation under the GOE ensemble of size $N \times N$.

Let $\lambda_0^N \le \lambda_1^N \le \cdots \le \lambda_{N-1}^N$ be the ordered eigenvalues of M^N. We will denote by $L_N = \frac{1}{N}\sum_{i=0}^{N-1}\delta_{\lambda_i^N}$ the (random) spectral measure of M^N, and by $\rho_N(x)$ the density of the (non-random) probability measure $\mathbb{E}_{\mathrm{GOE}}(L_N)$. The function $\rho_N(x)$ is usually called the (normalised) one-point correlation function and satisfies

$$\int_{\mathbb{R}} f(x)\rho_N(x)dx = \frac{1}{N}\mathbb{E}_{\mathrm{GOE}}^N\Big[\sum_{i=0}^{N-1}f(\lambda_i^N)\Big]. \qquad (2.7)$$

We now state our main identity

Theorem 2.1. *The following identity holds for all N, $p \ge 2$, $k \in \{0,\dots,N-1\}$, and for all Borel sets $B \subset \mathbb{R}$,*

$$\mathbb{E}[\mathrm{Crt}_{N,k}(B)] = 2\sqrt{\frac{2}{p}}(p-1)^{\frac{N}{2}}\mathbb{E}_{GOE}^N\Big[e^{-N\frac{p-2}{2p}(\lambda_k^N)^2}\mathbf{1}\Big\{\lambda_k^N \in \sqrt{\frac{p}{2(p-1)}}B\Big\}\Big]. \qquad (2.8)$$

Summing the preceding identities for $0 \le k \le N - 1$, we easily find the mean total number of critical points given a level of energy and relate it to the one-point function.

Theorem 2.2. *The following identity holds for all N, $p \ge 2$, and for all Borel sets $B \subset \mathbb{R}$,*

$$\mathbb{E}[\mathrm{Crt}_N(B)] = 2N \sqrt{\frac{2}{p}} (p-1)^{\frac{N}{2}} \int_{\sqrt{\frac{p}{2(p-1)}} B} \exp \left\{ - \frac{N(p-2)x^2}{2p} \right\} \rho_N(x) \, \mathrm{d}x. \tag{2.9}$$

Remark 2.3. We want to discuss here very briefly the trivial case where $p = 2$. This case is easier to understand since there is a simpler connection between the 2-spin spherical model and the random matrix theory. In fact, if M is a $N \times N$ GOE matrix, the Hamiltonian of the 2-spin spherical model can be viewed as the quadratic form defined by the symmetric matrix M restricted to the sphere $S^{N-1}(\sqrt{N})$,

$$H_{N,2}(x) = (Mx, x). \tag{2.10}$$

Therefore, if $p = 2$, the $2N$ critical points of the Hamiltonian are simply the eigenvectors of M, while the critical values are the eigenvalues of M. Our formulas (2.8), (2.9) simplify greatly for $p = 2$ and are compatible with this obvious observation. Indeed, for $p = 2$, these formulas read

$$\mathbb{E}[\mathrm{Crt}_{N,k}(B)] = 2P_{\mathrm{GOE}}^N[\lambda_k^N \in B] \tag{2.11}$$

and

$$\mathbb{E}[\mathrm{Crt}_N(B)] = 2N\rho_N(B). \tag{2.12}$$

We are now in a position to give our main results about the asymptotic complexity of spherical spin-glass models. We will see that the following number is an important threshold

$$E_c = E_c(p) = 2\sqrt{\frac{p-1}{p}}. \tag{2.13}$$

Let $I_1 : (-\infty, -E_c] \to \mathbb{R}$ be given by

$$I_1(u) = \frac{2}{E_c^2} \int_u^{-E_c} (z^2 - E_c^2)^{1/2} \mathrm{d}z = -\frac{u}{E_c^2} \sqrt{u^2 - E_c^2} - \log\left(-u + \sqrt{u^2 - E_c^2}\right) + \log E_c. \tag{2.14}$$

Remark 2.4. In [BDG01], it is proved that $I_1(u)$ is the rate function of the LDP for the smallest eigenvalue of the GOE with the proper normalization of the variance of the entries (more precisely, on its domain $I_1(u) = I_1(-u; E_c/2)$, see (A.2)).

We now define the following important functions which will describe the asymptotic complexity of the p-spin spherical spin-glass models.

$$\theta_p(u) = \begin{cases} \frac{1}{2} \log(p-1) - \frac{p-2}{4(p-1)} u^2 - I_1(u), & \text{if } u \le -E_c, \\ \frac{1}{2} \log(p-1) - \frac{p-2}{4(p-1)} u^2, & \text{if } -E_c \le u \le 0, \\ \frac{1}{2} \log(p-1), & \text{if } 0 \le u, \end{cases} \tag{2.15}$$

and, for any integer $k \ge 0$,

$$\theta_{k,p}(u) = \begin{cases} \frac{1}{2} \log(p-1) - \frac{p-2}{4(p-1)} u^2 - (k+1)I_1(u), & \text{if } u \le -E_c, \\ \frac{1}{2} \log(p-1) - \frac{p-2}{p}, & \text{if } u \ge -E_c. \end{cases} \tag{2.16}$$

We note that $\theta_p(u), \theta_{k,p}(u)$ are non-decreasing, continuous functions on $\mathbb{R}$, with maximal values $\frac{1}{2} \log(p-1)$, $\frac{1}{2} \log(p-1) - \frac{p-2}{p}$, respectively (see Figure 1).

We now give the logarithmic asymptotics of the complexity of spherical spin glasses. To simplify the statement, we fix $B = (-\infty, u)$, $u \in \mathbb{R}$, and we write $\mathrm{Crt}_{N,k}(u) = \mathrm{Crt}_{N,k}(B)$, $\mathrm{Crt}_N(u) = \mathrm{Crt}_N(B)$.

Theorem 2.5. *For all $p \geq 2$ and $k \geq 0$ fixed,*

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \operatorname{Crt}_{N,k}(u) = \theta_{k,p}(u). \tag{2.17}$$
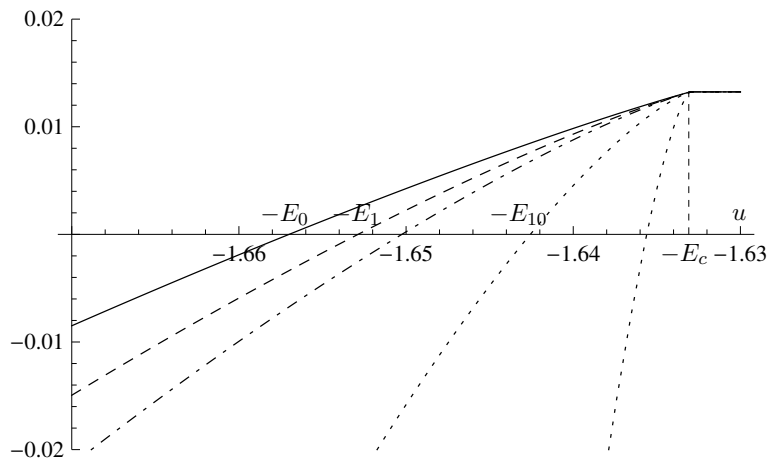

FIGURE 1. The functions $\theta_{k,p}$ for $p = 3$ and $k = 0$ (solid), $k = 1$ (dashed), $k = 2$ (dash-dotted), $k = 10$, $k = 100$ (both dotted). All these functions agree for $u \geq -E_c$.

Remark 2.6. It is straightforward to extend the last theorem to general Borel sets B (see Remark 4.1). Furthermore, by symmetry, Theorem 2.5 also holds as stated for the random variables $\operatorname{Crt}_{N,N-l}((u,\infty))$, with $l \geq 1$ fixed, if one replaces $\theta_{k,p}(u)$ by $\theta_{l-1,p}(-u)$.

Remark 2.7. For the local minima, i.e. when $k = 0$, the limit formula given by Theorem 2.5 is precisely the formula given by physicists in [CS95], [CLR03]. Arguing via a TAP approach (to be described below in Section 6), they derive the following asymptotic complexity of local minima,

$$g(E) = \frac{1}{2} \left\{ \frac{2-p}{p} - \log\left(\frac{pz^2}{2}\right) + \frac{p-1}{2} z^2 - \frac{2}{p^2 z^2} \right\}, \tag{2.18}$$

where $z = \frac{1}{p-1} \left(-E - (E^2 - \frac{2(p-1)}{p})^{1/2} \right)$. In Section 6, we show that, in fact, $g(E) = \theta_{0,p}(2^{-1/2} E)$. The factor $2^{-1/2}$ comes from the fact that in [CS95] the Hamiltonian H has a different normalization.

We also provide an exponential asymptotic for the expected total number of critical values below level Nu.

Theorem 2.8. *For all $p \geq 2$,*

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \operatorname{Crt}_N(u) = \theta_p(u). \tag{2.19}$$

Remark 2.9. As a simple consequence of Theorems 2.5 and 2.8, one can easily compute the logarithmic asymptotics of the mean total number of critical points $\mathbb{E}(\operatorname{Crt}_N(\mathbb{R}))$ and the mean total number of critical points of index k, $\mathbb{E}(\operatorname{Crt}_{N,k}(\mathbb{R}))$.

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}(\operatorname{Crt}_N(\mathbb{R})) = \frac{1}{2} \log(p-1), \tag{2.20}$$

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}(\operatorname{Crt}_{N,k}(\mathbb{R})) = \frac{1}{2} \log(p-1) - \frac{p-2}{p}. \tag{2.21}$$

This agrees with formula (13) of [CS95]. Note that the remarkable fact that the mean number of critical points of index k is independent of k (at least in these logarithmic estimates)

Remark 2.10. Theorems 2.5 and 2.8 are simple consequences of large deviation properties for random matrices, using the results of [BDG01] and of [BG97]. In Appendix A, we recall the LDP for the empirical spectral measure of the GOE proved in [BG97], and we prove a LDP for the k-th largest eigenvalue of a GOE matrix, extending the results of [BDG01].

Remark 2.11. The case $p = 2$ is particular since the total complexity is then always non positive. But for any $p \geq 3$ the complexity is positive.

Using our results about complexity, we now want to extract some information about the geometry of the bottom of the energy landscape $H_{N,p}$. For any integer $k \geq 0$, we introduce $E_k = E_k(p) > 0$ as the unique solution to (see Figure 1 again).

$$\theta_{k,p}(-E_k(p)) = 0. \tag{2.22}$$

These numbers will be crucial in the description of the ground state and of the low-lying critical values of the Hamiltonian $H_{N,p}$. It is important to note that, for any fixed $p \geq 3$, the sequence $(E_k(p))_{k \in \mathbb{N}}$ is strictly decreasing, and converges to $E_c(p)$ as $k \to \infty$. The first result we want to derive is about the ground state energy, which we define as the (normalised) minimum of the Hamiltonian $H_{N,p}$

$$GS^N = \frac{1}{N} \inf_{\boldsymbol{\sigma} \in S^{N-1}(\sqrt{N})} H_{N,p}(\boldsymbol{\sigma}). \tag{2.23}$$

Theorem 2.12. *For every $p \geq 3$*

$$\liminf_{N \to \infty} GS^N \geq -E_0(p)$$

Moreover for $p \geq 4$ even,

$$\lim_{N \to \infty} GS^N = -E_0(p) \qquad in\ probability. \tag{2.24}$$

Remark 2.13. The lower bound on the Ground State follows from our complexity estimates and holds for all $p \geq 3$. To obtain a matching upper bound we use the Parisi formula and the one step replica symmetry breaking as proved by Talagrand [Tal06]. The Parisi formula is proven there for every p but the one step replica symmetry breaking is proven only for even p's. It might be noteworthy that if we could go beyond our annealed estimates of the complexity we would be in a position to prove directly the one-step replica symmetry breaking at zero temperature

By Theorem 2.12, it is improbable to find a critical value below the level $-NE_0(p)$. The next interesting phenomenon is the role of the threshold $E_c(p)$. Namely, it is (even more) improbable to find, above the threshold $-NE_c(p)$, a critical value of the Hamiltonian of a fixed index k, when $N \to \infty$. Otherwise said, above the threshold $-NE_c(p)$, all critical values of the Hamiltonian must be of diverging index, with overwhelming probability.

Theorem 2.14. *Let for an integer $k \geq 0$ and $\varepsilon > 0$, $B_{N,k}(\varepsilon)$ be the event "there is a critical value of index k of the Hamiltonian $H_{N,p}$ above the level $-N(E_c(p) - \varepsilon)$", that is $B_{N,k}(\varepsilon) = \{\mathrm{Crt}_{N,k}((-E_c(p) + \varepsilon, \infty)) > 0\}$. Then for all $k \geq 0$ and $\varepsilon > 0$,*

$$\limsup_{N \to \infty} \frac{1}{N^2} \log \mathbb{P}(B_{N,k}(\varepsilon)) < 0. \tag{2.25}$$

By the last theorem, all critical values of the Hamiltonian of fixed index (non diverging with N) must be found in the band $(-NE_0(p), -NE_c(p))$. We now explain the role of the thresholds $E_k(p)$. Namely, it is improbable to find critical value of index larger or equal to k below the threshold $-NE_k(p)$, for any fixed integer k.

Theorem 2.15. *For $k \geq 0$ and $\varepsilon > 0$, let $A_{N,k}(\varepsilon)$ to be the event "there is a critical value of the Hamiltonian $H_{N,p}$ below the level $-N(E_k(p) + \varepsilon)$ and with index larger or equal to k", that is $A_{N,k}(\varepsilon) = \{\sum_{i=k}^{\infty} \mathrm{Crt}_{N,i}(-E_k(p) - \varepsilon) > 0\}$. Then for all $k \geq 0$ and $\varepsilon > 0$,*

$$\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}(A_{N,k}(\varepsilon)) < 0. \tag{2.26}$$

Theorem 2.15 describes an interesting layered structure for the lowest critical values of the Hamiltonian $H_{N,p}$. It says that the lowest critical values above the ground state energy (asymptotically $-NE_0(p)$) are (with an overwhelming probability again) only local minima, this being true up to the value $-NE_1(p)$, and that in a layer above, $(-NE_1(p), -NE_2(p))$, one finds only critical values with index 0 (local minima) or saddle point with index 1, and above this layer one finds only critical values with index 0,1 or 2, etc. This picture was already predicted by physicists for minima [CS95], [CLR03] and for critical points of finite indices [KL96]. In particular, this says that the energy barrier to cross when starting from the ground state in order to reach another local minima diverges with N, since it is bounded below by the energy difference between an index-one saddle point and the ground state, i.e. by $N(E_0(p) - E_1(p))$.

Remark 2.16. Even though it does not follow immediately from Theorem 2.15 and from our results on complexity, it is tempting to conjecture that the minimum possible energy of a critical point of index k, normalised by N, should converge to $-E_k(p)$ (For $k = 0$ this is the statement of Theorem 2.12), while likewise the maximum energy of a critical point of index k, once normalised by N, should converge to $-E_c$. It is also tempting to conjecture that the main contribution to the number of critical points of a finite index k is given by those whose energy is asymptotically $-NE_c$. That is, the number of critical points of any finite index with energy strictly below $-NE_c$ should be negligible with respect to those with energy near $-NE_c$ (with probability going to one, as N tends to infinity). However, near any energy value in $E \in (-E_k(p), -E_c)$ there are still an exponentially many critical values of index k. We cannot reach those statements at this point because our complexity results concern only the first moment of $\mathrm{Crt}_{N,k}(u)$. We would need to control the concentration of these random variables.

In Section 7, we show that the precision of Theorem 2.8 can be improved and we derive, using asymptotic properties of orthogonal polynomials, the following sharp asymptotics of $\mathbb{E}(\mathrm{Crt}_N(u))$.

Theorem 2.17. *For $p \geq 3$, the following holds as $N \to \infty$:*

(a) *For $u < -E_c$*

$$\mathbb{E}\,\mathrm{Crt}_N(u) = \frac{h(v)}{(2p\pi)^{1/2}} \frac{e^{\Psi(v) - \frac{v}{2}\Psi'(v)}}{F_p'(v) + \Psi'(v)} N^{-1/2} e^{N\theta_p(u)}(1 + o(1)), \tag{2.27}$$

where $v = -u\sqrt{\frac{p}{2(p-1)}}$ and the functions h, F_p and Ψ are given in (7.1), (7.10).

(b) *For $u = -E_c$*

$$\mathbb{E}\,\mathrm{Crt}_N(-E_c) = \frac{2\,\mathrm{Ai}(0)\sqrt{2p}}{3(p-2)} N^{-1/3} e^{N\theta_p(-E_c)}(1 + o(1)). \tag{2.28}$$

(c) *For $u \in (-E_c, 0)$*

$$\mathbb{E}\,\mathrm{Crt}_N(u) = \frac{2\sqrt{2p(E_c^2 - u^2)}}{(2 - p)\pi u} e^{N\theta_p(u)}(1 + o(1)). \tag{2.29}$$

(d) *For $u > 0$*

$$\mathbb{E}\,\mathrm{Crt}_N(u) = 2\mathbb{E}\,\mathrm{Crt}_N(0)(1 + o(1)) = \frac{4\sqrt{2}}{\sqrt{\pi}(p-2)} N^{1/2} e^{N\theta_p(0)}(1 + o(1)). \tag{2.30}$$

Since $\theta_k(u) < \theta_0(u)$ for all $k > 0$ and $u < -E_c$, we obtain as an easy consequence of Theorem 2.5 the following sharp asymptotics for the mean number of minima.

Corollary 2.18. *For $u < -E_c$,*

$$\mathbb{E}\operatorname{Crt}_{N,0}(u) = \frac{h(v)}{(2p\pi)^{1/2}} \frac{e^{\Psi(v) - \frac{v}{2}\Psi'(v)}}{F_p'(v) + \Psi'(v)} N^{-1/2} e^{N\theta_p(u)}(1 + o(1)), \qquad (2.31)$$

where $v = -u\sqrt{\frac{p}{2(p-1)}}$ and the functions h, F_p and Ψ are given in (7.1), (7.10).

3. Proof of the central identity

In this section we prove Theorems 2.1 and 2.2. For the proofs, we find it more convenient to work with processes of variance one on the unit sphere $S^{N-1} \subset \mathbb{R}^N$ rather than to work with $H_{N,p}$. Hence, for $\boldsymbol{\sigma} \in S^{N-1}$ we define,

$$f_{N,p}(\boldsymbol{\sigma}) = \frac{1}{\sqrt{N}} H_{N,p}(\sqrt{N}\boldsymbol{\sigma}). \qquad (3.1)$$

We will regularly omit the subscripts N and p to save on notations, $f = f_{N,p}$. The function f is again centered Gaussian process whose covariance satisfies

$$\mathbb{E}(f(\boldsymbol{\sigma})f(\boldsymbol{\sigma}')) = \Big(\sum_{i=1}^N \sigma_i \sigma_i'\Big)^p = R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')^p, \qquad \boldsymbol{\sigma}, \boldsymbol{\sigma}' \in S^{N-1}. \qquad (3.2)$$

To estimate the mean number of critical points in a certain level we will use the *Kac-Rice* formula as it appears in the recent book of Adler and Taylor [AT07] which we now formulate as a lemma. We use $\langle x, y \rangle$ to denote the usual Euclidean scalar product, as well as the scalar product on any tangent space $T_{\boldsymbol{\sigma}} S^{N-1}$. Let $\nabla^2 f$ be the covariant Hessian of f on S^{N-1} defined, e.g., by $\nabla^2 f(X, Y) = XYf - \nabla_X Y f$. Here $\nabla_X Y$ is the usual Riemann connection and $X, Y \in TS^{N-1}$ are tangent vectors. On S^{N-1} we fix an arbitrary orthonormal frame field $(E_i)_{1 \le i < N}$, that is a set of $N-1$ vector fields E_i on S^{N-1} such that $\{E_i(\boldsymbol{\sigma})\}$ is an orthonormal basis of $T_{\boldsymbol{\sigma}} S^{N-1}$. We write $\phi_{\boldsymbol{\sigma}}$ for the density of the gradient vector $(E_i f(\boldsymbol{\sigma}))_{1 \le i < N}$ and $\det \nabla^2 f(\boldsymbol{\sigma})$ for the determinant of the matrix $(\nabla^2 f(E_i, E_j)(\boldsymbol{\sigma}))_{1 \le i, j < N}$.

Lemma 3.1. *Let f be a centered Gaussian field on S^{N-1} and let $\mathcal{A} = (U_\alpha, \Psi_\alpha)_{\alpha \in I}$ be a finite atlas on S^{N-1}. Set $f^\alpha = f \circ \Psi_\alpha^{-1} : \Psi_\alpha(U_\alpha) \subset \mathbb{R}^{N-1} \to \mathbb{R}$ and define $f_i^\alpha = \partial f^\alpha / \partial x_i$, $f_{ij}^\alpha = \partial^2 f^\alpha / \partial x_i \partial x_j$. Assume that for all $\alpha \in I$ and all $x, y \in \Psi_\alpha(U_\alpha)$ the joint distribution of $(f_i^\alpha(x), f_{ij}^\alpha(x))_{1 \le i \le j < N}$ is non-degenerate, and*

$$\max_{i,j} \big| \operatorname{Var}(f_{ij}^\alpha(x)) + \operatorname{Var}(f_{ij}^\alpha(y)) - 2\operatorname{Cov}(f_{ij}^\alpha(x), f_{ij}^\alpha(y)) \big| \le K_\alpha |\ln|x - y||^{-1-\beta} \qquad (3.3)$$

for some $\beta > 0$ and $K_\alpha > 0$. For a Borel set $B \subset \mathbb{R}$, let

$$\operatorname{Crt}_{N,k}^f(B) = \sum_{\boldsymbol{\sigma}: \nabla f(\boldsymbol{\sigma}) = 0} \mathbf{1}\{i(\nabla^2 f(\boldsymbol{\sigma})) = k, f(\boldsymbol{\sigma}) \in B\}. \qquad (3.4)$$

Then, using $\mathrm{d}\boldsymbol{\sigma}$ to denote the usual surface measure on S^{N-1},

$$\mathbb{E}\operatorname{Crt}_{N,k}^f(B) = \int_{S^{N-1}} \mathbb{E}\big[|\det \nabla^2 f(\boldsymbol{\sigma})|\mathbf{1}\{f(\boldsymbol{\sigma}) \in B, i(\nabla^2 f(\boldsymbol{\sigma})) = k\} \,\big|\, \nabla f(\boldsymbol{\sigma}) = 0\big]\phi_{\boldsymbol{\sigma}}(0)\mathrm{d}\boldsymbol{\sigma}. \qquad (3.5)$$

Proof. Assumptions of the lemma, which are taken from Corollaries 11.3.2 and 11.3.5 of [AT07], assure that f is a.s. a Morse function and its gradient and Hessian exist in L^2 sense. The lemma can be then proved using the same procedure as Theorem 12.4.1 of [AT07]. Our formula (3.5) is analogous to the display just following formula (12.4.4) of [AT07], modulo the term $(-1)^k$ which is missing in our settings since we are interested in

the number of critical points of f in B and not in the Euler characteristic of the excursion set. $\qquad\square$

An application of Lemma 3.1 is made possible due to the following lemma which describes the joint law of Gaussian vector $(f(\boldsymbol{\sigma}), \nabla f(\boldsymbol{\sigma}), \nabla^2 f(\boldsymbol{\sigma})))$.

Lemma 3.2. *(a) Let $(f_i(\boldsymbol{\sigma}))_{1 \leq i < N}$ be the gradient and $(f_{ij}(\boldsymbol{\sigma}))_{1 \leq i,j < N}$ the Hessian matrix at $\boldsymbol{\sigma} \in S^{N-1}$, that is $f_i = E_i \bar{f}(\boldsymbol{\sigma}), f_{ij} = \nabla^2 f(E_i, E_j)(\boldsymbol{\sigma})$. Then, for all $1 \leq i, j, k < N$, $f(\boldsymbol{\sigma}), f_i(\boldsymbol{\sigma}), f_{jk}(\boldsymbol{\sigma})$ are centered Gaussian random variables whose joint distribution is determined by*

$$\begin{aligned} \mathbb{E}[f(\boldsymbol{\sigma})^2] = 1, \qquad & \mathbb{E}[f(\boldsymbol{\sigma})f_i(\boldsymbol{\sigma})] = \mathbb{E}[f_i(\boldsymbol{\sigma})f_{jk}(\boldsymbol{\sigma})] = 0, \\ \mathbb{E}[f(\boldsymbol{\sigma})f_{ij}(\boldsymbol{\sigma})] = -p\delta_{ij}, \qquad & \mathbb{E}[f_i(\boldsymbol{\sigma})f_j(\boldsymbol{\sigma})] = p\delta_{ij}, \end{aligned} \tag{3.6}$$

and

$$\mathbb{E}[f_{ij}(\boldsymbol{\sigma})f_{kl}(\boldsymbol{\sigma})] = p(p-1)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + p^2 \delta_{ij}\delta_{kl}. \tag{3.7}$$

(b) Under the conditional distribution $\mathbb{P}[\cdot|f(\boldsymbol{\sigma}) = x]$, $x \in \mathbb{R}$, the random variables $f_{ij}(\boldsymbol{\sigma})$, $1 \leq i, j < N$, are independent Gaussian variables satisfying

$$\begin{aligned} \mathbb{E}[f_{ij}(\boldsymbol{\sigma})] &= -xp\delta_{ij}, \\ \mathrm{Var}\left[f_{ij}(\boldsymbol{\sigma})^2\right] &= (1 + \delta_{ij})p(p-1). \end{aligned} \tag{3.8}$$

Alternatively, the random matrix $(f_{ij}(\boldsymbol{\sigma}))$ has the same distribution as

$$M^{N-1}\sqrt{2(N-1)p(p-1)} - xpI, \tag{3.9}$$

where M^{N-1} is $(N-1) \times (N-1)$ GOE matrix given by (2.6) and I is the identity matrix.

Proof. Without loss of generality we can suppose that $\boldsymbol{\sigma}$ is the north pole of the sphere $\boldsymbol{n} = (0, \ldots, 0, 1)$. We define the function $\Psi : S^{N-1} \to \mathbb{R}^{N-1}$ by $\Psi(x_1, \ldots, x_N) = (x_1, \ldots, x_{N-1})$. It is a chart in some neighborhood U of $\boldsymbol{n}$. We set

$$\bar{f} = f \circ \Psi^{-1}, \tag{3.10}$$

which is a Gaussian process on $\Psi(U)$ with covariance

$$C(x, y) = \mathrm{Cov}(\bar{f}(x), \bar{f}(y)) = \left\{ \sum_{i=1}^{N-1} x_i y_i + \sqrt{\left(1 - \sum_{i=1}^{N-1} x_i^2\right)\left(1 - \sum_{i=1}^{N-1} y_i^2\right)} \right\}^p. \tag{3.11}$$

We choose the orthonormal frame field (E_i) such that it satisfies $E_i(\boldsymbol{n}) = \partial/\partial x_i$ with respect to the chart Ψ. Then the covariant Hessian $(f_{ij}(\boldsymbol{n}))$ agrees with the usual Hessian of $\bar{f}$ at 0, by noting that the Christoffel symbols $\Gamma_{kl}^i(\boldsymbol{n}) \equiv 0$. Hence, to check (3.6), (3.7), we should prove analogous identities for $\bar{f}(0)$, $\bar{f}_i(0) = \frac{\partial}{\partial x_i}\bar{f}(0)$ and $\bar{f}_{ij}(0) = \frac{\partial^2}{\partial x_i \partial x_j}\bar{f}(0)$.

The covariances of $\bar{f}, \bar{f}_i, \bar{f}_{ij}$, can be computed using a well-known formula (see e.g. [AT07] formula (5.5.4)),

$$\mathrm{Cov}\left(\frac{\partial^k \bar{f}(x)}{\partial x_{i_1} \ldots \partial x_{i_k}} \frac{\partial^\ell \bar{f}(y)}{\partial y_{j_1} \ldots \partial y_{j_\ell}}\right) = \frac{\partial^{k+\ell} C(x, y)}{\partial x_{i_1} \ldots \partial x_{i_k} \partial y_{j_1} \ldots \partial y_{j_\ell}}. \tag{3.12}$$

Straightforward algebra then gives (3.6), (3.7). Moreover, since the derivatives of a centered Gaussian field have centered Gaussian distribution, relations (3.6) and (3.7) determine uniquely the joint distribution of $f(\boldsymbol{\sigma})$, $f_i(\boldsymbol{\sigma})$, and $f_{ij}(\boldsymbol{\sigma})$. This completes the proof of the claim (a).

The well-known rules how Gaussian distributions transform under conditioning (see, e.g., [AT07], pages 10–11) then yield the claim (b) of the lemma. $\qquad\square$

The next tool in order to prove Theorem 2.1 is the following lemma which is an independent fact about the distribution of the eigenvalues of the GOE. It allows us to deal with the (usually rather unpleasant) absolute value of the determinant of the Hessian that appears in (3.5).

Lemma 3.3. *Let M^{N-1} be a $(N-1) \times (N-1)$ GOE matrix and X be an independent Gaussian random variable with mean m and variance t^2. Then, for any Borel set $G \subset \mathbb{R}$,*

$$\mathbb{E}\Big[\big|\det M^{N-1} - XI\big| \mathbf{1}\{i(M^{N-1} - XI) = k, X \in G\}\Big]$$
$$= \frac{\Gamma(\frac{N}{2})(N-1)^{-\frac{N}{2}}}{\sqrt{\pi t^2}} \mathbb{E}_{GOE}^N \left[\exp\left\{\frac{N(\lambda_k^N)^2}{2} - \frac{(\{\frac{N}{N-1}\}^{\frac{1}{2}}\lambda_k^N - m)^2}{2t^2}\right\} \mathbf{1}\Big\{\lambda_k^N \in \{\tfrac{N-1}{N}\}^{\frac{1}{2}} G\Big\}\right]. \tag{3.13}$$

Proof. The left-hand side of (3.13) can be written as

$$\frac{1}{\sqrt{2\pi t^2}} \int_G e^{-\frac{(x-m)^2}{2t^2}} \mathbb{E}_{GOE}^{N-1}\Big[\big|\det(M - xI)\big| \mathbf{1}\{i(M - xI) = k\}\Big] \mathrm{d}x. \tag{3.14}$$

Observe that $\mathbf{1}\{i(M - xI) = k\} = \mathbf{1}\{A_k^N\}$, where $A_k^N \subset \mathbb{R}^N$ is the set

$$\big\{(x, \lambda^{N-1}) : \lambda_0^{N-1} \leq \cdots \leq \lambda_{k-1}^{N-1} < x \leq \lambda_k^{N-1} \leq \ldots \leq \lambda_{N-2}^{N-1}\big\}. \tag{3.15}$$

Recall the explicit formula for the distribution Q_N of the eigenvalues λ_i^N of the GOE matrix M^N, (see, e.g., [BG97], p. 519),

$$Q_N(\mathrm{d}\lambda^N) = \frac{1}{Z_N} \prod_{i=0}^{N-1} e^{-\frac{N}{2}(\lambda_i^N)^2} \mathrm{d}\lambda_i^N \prod_{0 \leq i < j < N} |\lambda_i^N - \lambda_j^N| \mathbf{1}\{\lambda_0^N \leq \cdots \leq \lambda_{N-1}^N\}, \tag{3.16}$$

where the normalisation Z_N can be computed from Selberg's integral (cf. [BG97], p. 529)

$$Z_N = \frac{1}{N!}(2\sqrt{2})^N N^{-N(N+1)/4} \prod_{i=1}^N \Gamma\Big(1 + \frac{i}{2}\Big). \tag{3.17}$$

Using this notation, (3.14) becomes

$$\frac{1}{\sqrt{2\pi t^2}} \int_G e^{-\frac{(x-m)^2}{2t^2}} \int_{A_k^N} \prod_{i=0}^{N-2} |\lambda_i^{N-1} - x| Q_{N-1}(\mathrm{d}\lambda^{N-1}). \tag{3.18}$$

Comparing the product in the integrand with the van der Monde determinant in (3.16) suggests considering x as the $k+1$-th smallest eigenvalue of a $N \times N$ GOE matrix. Indeed, if we substitute $\lambda_i^{N-1} = \{\frac{N}{N-1}\}^{1/2}\lambda_i^N$, for $i \in \{0, \ldots, k-1\}$, $\lambda_i^{N-1} = \{\frac{N}{N-1}\}^{1/2}\lambda_{i+1}^N$ for $i \in \{k, \ldots, N-2\}$ and $x = \{\frac{N}{N-1}\}^{1/2}\lambda_k^N$ and perform the change of variables, then we obtain

$$\frac{Z_N}{Z_{N-1}\sqrt{2\pi t^2}} \Big(\frac{N}{N-1}\Big)^{\frac{(N+2)(N+1)}{4}}$$
$$\times \mathbb{E}_{GOE}^N \left[\exp\left(\frac{N(\lambda_k^N)^2}{2} - \frac{(\{\frac{N}{N-1}\}^{1/2}\lambda_k^N - m)^2}{2t^2}\right) \mathbf{1}\Big\{\lambda_k^N \in \{\tfrac{N-1}{N}\}^{1/2} G\Big\}\right]. \tag{3.19}$$

Plugging (3.17) into (3.19) completes the proof of the lemma. $\qquad\square$

The three above lemmas yield the proof of Theorem 2.1 as follows.

Proof of Theorem 2.1. We first verify the conditions of Lemma 3.1. We use the same chart and orthogonal frame as in the proof of Lemma 3.2. From (3.6), (3.7), it is not difficult to check that the joint distribution of $(f_i(\boldsymbol{\sigma}), f_{ij}(\boldsymbol{\sigma}))$ is non-degenerate for $\boldsymbol{\sigma} = \boldsymbol{n}$. By the continuity of the covariances, it is then non-degenerate for all $\boldsymbol{\sigma} \in U$, if U is small enough. Similarly, using (3.12) we can verify that (3.3) is satisfied on U. Since S^{N-1} can be covered by a finite number of copies of U obtained by rotations around the center of sphere, the conditions of Lemma 3.1 are satisfied.

We can thus apply formula (3.5). First note that, due to the rotational symmetry again, the integrand does not depend on $\boldsymbol{\sigma}$. Hence, recalling that $\boldsymbol{n}$ denotes the north pole and using (2.5),(3.1),

$$
\begin{aligned}
&\mathbb{E}\,\mathrm{Crt}_{N,k}(B) \\
&= \omega_N \mathbb{E}\Big[\,\big|\det \nabla^2 f(\boldsymbol{n})\big|\,\mathbf{1}\{i(\nabla^2 f_N(\boldsymbol{\sigma})) = k\}\mathbf{1}\{f(\boldsymbol{n}) \in \sqrt{N}B\}\big|\nabla f(\boldsymbol{n}) = 0\Big]\phi_{\boldsymbol{n}}(0),
\end{aligned}
\tag{3.20}
$$

where ω_N is the volume of the sphere S^{N-1},

$$
\omega_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}.
\tag{3.21}
$$

The density of gradient $\nabla f(\boldsymbol{n})$ is same as the density of $(\bar{f}_i(0))_{1 \le i < N}$. Hence, using (3.6),

$$
\phi_{\boldsymbol{n}}(0) = (2\pi p)^{-(N-1)/2}.
\tag{3.22}
$$

To compute the expectation in (3.20), we condition on $f(\boldsymbol{n})$ and use the fact that, by (3.6), both f and its Hessian are independent of the gradient,

$$
\begin{aligned}
&\mathbb{E}\Big[|\det \nabla^2 f_N(\boldsymbol{n})|\mathbf{1}\{i(\nabla^2 f_N(\boldsymbol{\sigma})) = k, f(\boldsymbol{n}) \in \sqrt{N}B\}\big|\nabla f(\boldsymbol{n}) = 0\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\big[|\det \nabla^2 f_N(\boldsymbol{n})|\mathbf{1}\{i(\nabla^2 f_N(\boldsymbol{\sigma})) = k, f(\boldsymbol{n}) \in \sqrt{N}B\}\big|f(\boldsymbol{n})\big]\Big].
\end{aligned}
\tag{3.23}
$$

By Lemma 3.2, the interior expectation satisfies

$$
\begin{aligned}
&\mathbb{E}\big[|\det \nabla^2 f_N(\boldsymbol{n})|\mathbf{1}\{i(\nabla^2 f_N(\boldsymbol{\sigma})) = k, f(\boldsymbol{n}) \in \sqrt{N}B\}\big|f(\boldsymbol{n})\big] \\
&= (2(N-1)p(p-1))^{\frac{N-1}{2}}\mathbb{E}_{\mathrm{GOE}}^{N-1}\Big[\big|\det\big(M^{N-1} - p^{1/2}(2(N-1)(p-1))^{-1/2}f(\boldsymbol{n})I\big)\big| \\
&\quad\times \mathbf{1}\Big\{i(M^{N-1} - p^{1/2}(2(N-1)(p-1))^{-1/2}f(\boldsymbol{n})I) = k, f(\boldsymbol{n}) \in \sqrt{N}B\Big\}\Big].
\end{aligned}
\tag{3.24}
$$

Inserting (3.24) into (3.23), we can apply Lemma 3.3 with $m = 0$, $t^2 = \frac{p}{2(N-1)(p-1)}$, and $G = \sqrt{\frac{Np}{2(N-1)(p-1)}}B$. Using (3.21) and (3.22), we get after a little straightforward algebra,

$$
\mathbb{E}[\mathrm{Crt}_{N,k}(B)] = 2\sqrt{\frac{2}{p}}(p-1)^{\frac{N}{2}}\mathbb{E}_{\mathrm{GOE}}^N\Big[e^{-N\frac{p-2}{2p}(\lambda_k^N)^2}\mathbf{1}\Big\{\lambda_k^N \in \sqrt{\frac{p}{2(p-1)}}B\Big\}\Big].
\tag{3.25}
$$

This completes the proof of Theorem 2.1. $\qquad\square$

Proof of Theorem 2.2. Theorem 2.2 follows from Theorem 2.1 by summing over $k \in \{0, \dots, N-1\}$. The additional N in the prefactor comes from the fact that ρ_N is *normalised* one-point correlation function, cf. (2.7). $\qquad\square$

4. Logarithmic asymptotics of the complexity

In this section we apply the LDP for the k-th largest eigenvalue of GOE, Theorem A.1, to study the logarithmic asymptotics of the complexity of the spherical spin glass, that is we show Theorems 2.5 and 2.8. Observe, that comparing (2.6) with (A.1), we must use Theorem A.1 with $\sigma = 2^{-1/2}$. Here and later we write λ_i for λ_i^N.

4.1. **Proof of Theorem 2.5.** We analyse the exact formula given in Theorem 2.1, or equivalently in (3.25). By Theorem A.1 and the obvious symmetry between the largest and the smallest eigenvalues, the $(k+1)$-th smallest eigenvalue λ_k^N of M^N satisfies the LDP with the good rate function $J_k(u) = (k+1)I_1(-u; 2^{-1/2})$, where I_1 is defined in (A.2). Set $t = u\sqrt{\frac{p}{2(p-1)}}$ and $\phi(x) = -\frac{p-2}{2p}x^2$. Then, by Theorem 2.1,

$$
\lim_{N\to\infty}\frac{1}{N}\log \mathbb{E}\,\mathrm{Crt}_{N,k}(u) = \frac{1}{2}\log(p-1) + \lim_{N\to\infty}\frac{1}{N}\log \mathbb{E}_{\mathrm{GOE}}^N\big[e^{N\phi(\lambda_k^2)}\mathbf{1}_{\lambda_k \le t}\big].
\tag{4.1}
$$

By Varadhan's Lemma (see e.g. [DZ98], Theorem 4.3.1 and Exercise 4.3.11, observe that ϕ is bounded from above, so that condition (4.3.2) of [DZ98] is obviously satisfied)

$$\sup_{x\in(-\infty,t)}(\phi(x)-J_k(x)) \le \liminf_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N_{\mathrm{GOE}}\big[e^{N\phi(\lambda_k^2)}\mathbf{1}_{\lambda_k<t}\big]$$

$$\le \limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N_{\mathrm{GOE}}\big[e^{N\phi(\lambda_k^2)}\mathbf{1}_{\lambda_k\le t}\big] \le \sup_{x\in(-\infty,t]}(\phi(x)-J_k(x)). \tag{4.2}$$

It can be seen easily that for $t \le -\sqrt{2}$ the both suprema in (4.2) equal $\phi(t)-J_k(t)$. On the other hand, if $t > -\sqrt{2}$, these suprema equal $\phi(\sqrt{2})$. Hence, using the definitions of ϕ, t, J_k, E_c, and the scaling relation (A.5),

$$(4.1) = \begin{cases} \frac{1}{2}\log(p-1) - \frac{p-2}{p}, & \text{if } u > -E_c, \\ \frac{1}{2}\log(p-1) - \frac{(p-2)u^2}{4(p-1)} - (k+1)I_1(-u;E_c/2), & \text{if } u \le -E_c. \end{cases} \tag{4.3}$$

Using Remark 2.4, this completes the proof of Theorem 2.5.

Remark 4.1. The proof of Theorem 2.5 clearly extends to a Borel set B. In fact, one just need to apply Varadhan's Lemma and find the supremum of $\phi(x)-J_k(x)$ on a appropriate domain.

4.2. **Proof of Theorem 2.8.** Let t and ϕ as in the previous proof. By Theorem 2.2, we have to study

$$\lim_{N\to\infty}\frac{1}{N}\log 2N\sqrt{\frac{2}{p}}(p-1)^{\frac{N}{2}}\mathbb{E}\int_{-\infty}^{t}e^{N\phi(x)}L_N(\mathrm{d}x). \tag{4.4}$$

where L_N is the empirical spectral measure of the GOE matrix (see below (2.6)). The constant in front the integral gives the term $\frac{1}{2}\log(p-1)$ of $\theta_p(u)$. We need to evaluate the contribution of the integral. For $t \le -\sqrt{2}$, using $\mathbf{1}_{\lambda_0\le t} \ge \mathbf{1}_{\lambda_i\le t}$ for all i, we write

$$\frac{1}{N}\mathbb{E}\big[e^{N\phi(\lambda_0)}\mathbf{1}_{\lambda_0\le t}\big] \le \mathbb{E}\int_{-\infty}^{t}e^{N\phi(x)}L_N(\mathrm{d}x) \le \mathbb{E}\big[e^{N\phi(t)}\mathbf{1}_{\lambda_0\le t}\big]. \tag{4.5}$$

Taking the logarithm and dividing by N, the both sides of this inequality converge to $\phi(t)-I_1(-t;2^{-1/2})$, by the same argument as in the previous proof. Using the values of t and ϕ, this proves the theorem for $u \le -E_c$.

For $t \in (-\sqrt{2},0]$, we cannot use the smallest eigenvalue λ_0 in the lower bound. Therefore we write, for $\varepsilon > 0$,

$$\frac{1}{N}\mathbb{E}\big[e^{N\phi(t-\varepsilon)}\mathbf{1}_{L_N((t-\varepsilon,t))>0}\big] \le \mathbb{E}\int_{-\infty}^{t}e^{N\phi(x)}L_N(\mathrm{d}x) \le \mathbb{E}\big[e^{N\phi(t)}\mathbf{1}_{\lambda_0\le t}\big]. \tag{4.6}$$

By the LDP for λ_0, $\mathbb{P}[\lambda_0 \ge t] \to 1$ as $N \to \infty$. Similarly, by the convergence of L_N to the semi-circle distribution, we have $\mathbb{P}[L_N((t-\varepsilon,t)) > 0] \to 1$ as $N \to \infty$. Therefore, after taking the logarithm and dividing by N, we find

$$\phi(t-\varepsilon) \le \lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}\int_{-\infty}^{t}e^{N\phi(x)}L_N(\mathrm{d}x) \le \phi(t). \tag{4.7}$$

Since ϕ is continuous, the claim of the theorem follows for $t \in (-\sqrt{2},0]$, or equivalently for $u \in (-E_c,0]$. The proof in the case $u > 0$ is analogous and we left it to the reader.

5. The geometry of the bottom of the energy landscape

In this section, we use our complexity estimates to obtain information about the bottom of the energy landscape. We first prove Theorems 2.14 and 2.15. We then prove Theorem 2.12, that is we show that the normalised energy of the ground state

$$GS^N = N^{-1}\inf\{H_{N,p}(\boldsymbol{\sigma}) : \boldsymbol{\sigma} \in S^{N-1}(\sqrt{N})\} \tag{5.1}$$

converges to $-E_0(p)$ in probability.

5.1. **Proof of Theorem 2.14.** We want to show that there are no critical points of a finite index of the Hamiltonian above the level $N(-E_c + \varepsilon)$. Let k and ε be as in the statement of the theorem. Then, by Markov's inequality and Theorem 2.1,

$$\mathbb{P}\big[\operatorname{Crt}_{N,k}((-E_c + \varepsilon, \infty)) > 0\big] \leq \mathbb{E}\big[\operatorname{Crt}_{N,k}((-E_c + \varepsilon, \infty))\big]$$

$$\leq c(p)(p-1)^{N/2}\mathbb{E}\left[\exp\left\{-\frac{N(p-2)(\lambda_k)^2}{2p}\right\}\mathbf{1}\{\lambda_k \geq -\sqrt{2} + \varepsilon'\}\right] \tag{5.2}$$

$$\leq c(p)(p-1)^{N/2}\mathbb{P}\left[\lambda_k \geq -\sqrt{2} + \varepsilon'\right],$$

where $\varepsilon' = \varepsilon\sqrt{2}/E_c$. By the LDP for the empirical spectral measure L_N (see Theorem 1.1 of [BG97]), we know that for some $C(\varepsilon') > 0$

$$\mathbb{P}\left[\lambda_k \geq -\sqrt{2} + \varepsilon\right] \leq e^{-C_\varepsilon N^2}. \tag{5.3}$$

Combining this estimate with (5.2) completes the proof of Theorem 2.14.

5.2. **Proof of Theorem 2.15.** We want to prove that there are no critical values of index k of the Hamiltonian below $-N(E_k + \varepsilon)$. Using Theorem 2.5, we have

$$\mathbb{E}\big[\operatorname{Crt}_{N,k}(-E_k - \varepsilon)\big] \leq \exp\big\{N\theta_{k,p}(-E_k - \varepsilon) + o(N)\big\}. \tag{5.4}$$

The function $\theta_{k,p}$ is strictly increasing on $(-\infty, -E_c)$. The constant $E_k > E_c$ is defined by $\theta_{k,p}(-E_k) = 0$. Therefore, $\theta_{k,p}(-E_k - \varepsilon) = c(k, p, \varepsilon) < 0$. An application of Markov's inequality as before completes the proof of Theorem 2.15.

5.3. **Proof of Theorem 2.12.** Note that, by Theorem 2.15 there are no minima of the Hamiltonian below $-N(E_0(p) + \varepsilon)$, with high probability. This implies that for all $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}[GS^N \geq -E_0(p) - \varepsilon] = 1. \tag{5.5}$$

To find a matching upper bound on GS^N we use known results about the free energy at positive temperature, more precisely the Parisi formula as proved by Talagrand [Tal06]. Recall that the partition function of the p-spin spin glass is given by

$$Z_{N,p}(\beta) = \int_{S^{N-1}(\sqrt{N})} e^{-\beta H_{N,p}(\boldsymbol{\sigma})}\Lambda_N(\mathrm{d}\boldsymbol{\sigma}), \tag{5.6}$$

where Λ_N is the normalised surface measure on the sphere $S^{N-1}(\sqrt{N})$. By Theorem 1.1 of [Tal06],

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\log Z_{N,p}(\beta) = F_p(\beta), \tag{5.7}$$

where $F(\beta)$ is given by the variational principle as in (1.11) of [Tal06]. Furthermore, it is known that in the case of the spherical p-spin model, the $F_p(\beta)$ can be computed using the following simpler variational problem (see (1.16) in [PT07]),

$$F_p(\beta) = \inf_{q,m\in[0,1]} \frac{1}{2}\left\{\beta^2\big(1 + (m-1)q^p\big) + \left(1 - \frac{1}{m}\right)\log(1-q) + \frac{1}{m}\log\big(1 - q(1-m)\big)\right\}. \tag{5.8}$$

We now analyse this variational problem. We replace $q = 1 - d\beta^{-1}$ and $m = c\beta^{-1}$ in the last equation, and we denote the function inside the infimum by $P(\beta, c, d)$,

$$P(\beta,c,d) = \frac{1}{2}\left\{\beta^2\big(1 + (c\beta^{-1} - 1)(1 - d\beta^{-1})^p\big)\right.$$

$$\left. + \big(1 - c^{-1}\beta\big)\log\big(1 - (1 - d\beta^{-1})\big) + c^{-1}\beta\log\big[1 - (1 - d\beta^{-1})(1 - c\beta^{-1})\big]\right\}. \tag{5.9}$$

The following lemma shows that this is the right scaling as β tends to infinity.

Lemma 5.1. *There exist constants $0 < \varepsilon < M < \infty$, such that, for all β large enough,*

$$F_p(\beta) = \inf_{c,d\in[0,\beta]} P(\beta, c, d) = \inf_{c,d\in[\varepsilon,M]} P(\beta, c, d). \tag{5.10}$$

Proof. It follows directly from Lemma 3 in [PT07]. □

As $\beta \to \infty$, the function $\beta^{-1}P(\beta, c, d)$ converges uniformly on the compact set $c, d \in [\varepsilon, M]$. Therefore the last lemma implies that

$$\lim_{\beta \to \infty} \beta^{-1}F_p(\beta) = \inf_{c,d \in [\varepsilon, M]} \frac{1}{2}\left\{c + pd + \frac{1}{c}\left(\log(c + d) - \log d\right)\right\} =: \inf_{c,d \in [\varepsilon, M]} P(c, d). \quad (5.11)$$

Lemma 5.2. *We have that*

$$\gamma \equiv \inf_{c,d \in [\varepsilon, M]} P(c, d) = E_0(p). \quad (5.12)$$

Proof. The constant $E_0(p)$ is the unique solution of the equation $\theta_{0,p}(-x) = 0$ (see (2.16), (2.22)). Therefore, to prove the lemma it suffices to show

$$\theta_{0,p}(-\gamma) = 0. \quad (5.13)$$

A critical point of $P(c, d)$ satisfies

$$p^{-1} = d(c + d) \quad (5.14)$$

$$c^2(c + d) + c = (c + d)(\log(c + d) - \log(d)). \quad (5.15)$$

Writing $y = c + d$, it follows from (5.14) that $d = (py)^{-1}$, $c = y - d = (py^2 - 1)/py$. From (5.14) and $d \geq \varepsilon$, we further obtain $d = \frac{1}{2}(-c + \sqrt{c^2 + 4p^{-1}})$. Therefore $c, d \geq \varepsilon$ implies

$$y = c + d > p^{-1/2}. \quad (5.16)$$

Inserting these computations into (5.15), we obtain that y is a solution of

$$\left(\frac{py^2 - 1}{py}\right)^2 y + \frac{py^2 - 1}{py} = y \log(py^2), \qquad y > p^{-1/2}. \quad (5.17)$$

Setting $a = py^2$, this is equivalent to

$$g_p(a) := (a - 1)^2 + p(a - 1) - pa \log a = 0, \qquad a > 1. \quad (5.18)$$

The function g satisfies $g_p(1) = g_p'(1) = 0$, $g_p''(1) < 0$ and g'' is increasing on $[1, \infty)$. Therefore, g_p has a unique minimum a_0 on $[1, \infty)$ and is strictly increasing on $[a_0, \infty)$. Moreover, since $g_p(p - 1) < 0$, the equation (5.17) has a unique solution satisfying

$$a = py^2 > p - 1 \qquad \text{and thus} \qquad y \geq \sqrt{(p - 1)/p}. \quad (5.19)$$

Using the definition of γ and (5.17)

$$\gamma = \frac{1}{2}\left(\frac{py^2 - 1}{py} + \frac{1}{y} + \frac{py}{py^2 - 1}\log(py^2)\right) = y + \frac{p - 1}{yp}. \quad (5.20)$$

To compute $\theta_{0,p}(-\gamma)$ observe that, by (5.20), $\gamma^2 - E_c^2 = \left(y - \frac{p-1}{py}\right)^2$. Hence, (5.19) implies

$$\gamma + \sqrt{\gamma^2 - E_c^2} = 2y. \quad (5.21)$$

Inserting these results into definition (2.16) of $\theta_{0,p}$ and using equation (5.17) yields after a little algebra that $\theta_{0,p}(-\gamma) = 0$. This completes the proof of Lemma 5.2 □

We can now prove the upper bound on GS^N. Note that

$$\frac{1}{\beta N}\mathbb{E}\log Z_N = \frac{1}{\beta N}\mathbb{E}\log\int_{S^N(\sqrt{N})} e^{-\beta H_{N,p}(\boldsymbol{\sigma})}\Lambda_N(d\boldsymbol{\sigma}) \leq -\mathbb{E}GS^N. \quad (5.22)$$

Taking the limits $N \to \infty$ and then $\beta \to \infty$, using (5.11) and Lemma 5.2, we obtain

$$\mathbb{E}GS^N \leq -E_0(p) + \varepsilon \qquad \text{for } N \text{ large enough.} \quad (5.23)$$

By Borell-TIS inequality (see Theorem 2.7 in [AW09]),

$$\mathbb{P}\left[|GS^N + \mathbb{E}GS^N| > \varepsilon\right] \leq e^{-N\varepsilon^2}. \quad (5.24)$$

Combining (5.23) and (5.24), we get that for all $\varepsilon > 0$, for N large enough,

$$\mathbb{P}\big[GS^N \leq -E_0(p) + 2\varepsilon\big] \geq 1 - \varepsilon. \tag{5.25}$$

This combined with the lower bound (5.5) completes the proof of Theorem 2.12.

6. The Thouless-Anderson-Palmer complexity

In this section, we study the mean number of solutions of the Thouless-Anderson-Palmer (TAP) equation (6.4), called the TAP complexity. Using the previous results of this paper, we give, in Theorem 6.1, a formula for the TAP complexity at any finite temperature. In physics literature [CS95], the TAP complexity was predicted and used to derive a formula for the complexity of the spherical p-spin model. In Lemma 6.3, we show that the formula of [CS95] agrees with our Theorem 2.5. For further physics interpretation of the TAP solutions, such as connections to metastable states, the reader is invited to check [KPV93, CS95] and the references therein.

Let $B(0, \sqrt{N}) \subset \mathbb{R}^N$ be the open ball of radius $\sqrt{N}$ centered at 0. We define the TAP functional as Kurchan, Parisi and Virasoro in [KPV93]. For $\boldsymbol{m} = (m_1, \ldots, m_N)$ in $B(0, \sqrt{N})$ and $q = \frac{1}{N}\sum_i m_i^2 \in [0, 1)$, let

$$F_{\text{TAP}}(\boldsymbol{m}) = \frac{1}{2^{1/2}N^{(p+1)/2}} \sum_{i_1,\ldots,i_p=1}^N J_{i_1,\ldots i_p} m_{i_1} \cdots m_{i_p} + B_{p,\beta}(q), \tag{6.1}$$

where, as usual, $J_{i_1,\ldots i_p}$ are independent standard normal random variables and

$$B(q) = B_{p,\beta}(q) = -\frac{1}{2\beta}\log(1-q) - \frac{\beta}{4}\big(1 + (p-1)q^p - pq^{p-1}\big). \tag{6.2}$$

Notice that the TAP functional F_{TAP} can be written in spherical coordinates: Defining $\boldsymbol{\sigma} = q^{-1/2}\boldsymbol{m} \in S^{N-1}(\sqrt{N})$ and $h_{N,p}(\boldsymbol{\sigma}) = N^{-1}H_{N,p}(\boldsymbol{\sigma})$,

$$F_{\text{TAP}}(\boldsymbol{m}) = f_{\text{TAP}}(q, \boldsymbol{\sigma}) = 2^{-1/2}q^{p/2}h_{N,p}(\boldsymbol{\sigma}) + B_{p,\beta}(q). \tag{6.3}$$

The TAP equations are equations for critical points of the TAP functional,

$$\frac{\partial}{\partial m_i}F_{\text{TAP}}(\boldsymbol{m}) = 0. \tag{6.4}$$

Since $q = 0$ is not a critical point of F_{TAP}, the equations (6.4) are equivalent with

$$\frac{\partial}{\partial \sigma_i}f_{\text{TAP}}(q, \boldsymbol{\sigma}) = 0 \quad \Longleftrightarrow \quad \frac{\partial}{\partial \sigma_i}H_{N,p}(\boldsymbol{\sigma}) = 0, \tag{6.5}$$

$$\frac{\partial}{\partial q}f_{\text{TAP}}(q, \boldsymbol{\sigma}) = 2^{-3/2}pq^{p/2-1}h_{N,p}(\boldsymbol{\sigma}) + B'(q) = 0. \tag{6.6}$$

Therefore, a solution of the TAP equation (6.4) must be a critical point of the Hamiltonian $H_{N,p}$ such that q satisfies (6.6). A critical point $(q, \boldsymbol{\sigma})$ of the TAP functional of index k is thus either a critical point of $H_{N,p}$ of index k satisfying (6.6) and $\frac{\partial^2}{\partial q^2}f_{\text{TAP}}(q, \boldsymbol{\sigma}) > 0$, or a critical point of $H_{N,p}$ of index $k-1$ satisfying (6.6) and $\frac{\partial^2}{\partial q^2}f_{\text{TAP}}(q, \boldsymbol{\sigma}) < 0$.

Let $\mathcal{N}_k(u, \beta)$ represent the number of critical points of index k of the TAP functional with normalised energy $h_{N,p}$ smaller than u at temperature β^{-1},

$$\mathcal{N}_k(u, \beta) = \sum_{(q, \boldsymbol{\sigma}):\nabla f_{\text{TAP}}(q, \boldsymbol{\sigma})=0} \mathbf{1}\big\{h_{N,p}(\boldsymbol{\sigma}) \in (-\infty, u], i(\nabla^2 f_{\text{TAP}}(q, \boldsymbol{\sigma})) = 0\big\}. \tag{6.7}$$

For each $u \in (-\infty, -E_c]$, $p \geq 3$, and $\beta > 0$, we further define

$$\beta(u) = \frac{p}{2^{3/2}(p-1)}\Big(\frac{p}{p-2}\Big)^{\frac{p-2}{2}}\big(-u - \sqrt{u^2 - E_c^2}\big) \tag{6.8}$$

$$u^\star(\beta) = \sup\big\{v \in (-\infty, -E_c] : \beta(v) < \beta\big\}. \tag{6.9}$$

Observe that $\beta(\cdot)$ is an increasing function with $\lim_{u \to -\infty} \beta(u) = 0$.

Theorem 6.1. *For all $\beta > 0$ and $p \geq 3$,*

(a) *If $u \leq -E_c$, $k = 0$*

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}\mathcal{N}_0(u, \beta) = \theta_{0,p}(u^*(\beta) \wedge u). \tag{6.10}$$

(b) *If $u \leq -E_c$, $k > 0$,*

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}\mathcal{N}_k(u, \beta) = \theta_{k-1,p}(u^*(\beta) \wedge u). \tag{6.11}$$

(c) *If $\beta < \beta(-E_{(k-1)\vee 0}(p))$ or if $u < -E_{(k-1)\vee 0}(p)$, then $\mathcal{N}_k(u, \beta)$ tends to zero in probability as $N \to \infty$.*

Proof. We start by proving parts (a) and (b). Computing $B'(q)$ and multiplying the both sides of (6.6) by $2^{3/2}\beta(1-q)/p$, (6.6) is equivalent to

$$2^{-1/2}(p-1)\beta^2\big((1-q)q^{p/2-1}\big)^2 + h_{N,p}(\boldsymbol{\sigma})\beta(1-q)q^{p/2-1} + \frac{2^{1/2}}{p} = 0. \tag{6.12}$$

Setting

$$z = z(q) = (1-q)q^{p/2-1} \tag{6.13}$$

we obtain

$$\beta z = \frac{1}{2^{1/2}(p-1)}\Big(-h_{N,p}(\boldsymbol{\sigma}) \pm \sqrt{h_{N,p}^2(\boldsymbol{\sigma}) - E_c^2}\Big). \tag{6.14}$$

Thus, a solution $(\boldsymbol{\sigma}, q)$ to the TAP equation (6.4) is a critical point $\boldsymbol{\sigma}$ of the Hamiltonian that satisfies (6.13) and (6.14).

The next lemma counts the number of solutions of equation (6.13).

Lemma 6.2. *The function $f(q) = (1-q)q^{p/2-1} - \lambda$, has exactly two, one or no zeros in $q \in [0,1)$ if $0 < \lambda < \frac{2}{p}(\frac{p-2}{p})^{\frac{p-2}{2}}$, $\lambda = \frac{2}{p}(\frac{p-2}{p})^{\frac{p-2}{2}}$ and $\lambda > \frac{2}{p}(\frac{p-2}{p})^{\frac{p-2}{2}}$, respectively.*

Proof. If $\lambda = 0$, then clearly 0 and 1 are the only zeros. Since $f'(q) = q^{p/2-2}(p-pq-2))/2$, there is a unique critical point of f in $(0,1)$, a maxima at $q = \frac{p-2}{p}$. Now, varying λ gives us the result of the lemma. $\square$

Applying the lemma to equations (6.13), (6.14), we see that in order to $(\boldsymbol{\sigma}, q)$ be a critical point, $h_{N,p}(\boldsymbol{\sigma})$ must satisfy (ignoring the equalities which have zero probability)

$$0 < \frac{1}{2^{1/2}\beta(p-1)}\Big(-h_{N,p}(\boldsymbol{\sigma}) \pm \sqrt{h_{N,p}^2(\boldsymbol{\sigma}) - E_c^2}\Big) < \frac{2}{p}\Big(\frac{p-2}{p}\Big)^{\frac{p-2}{2}}, \tag{6.15}$$

which is equivalent to (cf. (6.8), (6.9))

$$h_{N,p}(\boldsymbol{\sigma}) < u^\star(\beta). \tag{6.16}$$

Furthermore, if (6.16) is satisfied, the equations (6.13), (6.14) have two solutions q_1, q_2 such that $0 < q_1 < (p-2)/p < q_2 < 1$.

At a critical point, using (6.12) to compute $h_{N,p}$, and (6.13) to simplify the terms containing q^p,

$$\frac{\partial^2}{\partial q^2} f_{\mathrm{TAP}}(q, \boldsymbol{\sigma})$$

$$= \frac{p}{2}\Big(\frac{p}{2} - 1\Big)q^{\frac{p}{2}-2}h_{N,p}(\boldsymbol{\sigma}) + \frac{1}{2\beta(1-q)^2} - \frac{\beta q^{p-3}p(p-1)}{4}((p-1)q - p + 2) \tag{6.17}$$

$$= \frac{p(p-1)}{8q\beta(1-q)^2}\Big(q - \frac{p-2}{p}\Big)\Big(\beta^2 z^2 - \frac{2}{p(p-1)}\Big).$$

Since, $z(q_1) = z(q_2)$, the above expression is positive for one of q_1, q_2 and negative for the remaining one. Hence, every critical point $\boldsymbol{\sigma}$ of index k of the Hamiltonian with $h_{N,p}(\boldsymbol{\sigma})$

as in (6.16) contributes one critical point of f_{TAP} with index k and one of index $k+1$. Therefore, for every $k \geq 0$, $\beta > 0$ and $u < -E_c$, $\mathbb{P}$-a.s.,

$$\frac{\mathcal{N}_k(u, \beta)}{\text{Crt}_{N,k-1}(u \wedge u^\star(\beta)) + \text{Crt}_k(u \wedge u^\star(\beta))} \in [1, 2], \tag{6.18}$$

where we defined $\text{Crt}_{N,-1}(-\infty, c) = 0$. Claims (a) and (b) then follows directly from Theorem 2.5, using the observation $\theta_{k,p}(u) \geq \theta_{k+1,p}(u)$ for all $k \geq 0$ and $u \leq -E_c$.

Claim (c) of the theorem then follows from (a), (b) using the Markov inequality. $\qquad \square$

6.1. Complexity of the minima: derivation of formula (11) of [CS95].

In this short section, we verify that the results obtained in Theorem 2.5 for minima, resp. maxima, agree with the formula proposed by A. Crisanti and H.-J. Sommers in [CS95].

Lemma 6.3. Let $z = \frac{1}{p-1}(-u - \sqrt{u^2 - \frac{2(p-1)}{p}})$ and $u \leq -\sqrt{\frac{2(p-1)}{p}}$, then

$$\theta_{0,p}(\sqrt{2}u) = \frac{1}{2}\Big(\frac{2-p}{p} - \log(\frac{pz^2}{2}) + \frac{p-1}{2}z^2 - \frac{2}{p^2z^2}\Big). \tag{6.19}$$

Proof. First, note that for $u = -\sqrt{\frac{2(p-1)}{p}}$, we have that, by (2.16),

$$\theta_{0,p}(\sqrt{2}u) = \frac{1}{2}\log(p-1) - \frac{p-2}{p}, \tag{6.20}$$

which agrees with the left-hand side of (6.19), taking $z = -\sqrt{\frac{2}{p(p-1)}}$.

Second, the derivative with respect to u of the left-hand side of (6.19) is given by

$$\theta'_{0,p}(\sqrt{2}u) = (p-1)^{-1}\big\{(2-p)u + \sqrt{p(pu^2 - 2p + 2)}\big\}. \tag{6.21}$$

On the other hand, the derivative of the right-hand side of (6.19) equals

$$\begin{aligned}
\frac{1}{2}\Big(&(4(p-1)^2(-1 - u/\sqrt{u^2 - (2(p-1))/p}))/(p^2(-u - \sqrt{u^2 - (2(p-1))/p})^3)\\
&- (2(-1 - u/\sqrt{u^2 - (2(p-1))/p}))/(-u - \sqrt{u^2 - (2(p-1))/p})\\
&+ ((-1 - u/\sqrt{u^2 - (2(p-1))/p})(-u - \sqrt{u^2 - (2(p-1))/p}))/(p-1)\Big).
\end{aligned} \tag{6.22}$$

After a lengthy, but straightforward, simplification, the last expression coincides with $\theta'_{0,p}(\sqrt{2}u)$. This completes the proof. $\qquad \square$

7. Sharper estimates of the mean number of critical points

In this section we prove the sharp asymptotic results on the complexity of spherical p-spin spin glass, that is Theorem 2.17. To this end we analyse the exact formula (2.9) proved in Theorem 2.2. For $u > -E_c$, a simple application of Laplace's method is sufficient. For $u \leq -E_c$, more care is needed. We write the one-point correlation function $\rho_N(x)$ appearing in (2.9) as a function of the Hermite polynomials and to use the asymptotics of these polynomials given by the Plancherel-Rotach formula.

7.1. Sharp asymptotics in the bulk.

We start with the case $u > -E_c$ that is parts (c), (d) of Theorem 2.17. Setting

$$C_{N,p} = 2N\sqrt{\frac{2}{p}}(p-1)^{N/2}, \qquad v = -u\sqrt{\frac{p}{2(p-1)}}, \qquad F_p(x) = \frac{(p-2)x^2}{2p}, \tag{7.1}$$

we rewrite the expectation of the global complexity (2.9) as

$$\mathbb{E}\,\text{Crt}_N(u) = C_{N,p}\int_{-\infty}^{-v} e^{-NF_p(x)}\rho_N(x)\,\mathrm{d}x = C_{N,p}\int_{v}^{\infty} e^{-NF_p(x)}\rho_N(x)\,\mathrm{d}x. \tag{7.2}$$

For the last equality we used the fact that F_p and ρ_N are even functions. The case $u > -E_c$ corresponds to $v < \sqrt{2}$. Since ρ_N converges uniformly to the density of the semi-circle law $\rho(x) = \pi^{-1}\sqrt{2 - x^2}\mathbf{1}_{|x|<\sqrt{2}}$, by Laplace's method, the principal contribution to the integral (7.2) comes from the boundary point $x = v$. Since $F'(v) = 0$ iff $v = 0$, we have

$$\mathbb{E}\operatorname{Crt}_N(0) = \frac{1}{2}C_{N,p}e^{-NF_p(0)}\rho(0)\sqrt{\frac{2\pi}{NF_p''(0)}}(1 + o(1)),$$
$$\tag{7.3}$$

$$\mathbb{E}\operatorname{Crt}_N(u) = C_{N,p}e^{-NF_p(v)}\rho(v)\big(NF_p'(v)\big)^{-1}(1 + o(1)), \quad \text{for } u \in (-E_c, 0).$$

For $u > 0$, $\mathbb{E}\operatorname{Crt}_N(u) = 2\mathbb{E}\operatorname{Crt}_N(0)(1 + o(1))$, obviously. Inserting back the definitions (7.1) completes the proof of Theorem 2.17(c,d).

7.2. **Sharp asymptotics at and beyond the edge.** We first rewrite the one-point correlation function ρ_N using the Hermite functions ϕ_j, $j \in \mathbb{N}$, given by

$$\phi_j(x) = (2^j j!\sqrt{\pi})^{-1/2}H_j(x)e^{-\frac{x^2}{2}}, \tag{7.4}$$

where H_j, $j \in \mathbb{N}$ are Hermite polynomials, $H_0(x) = 1$ and $H_j(x) = e^{x^2}(-\frac{\mathrm{d}}{\mathrm{d}x})^j e^{-x^2}$. The Hermite functions are orthonormal functions in $\mathbb{R}$ with respect to Lebesgue measure. From [Meh91], pp. 128 and 135, it follows that

$$\rho_N(x) = N^{-1/2}\big(S_N(x) + \alpha_N(x)\big), \tag{7.5}$$

where

$$S_N(x) = \sum_{i=0}^{N-1} \phi_i^2(\sqrt{N}x) + \left(\frac{N}{2}\right)^{1/2}\phi_{N-1}(\sqrt{N}x)\int_{-\infty}^{\infty}\varepsilon(\sqrt{N}x - t)\phi_N(t)\,\mathrm{d}t, \tag{7.6}$$

$$\varepsilon(x) = \operatorname{sign}(x)/2, \tag{7.7}$$

$$\alpha_N(x) = \begin{cases} \phi_{2m}(\sqrt{N}x)\big\{\int_{-\infty}^{\infty}\phi_{2m}(t)dt\big\}^{-1}, & \text{if } N = 2m + 1, \\ 0, & \text{if } N \text{ is even.} \end{cases} \tag{7.8}$$

The factor $N^{-1/2}$ in (7.5) comes from a change of variables and the fact that the one-point correlation function R_1 in [Meh91] is not normalised to be a probability density.

Using the Chistoffel-Darboux formula ([Meh91], p. 420), the first term of (7.6) satisfies

$$\sum_{i=0}^{N-1}\phi_i^2(\sqrt{N}x) = N\phi_N^2(\sqrt{N}x) - \sqrt{N(N+1)}\phi_{N-1}(\sqrt{N}x)\phi_{N+1}(\sqrt{N}x), \tag{7.9}$$

so that $\rho_N(x)$ depends on ϕ_{N-1}, ϕ_N and ϕ_{N+1} only. We now state the Plancherel-Rotach asymptotics of the Hermite functions in domains of our interest as a lemma since it will be our main tool from now on. We use ψ, Ψ and h to denote the functions

$$\psi(x) = |x^2 - 2|^{1/2}, \qquad \Psi(x) = \int_{\sqrt{2}}^{x}\psi(y)\mathrm{d}y,$$
$$\tag{7.10}$$
$$h(x) = \left|\frac{x - \sqrt{2}}{x + \sqrt{2}}\right|^{1/4} + \left|\frac{x + \sqrt{2}}{x - \sqrt{2}}\right|^{1/4}.$$

Lemma 7.1. *There exists a $\delta_0 > 0$, such that for all $0 < \delta < \delta_0$ the following holds uniformly for x in the given domains.*

(a) For $\sqrt{2} - \delta < x < \sqrt{2} + \delta$,

$$\phi_N(\sqrt{N}x) = \frac{1}{(2N)^{1/4}}\Bigg\{\left|\frac{x + \sqrt{2}}{x - \sqrt{2}}\right|^{1/4}|f_N(x)|^{1/4}\operatorname{Ai}(f_N(x))(1 + O(N^{-1}))$$
$$- \left|\frac{x - \sqrt{2}}{x + \sqrt{2}}\right|^{1/4}\frac{1}{|f_N(x)|^{1/4}}\operatorname{Ai}'(f_N(x))(1 + O(N^{-1}))\Bigg\}, \tag{7.11}$$

where $f_N(x) = N^{2/3}\{\frac{3}{2}\Psi(x)\}^{2/3}$, *and* $\mathrm{Ai}(x)$ *is the Airy function of first kind,*
$\mathrm{Ai}(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\cos\left(\frac{t^3}{3} + tx\right)\mathrm{d}t.$
  (b) *For* $x > \sqrt{2} + \delta,$

$$\phi_N(\sqrt{N}x) = \frac{e^{-N\Psi(x)}h(x)}{\sqrt{4\pi\sqrt{2N}}}(1 + O(N^{-1})). \tag{7.12}$$

Proof. The lemma follows from [DKM$^+$99, DG07]. Formulas closest to our formulation can be found in [DG07], pp. 20–22. Under their notation, our case has $c_N = \sqrt{2N}$, $h_N(x) = 4$ and $m = 1$. For definitions of f see (2.15), (2.16) of [DKM$^+$99]. For the original Plancherel-Rotach asymptotics for the Hermite polynomials, the reader can also check [PR29] or [Sze81]. $\qquad\square$

For the rest of this section, we suppose that δ is small enough so that Lemma 7.1 holds. We write $a_N \sim b_N$, if $a_N = b_N(1 + o(1))$ as $N \to \infty$. For sequences $a_N(\delta)$, $b_N(\delta)$ which depend on δ, we write $a_N \overset{\delta}{\sim} b_N$ if $\lim_{\delta\to 0}\lim_{N\to\infty} a_N(\delta)/b_N(\delta) = 1$. Finally, if $a_N \leq C(\delta)b_N$ for some $C(\delta) < \infty$, we write $a_N = O_\delta(b_N)$.
  We first analyse the integrals appearing in (7.6) and (7.8). We define

$$J_N(x) := \int_{-\infty}^{\infty} \varepsilon(\sqrt{N}x - t)\phi_N(t)\,\mathrm{d}t = N^{1/2}\int_{-\infty}^{\infty}\varepsilon(x - s)\phi_N(s\sqrt{N})\,\mathrm{d}s. \tag{7.13}$$

Lemma 7.2.   (a) *As* $N \to \infty$

$$\int_{-\infty}^{\infty}\phi_N(x)\,\mathrm{d}x = 2\int_0^{\infty}\phi_N(x)\,\mathrm{d}x \sim 2(2N)^{-1/4}. \tag{7.14}$$

  (b) *If* $x > \sqrt{2}$ *and* $N \to \infty$ *over odd integers, then* $J_N(x) = O(N^c e^{-N\Psi(x)}).$
  (c) *If* $x > \sqrt{2}$ *and* $N \to \infty$ *over even integers, then* $J_N(x) \sim (2N)^{-1/4}.$
  (d) *Let* $a_N \geq 0$ *be such that* $\lim_{N\to\infty} a_N N^{2/3} = 0$. *Then*

$$J_N(\sqrt{2} + a_N) \sim \begin{cases} \frac{2}{3}(2N)^{-1/4}, & \text{as } N \to \infty \text{ over even integers}, \\ -\frac{1}{3}(2N)^{-1/4}, & \text{as } N \to \infty \text{ over odd integers}. \end{cases} \tag{7.15}$$

Proof. Claim (a) can be derived directly from Proposition 4.3 of [DG07].
  The Hermite functions are even (odd) for N even (odd). Therefore,

$$J_N(x) = \begin{cases} N^{1/2}\operatorname{sign} x\int_0^{|x|}\phi_N(s\sqrt{N})\,\mathrm{d}s, & N \text{ even}, \\ -N^{1/2}\operatorname{sign} x\int_{|x|}^{\infty}\phi_N(s\sqrt{N})\,\mathrm{d}s, & N \text{ odd}. \end{cases} \tag{7.16}$$

From Lemma 7.1(b) it follows that for $x > \sqrt{2}$ and some $c, C < \infty$

$$\int_x^{\infty}\phi_N(s\sqrt{N})\,\mathrm{d}s \leq CN^c\int_x^{\infty}e^{-N\Psi(s)}\mathrm{d}s = O(N^c e^{-N\Psi(x)}). \tag{7.17}$$

To prove the last equality we used the fact that Ψ is strictly increasing on $(\sqrt{2}, \infty)$ and Laplace's method. Claims (b), (c) are then direct consequences of (7.17) and claim (a).
  In view of claim (a), (7.16) and (7.17), to prove (d) it suffices to show

$$\int_{\sqrt{2}+a_N}^{\sqrt{2}+\delta}\phi_N(s\sqrt{N})\mathrm{d}s \overset{\delta}{\sim} \frac{1}{3}2^{-1/4}N^{-3/4}. \tag{7.18}$$

To this end we use Lemma 7.1(a). We first linearise the function f_N appearing there. It can be proved easily from its definition that for all $N \geq 1$, uniformly over $x \in (\sqrt{2} - \delta, \sqrt{2} + \delta)$

$$\begin{aligned} f_N(x) &\overset{\delta}{\sim} 2^{1/2}N^{2/3}(x - \sqrt{2}), \\ f_N'(x) &\overset{\delta}{\sim} 2^{1/2}N^{2/3}. \end{aligned} \tag{7.19}$$

Hence, after substitution $f_N(x) = z$ we obtain using Lemma 7.1(a),

$$\int_{\sqrt{2}+a_N}^{\sqrt{2}+\delta} \phi_N(s\sqrt{N})\,\mathrm{d}s \overset{\delta}{\sim} \frac{1}{(2N)^{1/4}} \int_{f_N(\sqrt{2}+a_N)}^{f_N(\sqrt{2}+\delta)} \Big\{ |2^{3/2}|^{1/4}(N^{2/3}2^{1/2})^{1/4}\,\mathrm{Ai}(z) \tag{7.20}$$
$$- |2^{3/2}|^{-1/4}(N^{2/3}2^{1/2})^{-1/4}\,\mathrm{Ai}'(z)\Big\} 2^{-1/2}N^{-2/3}\,\mathrm{d}z.$$

The limits of the integral converge to 0 and ∞ respectively. Using the well-known identity $\int_0^\infty \mathrm{Ai}(z)\,\mathrm{d}z = \frac{1}{3}$, the claim (7.18) follows. This completes the proof of the lemma. $\qquad\square$

We can now compute $\mathbb{E}\,\mathrm{Crt}_N(u)$ for $u \le -E_c$. By (7.2), (7.5), (7.9),

$$\mathbb{E}\,\mathrm{Crt}_N(u) = C_{N,p}N^{-1/2}\sum_{i=1}^4 \int_v^\infty e^{-NF_p(x)}T_i(x)\,\mathrm{d}x =: C_{N,p}N^{-1/2}\sum_{i=1}^4 I_i(v), \tag{7.21}$$

where

$$T_1(x) = \alpha_N(x), \tag{7.22}$$
$$T_2(x) = N\phi_N^2(\sqrt{N}x), \tag{7.23}$$
$$T_3(x) = -\sqrt{N(N+1)}\,\phi_{N-1}(\sqrt{N}x)\phi_{N+1}(\sqrt{N}x), \tag{7.24}$$
$$T_4(x) = \left(\frac{N}{2}\right)^{1/2}\phi_{N-1}(\sqrt{N}x)J_N(x). \tag{7.25}$$

Proof of Theorem 2.17(a). We first estimate the four integrals $I_1(v), \ldots, I_4(v)$ for $v > \sqrt{2}$.

Lemma 7.3. *For N even $I_1(v) = 0$. When $N \to \infty$ over odd integers, then*

$$I_1(v) \sim \frac{h(v)}{4\pi^{1/2}N}\frac{e^{\Psi(v)-\frac{v}{2}\psi(v)}}{F_p'(v)+\Psi'(v)}\,e^{-N(F(v)+\Psi(v))}. \tag{7.26}$$

Proof. The first claim follows directly from the definition (7.8) of α_N. To prove the second claim we fix $\delta < v - \sqrt{2}$ and use Lemmas 7.1(b), 7.2(a). Then,

$$I_1(v) \sim \frac{(2N)^{1/4}}{2}\int_v^\infty \frac{h(x_+)e^{-NF_p(x)-(N-1)\Psi(x_+)}}{(4\pi)^{1/2}(2(N-1))^{1/4}}\,\mathrm{d}x, \tag{7.27}$$

where $x_+ = x\sqrt{N/(N-1)} \sim x(1 + 1/(2N))$. Expanding $\Psi(x_+) \sim \Psi(x) + x\Psi'(x)/(2N)$, we obtain the lemma by Laplace's method. $\qquad\square$

The integrals $I_2(v)$, $I_3(v)$ are negligible for $v > \sqrt{2}$:

Lemma 7.4. *There exists a $c < \infty$ such that $I_i(v) = O(N^c e^{-N(F_p(v)+2\Psi(v))})$ for $i = 2, 3$.*

Proof. Observe that T_2, T_3 contain a product of two Hermite functions. By Lemma 7.1(b), these behave like $O(N^c e^{-N\Psi(x)})$ for $x > \sqrt{2}$, which implies the claim. $\qquad\square$

Lemma 7.5. (a) *If $N \to \infty$ over odd integers, then $I_4(v) = O(N^c e^{-N(F_p(v)+2\Psi(v))})$ for some $c < \infty$.*
(b) *If $N \to \infty$ over even integers, then $I_4(v)$ behaves like the right-hand side of (7.26).*

Proof. Claim (a) follows from Lemma 7.2(b) and the same reasoning as in the previous proof. For claim (b), we find using Lemma 7.2(c) and Lemma 7.1(b) that I_4 is asymptotically equivalent to the right-hand side of (7.27) and the claim follows. $\qquad\square$

Theorem 2.17(a) follows directly from the previous three lemmas and definitions (7.1), (7.10). Observe that the dominant contribution comes from I_1 for N odd and from I_4 for N even. $\qquad\square$

Proof of Theorem 2.17(b). We need now compute $I_1(v), \ldots, I_4(v)$ for $v = \sqrt{2}$. We split all these integrals into two parts: over $(\sqrt{2}, \sqrt{2} + \delta)$ and over $(\sqrt{2} + \delta, \infty)$. For the second interval we can use Lemmas 7.3–7.5. We need thus analyse the integrals over the first interval, $I_i^\delta = \int_{\sqrt{2}}^{\sqrt{2}+\delta} T_i(x)\, dx$, $i = 1, \ldots, 4$.

Lemma 7.6. *For N even $I_1(\sqrt{2}) = 0$. When $N \to \infty$ over odd integers, then*

$$I_1(\sqrt{2}) \overset{\delta}{\sim} I_1^\delta \overset{\delta}{\sim} \frac{\mathrm{Ai}(0)}{2^{1/2} N^{5/6} F_p'(\sqrt{2})} e^{-NF_p(\sqrt{2})}. \tag{7.28}$$

Proof. Due to Lemma 7.3 it suffices to prove the second '$\overset{\delta}{\sim}$' relation only. By Lemma 7.1(a) and Lemma 7.2(a), setting $x_+ = x\sqrt{(N-1)/N}$,

$$I_1^\delta \overset{\delta}{\sim} \frac{1}{2} \int_{\sqrt{2}}^{\sqrt{2}+\delta} e^{-NF_p(x)} \left\{ \left| \frac{x_+ + \sqrt{2}}{x_+ - \sqrt{2}} \right|^{1/4} |f_N(x_+)|^{1/4} \mathrm{Ai}(f_N(x_+)) \right.$$
$$\left. + \left| \frac{x_+ - \sqrt{2}}{x_+ + \sqrt{2}} \right|^{1/4} |f_N(x_+)|^{-1/4} \mathrm{Ai}'(f_N(x_+)) \right\} dx. \tag{7.29}$$

If it were not for the $f_N(x_+)$ in the argument of the Airy function, this integral could be analysed trivially by Laplace's method, the main contribution coming from neighbourhoods of size $O(N^{-1})$ of $\sqrt{2}$. However, due to (7.19), $f_N(x_+)$ converges to 0 uniformly over such neighbourhoods. Hence, by a simple extension of Laplace's method, using (7.19) several times,

$$I_1^\delta \overset{\delta}{\sim} \frac{\mathrm{Ai}(0)}{2^{1/2} N^{5/6} F_p'(\sqrt{2})} e^{-NF_p(\sqrt{2})}. \tag{7.30}$$

This completes the proof. $\qquad\square$

Lemma 7.7. *As $N \to \infty$,*

$$I_2(\sqrt{2}) = O(N^{-1/6} e^{-NF_p(\sqrt{2})}) \qquad and \qquad I_3(\sqrt{2}) = -I_2(\sqrt{2})(1 + O(N^{-1})). \tag{7.31}$$

Proof. As before, $I_2(\sqrt{2}) \overset{\delta}{\sim} I_2^\delta$. To estimate I_2^δ we use Lemma 7.1(a) again. The same computation as in the proof of the previous lemma gives

$$I_2^\delta \overset{\delta}{\sim} \frac{\mathrm{Ai}^2(0)\sqrt{2}}{N^{1/6} F_p'(\sqrt{2})} e^{-NF_p(\sqrt{2})}. \tag{7.32}$$

To prove the second claim, it suffices to observe that on $(\sqrt{2}, \sqrt{2} + \delta)$, by Lemma 7.1(a),

$$\frac{\phi_N^2(x\sqrt{N})}{\phi_{N-1}(x\sqrt{N})\phi_{N+1}(x\sqrt{N})} = 1 + O(N^{-1}). \tag{7.33}$$

$\qquad\square$

Lemma 7.8. (a) *When $N \to \infty$ over even integers, then*

$$I_4(\sqrt{2}) \overset{\delta}{\sim} I_4^\delta \overset{\delta}{\sim} \frac{2}{3} G_N(p) \tag{7.34}$$

where $G_N(p)$ denotes the right-hand side of (7.28).
  (b) *When $N \to \infty$ over odd integers, then*

$$I_4(\sqrt{2}) \overset{\delta}{\sim} I_4^\delta \overset{\delta}{\sim} -\frac{1}{3} G_N(p). \tag{7.35}$$

Proof. The lemma follows by the same computations in Lemma 7.6. Again, the dominant contribution comes from neighbourhoods of size $O(N^{-1})$ of $\sqrt{2}$. On such neighbourhoods, using Lemma 7.2(d), the integral $J_N(x)$ appearing in $T_4(x)$ can be approximated by $cN^{-1/4}$, where c depends on N being even or odd. $\qquad\square$

Theorem 2.17(b) follows now directly from the previous three lemmas and definitions (7.1), (7.10). Observe that the dominant contribution comes from I_1 and I_4, since I_2 and I_3 mutually cancel. Moreover, the combined contributions of I_1 and I_4 do not depend on N being odd or even. $\square$

Proof of Corollary 2.18. By Theorem 2.5, for all $\varepsilon > 0$,

$$\mathbb{E}\,\mathrm{Crt}_N(u) \geq \mathbb{E}\,\mathrm{Crt}_{N,0}(u) \geq \mathbb{E}\,\mathrm{Crt}_N(u) - N\mathbb{E}\,\mathrm{Crt}_{N,1}(u)$$
$$\geq \mathbb{E}\,\mathrm{Crt}_N(u) - Ne^{N(\theta_{1,p}(u)+\varepsilon)}. \tag{7.36}$$

The corollary then follows from the fact that for $u < -E_c$, $\theta_{0,p}(u) > \theta_{1,p}(u)$. $\square$

APPENDIX A. LARGE DEVIATIONS FOR THE LARGEST EIGENVALUES OF THE GOE

In this appendix we extend Theorem 6.2 of [BDG01], proving a LDP for the k-th largest eigenvalue of the Gaussian Orthogonal Ensemble. This results might be of independent interest. Its proof follows the lines of [BDG01].

Let $X = X_N$ be a $N \times N$ real symmetric random matrix whose entries X_{ij} are independent (up to the symmetry) centered Gaussian random variables with the variance

$$\mathbb{E}X_{ij}^2 = \sigma^2 N^{-1}(1 + \delta_{ij}), \tag{A.1}$$

and let $\lambda_1 \leq \cdots \leq \lambda_N$ be the ordered eigenvalues of X. Note, that in this appendix we use the notation that is more usual in the random matrix theory, that is the eigenvalues are numbered from 1 to N, not from 0 to $N-1$ as in the rest of the paper. We show the following LDP.

Theorem A.1. *For each fixed $k \geq 1$, the k-th largest eigenvalue λ_{N-k+1} of X satisfies a LDP with speed N and a good rate function*

$$I_k(x; \sigma) = kI_1(x; \sigma) = \begin{cases} k \int_{2\sigma}^x \sigma^{-1}\sqrt{(\frac{z}{2\sigma})^2 - 1}\,\mathrm{d}z, & \text{if } x \geq 2\sigma, \\ \infty, & \text{otherwise.} \end{cases} \tag{A.2}$$

Proof. We first recall some know fact about the distribution of λ_i's and introduce some notation. The joint law of λ_i, $1 \leq i \leq N$, is given by

$$Q_N(\mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_N) = Z_N(\sigma)^{-1} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j| \prod_{i=1}^N \exp\left(-\frac{N}{4\sigma^2}\lambda_i^2\right)\mathrm{d}\lambda_i \mathbf{1}_{\lambda_1 \leq \cdots \leq \lambda_N}. \tag{A.3}$$

The distribution of unordered eigenvalues $\bar{Q}_N$ is given by the same formula without the final indicator function and with $Z_N(\sigma)$ replaced by $\bar{Z}_N(\sigma) = N!Z_N(\sigma)$. By Wigner's theorem, the spectral measure $L_N = N^{-1}\sum_{i=1}^N \delta_{\lambda_i}$ of X_N converges weakly in probability to the semi-circle distribution

$$\rho(\mathrm{d}x) = (2\pi\sigma^2)^{-1}\sqrt{(2\sigma)^2 - x^2}\,\mathbf{1}_{|x| \leq 2\sigma}\,\mathrm{d}x. \tag{A.4}$$

For $A \subset \mathbb{R}$ we denote by $\mathcal{P}(A)$ the space of all Borel probability measures on A endowed with the weak topology and a compatible metric d. By Theorem 1.1 of [BG97], the spectral measure L_N satisfies a LDP on $\mathcal{P}(\mathbb{R})$ with the speed N^2 and a good rate function whose unique minimiser is the semi-circle distribution (A.4).

We now start proving Theorem A.1. $I_k(x; \sigma)$ is obviously good rate function. Since $I_k(x, \sigma)$ is scale invariant, that is

$$I_k(x; \sigma) = I_k(x/\sigma; 1), \tag{A.5}$$

we can assume that $\sigma = 1$ as in [BDG01], and omit σ from the notation. Note that this value differs from the value $\sigma = 2^{-1/2}$ used in the rest of the present paper (cf.(2.6)). In

particular, for $\sigma = 1$ (cf. (3.17))

$$\bar{Z}_N = \bar{Z}_N(1) = (2\sqrt{2})^N \left(\frac{N}{2}\right)^{-N(N+1)/4} \prod_{i=1}^{N} \Gamma\left(1 + \frac{i}{2}\right). \tag{A.6}$$

To show Theorem A.1 it is sufficient to prove

$$\limsup_{N \to \infty} \frac{1}{N} \log Q_N\big(\lambda_{N-k+1} \leq x\big) = -\infty \qquad \text{for all } x < 2, \tag{A.7}$$

and, since $I_k(x; 1)$ is continuous and strictly increasing on $[2, \infty)$,

$$\limsup_{N \to \infty} \frac{1}{N} \log Q_N\big(\lambda_{N-k+1} \geq x\big) = -I_k(x; 1) \qquad \text{for all } x \geq 2. \tag{A.8}$$

We first prove (A.7). Suppose that $\lambda_{N-k+1} \leq x$ for some $x < 2$. Then $L_N((x, 2]) \leq (k-1)N^{-1}$. Since $\rho((x, 2]) > 0$, for N large enough there exists a closed set $A \subset \mathcal{P}(\mathbb{R})$ such that $\rho \notin A$ and $\{\lambda_{N-k+1} \leq x\} \subset A$. By the LDP for the spectral measure L_N, $Q_N(A) \leq e^{-cN^2}$ for some $c > 0$, which concludes the proof of (A.7).

We now prove the upper bound for (A.8). By Lemma 6.3 of [BDG01],

$$\bar{Q}_N\Big(\max_{i=1}^{N} |\lambda_i| \geq M \Big) \leq e^{-NM^2/9} \qquad \text{for all } M \text{ large enough and all } N. \tag{A.9}$$

Writing

$$Q_N(\lambda_{N-k+1} \geq x) \leq Q_N\big(\max_{i=1}^{N} |\lambda_i| \geq M \big) + Q_N\big(\lambda_{N-k+1} \geq x, \max_{i=1}^{N} |\lambda_i| < M\big), \tag{A.10}$$

the upper bound follows easily provided we show that for all $M > x > 2$

$$\limsup_{N \to \infty} \frac{1}{N} \log Q_N\big(\max_{i=1}^{N} |\lambda_i| \leq M, \lambda_{N-k+1} \geq x \big) \leq -I_k(x; 1). \tag{A.11}$$

To show (A.11) we introduce some additional notation. Let $\bar{Q}_{N-k}^N$ be a measure on $\mathbb{R}^{N-k}$ given by

$$\bar{Q}_{N-k}^N\big(\lambda \in \cdot\big) = \bar{Q}_{N-k}\big((1 - kN^{-1})^{1/2}\lambda \in \cdot\big). \tag{A.12}$$

We set

$$C_N^k = \left(1 - \frac{k}{N}\right)^{(N-k)(N-k+1)/4} \frac{\bar{Z}_{N-k}}{\bar{Z}_N}, \tag{A.13}$$

and for $x \in \mathbb{R}$ and $\mu \in \mathcal{P}(\mathbb{R})$ we define

$$\Phi(z, \mu) = \int_{\mathbb{R}} \log|z - y|\mu(\mathrm{d}y) - \frac{z^2}{4}. \tag{A.14}$$

It was shown in [BDG01, p. 50] that $\Phi(z, \mu)$ is upper semi-continuous on $[-M, M] \times \mathcal{P}([-M, M])$ and continuous on $[x, y] \times \mathcal{P}([-M, M])$ for all $M, x, y \in \mathbb{R}$ such that $y > x > M > 2$.

Using (A.3) and this notation, we can write

$$Q_N\big(\max_{i=1}^{N} |\lambda_i| \leq M, \lambda_{N-k+1} \geq x \big)$$

$$= Z_N^{-1} \int_{[x,M]^k} \prod_{i=N-k+1}^{N} e^{-N\lambda_i^2/4}\mathrm{d}\lambda_i \int_{[-M,M]^{N-k}} \prod_{i=1}^{N-k} e^{-N\lambda_i^2/4}\mathrm{d}\lambda_i$$

$$\times \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j| \mathbf{1}_{\lambda_1 \leq \cdots \leq \lambda_N} \tag{A.15}$$

$$\leq C_N^k \frac{N!}{(N-k)!} \int_{[x,M]^k} \prod_{N-k < i < j \leq N} |\lambda_i - \lambda_j|\mathrm{d}\lambda_{N-k+1}\ldots\mathrm{d}\lambda_N$$

$$\times \int_{[-M,M]^{N-k}} e^{(N-k)\sum_{i=N-k+1}^{N} \Phi(\lambda_i, L_{N-k})}\bar{Q}_{N-k}^N(\mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_{N-k}),$$

where the factor $N!/(N-k)!$ comes from replacing Q_N by $\bar{Q}_N$. Let $B(\rho, \delta)$ denote the open ball in $\mathcal{P}(\mathbb{R})$ of radius $\delta > 0$ and center ρ. We write $B_M(\rho, \delta) = B(\rho, \delta) \cap \mathcal{P}([-M, M])$. On the domain of the integration $|\lambda_i - \lambda_j| \leq 2M$ and $e^{(N-k)\Phi(\lambda_i, L_{N-k})} \leq (2M)^{N-k}$. Therefore by splitting the second integral, (A.15) is bounded from above by

$$
C_N^k \frac{N!}{(N-k)!} (2M)^{k(k-1)/2} \Big\{ \Big(\int_x^M e^{(N-k)\sup_{\mu \in B_M(\rho, \delta)} \Phi(z, \mu)} \, \mathrm{d}z \Big)^k
$$
$$
+ (2M)^{N-k} \bar{Q}_{N-k}^N (L_{N-k} \notin B(\rho, \delta)) \Big\}. \tag{A.16}
$$

To control the measure $\bar{Q}_{N-k}^N$, observe that for all functions $h : \mathbb{R} \to \mathbb{R}$ of Lipschitz norm at most 1 and $N \geq 2k$,

$$
\Big| (N-k)^{-1} \sum_{i=1}^{N-k} \big\{ h\big((1 - kN^{-1})^{1/2} \lambda_i \big) - h(\lambda_i) \big\} \Big| \leq cN^{-1} \max_{i=k}^{N-k} |\lambda_i| \tag{A.17}
$$

for some $c \in (0, \infty)$ independent of N and k. It follows from (A.9) that the laws of L_{N-k} under $\bar{Q}_{N-k}$ and $\bar{Q}_{N-k}^N$ are exponentially equivalent as $N \to \infty$. Therefore, by Theorem 4.2.13 of [DZ98], L_{N-k} under $\bar{Q}_{N-k}^N$ satisfies the same LDP as L_{N-k} under $\bar{Q}_{N-k}$. Hence, the second term in (A.16) is exponentially negligible for any $\delta > 0$ and $M < \infty$. This implies that

$$
\limsup_{N \to \infty} \frac{1}{N} \log Q_N \big(\max_{i=1}^N |\lambda_i| \leq M, \lambda_{N-k+1} \geq x \big)
$$
$$
\leq \limsup_{N \to \infty} \frac{1}{N} \log C_N^k + k \lim_{\delta \downarrow 0} \sup_{z \in [x, M], \mu \in B_M(\rho, \delta)} \Phi(z, \mu). \tag{A.18}
$$

The same reasoning as on p. 50 of [BDG01] implies that the second term in (A.18) equals $-k(1/2 + I_1(x; 1))$. From definition (A.13) of C_N^k and from (A.6), it is easy to obtain $\lim_{N \to \infty} N^{-1} \log C_N^k = k/2$. Combining these two claims, we can bound the left-hand side of (A.11) by $-k(1/2 + I_1(x; 1)) + k/2 = kI_1(x; 1)$. This completes the proof of (A.11) and thus of the upper bound for (A.8).

To prove the complementary lower bound we fix $y > x > r > 2$ and $\delta > 0$. By a similar computation as in (A.15) we obtain

$$
Q_N \big(\lambda_{N-k+1} \geq x \big) \geq \bar{Q}_N \big(\lambda_N \in [x, y], \dots, \lambda_{N-k+1} \in [x, y], \max_{i=1}^{N-k} |\lambda_i| \leq r \big)
$$

$$
= C_N^k \int_{[x,y]^k} \prod_{i=N-k+1}^N e^{-k\lambda_i^2/4} \mathrm{d}\lambda_i \prod_{N-k < i < j \leq N} |\lambda_i - \lambda_j|
$$
$$
\times \int_{[-r,r]^{N-k}} e^{(N-k)\sum_{i=N-k+1}^N \Phi(\lambda_i, L_{N-k})} \bar{Q}_{N-k}^N (\mathrm{d}\lambda_1, \dots, \mathrm{d}\lambda_{N-k}) \tag{A.19}
$$
$$
\geq K C_N^k \exp \Big(k(N-k) \inf_{z \in [x,y], \mu \in B_r(\rho, \delta)} \Phi(z, \mu) \Big) \bar{Q}_{N-k}^N \big(L_{N-k} \in B_r(\rho, \delta) \big),
$$

for some $K = K(k, x, y) > 0$.

Using the LDP for the measure L_{N-k} under $\bar{Q}_{N-k}^N$ we see that

$$
\bar{Q}_{N-k}^N \big(L_{N-k} \notin B_r(\sigma, \delta) \big) \xrightarrow{N \to \infty} 0. \tag{A.20}
$$

By symmetry of $\bar{Q}_N(\cdot)$ and by the upper bound in (A.8),

$$
\bar{Q}_{N-k}^N \big(L_{N-k} \notin \mathcal{P}((-r, r)) \big) \leq 2\bar{Q}_{N-k}^N (\lambda_{N-k} \geq r) \xrightarrow{N \to \infty} 0. \tag{A.21}
$$

Therefore, using the behaviour of C_N^k again,

$$
\liminf_{N \to \infty} \frac{1}{N} \log Q_N \big(\lambda_{N-k+1} \geq x \big) \geq \frac{k}{2} + k \inf_{z \in [x,y], \mu \in B_r(\rho, \delta)} \Phi(z, \mu). \tag{A.22}
$$

Letting now $\delta \to 0$ and then $y \searrow x$, using the continuity of $\Phi(z, \mu)$ in the used range of the parameters, we obtain the desired lower bound. $\square$

References

[AT07]   Robert J. Adler and Jonathan E. Taylor, *Random fields and geometry*, Springer Monographs in Mathematics, Springer, New York, 2007. MR2319516

[AW09]   Jean-Marc Azaïs and Mario Wschebor, *Level sets and extrema of random processes and fields*, John Wiley & Sons Inc., Hoboken, NJ, 2009. MR2478201

[BDG01]   G. Ben Arous, A. Dembo, and A. Guionnet, *Aging of spherical spin glasses*, Probab. Theory Related Fields **120** (2001), no. 1, 1–67. MR1856194

[BG97]   G. Ben Arous and A. Guionnet, *Large deviations for Wigner's law and Voiculescu's non-commutative entropy*, Probab. Theory Related Fields **108** (1997), no. 4, 517–542. MR1465640

[CLR03]   A. Crisanti, L. Leuzzi, and T. Rizzo, *The complexity of the spherical p-spin spin glass model, revisited*, The European Physical Journal B - Condensed Matter and Complex Systems **36** (2003), no. 1, 129–136.

[CS95]   A. Crisanti and H-J. Sommers, *Thouless-Anderson-Palmer approach to the spherical p-spin spin glass model*, J. Phys. I France **5** (1995), no. 7, 805.

[DKM$^+$99] P. Deift, T. Kriecherbauer, K. T-R McLaughlin, S. Venakides, and X. Zhou, *Strong asymptotics of orthogonal polynomials with respect to exponential weights*, Comm. Pure Appl. Math. **52** (1999), no. 12, 1491–1552. MR1711036

[DG07]   Percy Deift and Dimitri Gioev, *Universality in random matrix theory for orthogonal and symplectic ensembles*, Int. Math. Res. Pap. IMRP **2007** (2007), rpm004, 116. MR2335245

[DZ98]   Amir Dembo and Ofer Zeitouni, *Large deviations techniques and applications*, second ed., Applications of Mathematics, vol. 38, Springer-Verlag, New York, 1998. MR1619036

[FW07]   Yan V. Fyodorov and Ian Williams, *Replica symmetry breaking condition exposed by random matrix calculation of landscape complexity*, J. Stat. Phys. **129** (2007), no. 5-6, 1081–1116. MR2363390

[KL96]   J. Kurchan and L. Laloux, *Phase space geometry and slow dynamics*, J. Phys. A: Math. Gen. **29** (1996), 1929–1948.

[KPV93]   J. Kurchan, G. Parisi, and M. A. Virasoro, *Barriers and metastable states as saddle points in the replica approach*, J.Phys. I France **3** (1993), no. 8, 1819–1839.

[Meh91]   M. L. Mehta, *Random matrices*, second ed., Academic Press Inc., Boston, MA, 1991. MR1083764

[PT07]   Dmitry Panchenko and Michel Talagrand, *On the overlap in the multiple spherical SK models*, Ann. Probab. **35** (2007), no. 6, 2321–2355. MR2353390

[PR29]   M. Plancherel and W. Rotach, *Sur les valeurs asymptotiques des polynomes d'Hermite*, Comment. Math. Helv. **1** (1929), no. 1, 227–254. MR1509395

[Sze81]   Gábor Szegő, *Orthogonal polynomials*, fourth ed., American Mathematical Society, Providence, R.I., 1981, American Mathematical Society, Colloquium Publications, Vol. XXIII.

[Tal06]   Michel Talagrand, *Free energy of the spherical mean field model*, Probab. Theory Related Fields **134** (2006), no. 3, 339–382. MR2226885

A. Auffinger, Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012, USA
    E-mail address: `auffing@cims.nyu.edu`

G. Ben Arous, Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012, USA
    E-mail address: `benarous@cims.nyu.edu`

J. Černý, Department of Mathematics, ETH Zurich, Raemistr. 101, 8092 Zurich, Switzerland
    E-mail address: `jiri.cerny@math.ethz.ch`