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THE EXISTENCE OF TRANSLATION NON-INVARIANT MEASURE IN RANDOM-CLUSTER MODEL

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Contents

Introduction

The proof of existence of translation non-invariant measures in dimension $d \geq 3$ has been presented in the case of the Ising model by Dobrushin [3]. His approach was extended to a large class of models in [5]. Even though Potts model, strictly speaking, does not belong to this class, it can be rewritten so that the approach from [5] can be applied [11]. On the other side, the existence of translation non-invariant Gibbs state for random-cluster model has been doubted [4]. Given the equivalence with Potts model (at least for q integer) the existence of such a state for random-cluster model should follow, as was already pointed in [1], applying the methods from [5].

Our aim here is to present the details of such a proof directly for the random-cluster model. Even for Potts model, the proof was only sketched in [11]. There are some additional difficulties when formulating the claim directly in terms of random-cluster model. In particular, one has to prove that resulting state is actually DLR-state.

The outline of this diploma thesis is following. We define the Potts model in Section 1.1. Using the FK representation we convert the Potts model partition function into the partition function of random-cluster model defined in Section 1.2. In Section 1.3 we introduce the contours using the definition from [2], [6] or [10], which slightly differs from that in [9] and is more suitable for our purposes.

In Sections 1.4 and 1.5 we rewrite the partition function of random-cluster model into partition function of two contour models, one for ordered and one for disordered boundary condition. Following closely a procedure from [10] we get these models in the form suitable for a use of the Pirogov-Sinai theory, as it appears in [12], to describe translation invariant (i.e. both ordered and disordered) phases in Sections 1.6 and 1.8. We restrict ourselves only to the random-cluster model transition point where the ordered and disordered phases coexist.

In Section 1.7 we prove the uniqueness of random-cluster measure in some types of volumes using techniques from [4]. This will be used in the proof of existence of interface.

Chapters 2 and 3 contain the most important claims of this diploma thesis. In the first one we closely follow the methods of the article [5]. First, we rewrite the partition function for a special translation non-invariant boundary condition into the contour model partition function using similar techniques as in Chapter 1. Then we normalize it to have a possibility of rewriting it once more in terms of this time $(d-1)$-dimensional, contour model partition function. This model we will explore using the cluster expansion theory. As a result, we prove that there is, almost surely, an interface between ordered and disordered phases also in an infinite volume measure. Thus, the corresponding limiting measure is not translation invariant.

In Chapter 3 we prove that the constructed measure fulfills the DLR equation by using theorems from [4] and adapting their proofs. For reader convenience we recall some statements from the theory of contour models and cluster expansion in Appendix A.

Chapter 1

Translation invariant measures

1.1 Potts model

The Potts model is a classical lattice model. These models are characterized by random spins σ_i associated with sites i on hyper-cubic lattice $\mathbb{Z}^d$ (we consider $d > 2$). In Potts model we attach to each site a value from the finite set $S = \{1, \dots, q\}$, $q \in \mathbb{N}$. For every $\Lambda \subset \mathbb{Z}^d$ we denote $\Omega^P_\Lambda = S^\Lambda$ and, in particular, $\Omega^P = S^{\mathbb{Z}^d}$. Under the term configuration of Potts model we will understand one element σ of the set Ω^P. We use σ_Λ to denote the restriction of σ to Λ. Further, we introduce the sets of bonds $B_\Lambda = \{\langle i,j\rangle \mid i,j \in \Lambda\}$, $\mathcal{B}_\Lambda = \{\langle i,j\rangle \mid i \in \Lambda, j \notin \Lambda\}$ and $\mathbb{B}_\Lambda = B_\Lambda \cup \mathcal{B}_\Lambda$, where $\langle i,j\rangle$ denotes pairs of the nearest neighbours. For $\Lambda = \mathbb{Z}^d$ we will suppress the index. Using these definitions we introduce the Hamiltonian of the Potts model on a finite subset $\Lambda \subset \mathbb{Z}^d$ under a fixed boundary condition $\bar{\sigma}_{\Lambda^C}$ by

$$H_\Lambda(\sigma \mid \bar{\sigma}) = -\sum_{\langle i,j\rangle \in B_\Lambda} \delta_{\sigma_i \sigma_j} - \lambda \sum_{\langle i,j\rangle \in \mathcal{B}_\Lambda} \delta_{\sigma_i \bar{\sigma}_j}. \tag{1.1}$$

We will take λ to equal to 1 or 0. The latter case will be called free boundary condition.

If $\Lambda \in \mathbb{Z}^d$ is finite and nonempty, we introduce the partition function at inverse temperature β by

$$Z_\Lambda(\beta \mid \bar{\sigma}) = \sum_{\sigma_\Lambda} e^{-\beta H_\Lambda(\sigma \mid \bar{\sigma})}. \tag{1.2}$$

Using this we define the probability kernel $\mu^\beta_\Lambda(\cdot \mid \bar{\sigma})$ on Ω^P_Λ, called Gibbs state in Λ under the boundary condition $\bar{\sigma}$, by the formula

$$\int f(\sigma)\mu^\beta_\Lambda(d\sigma \mid \bar{\sigma}) = \sum_{\sigma \in X_\Lambda} f(\sigma_\Lambda \times \bar{\sigma}_{\Lambda^C}) \frac{\exp(-\beta H_\Lambda(\sigma \mid \bar{\sigma}))}{Z_\Lambda(\beta \mid \bar{\sigma})} \tag{1.3}$$

for every bounded measurable function f. We will usually skip the superscript β and write $\mu_\Lambda(\sigma \mid \bar{\sigma})$.

Let now $V \subset \mathbb{Z}^d$ be possibly infinite. We say that a probability measure μ on Ω^P (equipped with the σ-algebra generated by cylinder sets) is a Gibbs state of Potts model in volume V and at an inverse temperature β if

$$\mu(f) = \int \left[\int f(\sigma) \mu_{\Lambda}^{\beta}(d\sigma \mid \bar{\sigma}) \right] \mu(d\bar{\sigma}) \tag{1.4}$$

whenever $\Lambda \subset V$ is finite and f is a measurable bounded function.

It is well known that there are q different extremal, translation invariant Gibbs states in $\mathbb{Z}^d$ for Potts model at low temperatures and a unique state at high temperatures. The low-temperature states are usually called ordered because there exists one "colour" from S that is characteristic for each of them. The high-temperature state is then called disordered [8]. For q large enough there exists one temperature $T_c = 1/\beta_c$ (which we will call critical) where are $q+1$ equilibrium Gibbs states, q ordered and one disordered. In this diploma thesis we will try to find a translation non-invariant state and describe its properties. In particular, we will look for a state with an interface between an ordered and disordered state. In order be able to find such a state we will need to work at the critical temperature β_c for having $q+1$ coexisting translation invariant phases stable. In this chapter we will talk only about translation invariant states, therefore we can assume an arbitrary value of temperature. We will use the assumption that the temperature is critical later.

We can rewrite partition function (1.2) with the help of the FK (Fortuin-Kasteleyn) representation. Namely, using the equality

$$e^{c\delta_{\sigma_i\sigma_j}} = 1 + \delta_{\sigma_i\sigma_j}(e^c - 1) \tag{1.5}$$

we find

$$Z_{\Lambda}(\beta \mid \bar{\sigma}) = \sum_{\sigma_\Lambda} \prod_{\langle i,j\rangle \in B_\Lambda} [1 + \delta_{\sigma_i\sigma_j}(e^{\beta} - 1)] \prod_{\langle i,j\rangle \in \mathcal{B}_\Lambda} [1 + \delta_{\sigma_i\bar{\sigma}_j}(e^{\lambda\beta} - 1)] = \tag{1.6}$$

$$= \sum_{X \subset B_\Lambda} \sum_{\mathcal{X} \subset \mathcal{B}_\Lambda} \left(e^{\beta} - 1\right)^{|X|} \left(e^{\beta\lambda} - 1\right)^{|\mathcal{X}|} \sum_{\sigma_\Lambda} \prod_{\langle i,j\rangle \in X} \delta_{\sigma_i\sigma_j} \prod_{\langle i,j\rangle \in \mathcal{X}} \delta_{\sigma_i\bar{\sigma}_j}. \tag{1.7}$$

Here the sum runs over all subsets X ($\mathcal{X}$) of B_Λ ($\mathcal{B}_\Lambda$). Alternatively we can look at this sum as the sum over all subgraphs of the graph $(V(\mathbb{B}_\Lambda), \mathbb{B}_\Lambda)$ (where for any subset Y of $\mathbb{B}$ we introduce the set $V(Y)$ of all its vertices (sites)). Bonds in $X \cup \mathcal{X}$ (edges of the subgraph) will be called ordered and other bonds disordered. Similarly as for $\mathbb{B}$, we use $\mathbb{X}$ to denote $X \cup \mathcal{X}$.

With the help of the equality

$$\sum_{\sigma_\Lambda} \prod_{\langle i,j\rangle \in X} \delta_{\sigma_i\sigma_j} \prod_{\langle i,j\rangle \in \mathcal{X}} \delta_{\sigma_i\bar{\sigma}_j} = q^{\bar{D}_\Lambda(\mathbb{X})} \tag{1.8}$$

we can sum over all configurations σ_Λ. We denoted there $\bar{D}_\Lambda(\mathbb{X}) = D_\Lambda(\mathbb{X}) + E_\Lambda(\mathbb{X})$, with $D_\Lambda(\mathbb{X})$ being the number of connected components of $\mathbb{X}$ (counting all components connected to the boundary as one[1]) and $E_\Lambda(\mathbb{X})$ being the number of isolated sites in Λ, i.e. $E_\Lambda(\mathbb{X}) = |\{i \in \Lambda \mid i \notin V(\mathbb{X})\}|$.

Using equations (1.7) and (1.8) we get

$$Z_\Lambda(\beta \mid \bar{\sigma}) = \sum_{\mathbb{X}} (e^\beta - 1)^{|X|}(e^{\beta\lambda} - 1)^{|\mathcal{X}|} q^{\bar{D}_\Lambda(\mathbb{X})}. \tag{1.9}$$

1.2 Random-cluster model

In contrast to the Potts model, the random-cluster model is a process on the edges of a graph (lattice). To each edge of this graph we attach a value from the set $\{0, 1\}$. The bonds, to which we attach the value 1, are called ordered, the others disordered. We define a random cluster measure on a finite graph $G = (V_G, B_G)$ as follows.

Let $0 \leq p \leq 1$ and $q > 0$. In analogy with the previous section, the set X of ordered edges of G we will be called a configuration on G. Let us observe that the set of all subsets of B can be identified with the compact metric space $\Omega_G = \{0, 1\}^{B_G}$. Due to this identification it has no importance if we speak about X as the element of 2^B or Ω_B. Denoting by $\bar{D}_G(X)$ the number of components of graph (V_G, X), the probability measure μ_G on Ω_G will be called the random-cluster measure on G with parameters p and q if

$$\mu_G(X) = \frac{1}{Z_G} p^{|X|}(1-p)^{|B_G \setminus X|} q^{\bar{D}_G(X)} \tag{1.10}$$

with

$$Z_G = \sum_X p^{|X|}(1-p)^{|B_G \setminus X|} q^{\bar{D}_G(X)}. \tag{1.11}$$

The dependence on both p and q will not be denoted explicitly.

In addition, we will assume that G is a finite, connected subgraph of $(\mathbb{Z}^d, B)$. To be able to discuss the translation non-invariant measures, we need to define a random-cluster measure on a $G \subset (\mathbb{Z}^d, B)$ with a boundary condition. Let Y be a subset of B. Then, the probability of a configuration X on G with boundary condition Y outside G is defined by

$$\mu_G^Y(X) = \frac{1}{Z_G^Y} p^{|X|}(1-p)^{|B_G \setminus X|} q^{\bar{D}_G^Y(X)}. \tag{1.12}$$

Here $\bar{D}_G^Y(X)$ is the number of components of the graph $(\mathbb{Z}^d, X_{B_G} \circ Y_{B \setminus B_G})$ that intersect graph G and

$$Z_G^Y = \sum_X p^{|X|}(1-p)^{|B_G \setminus X|} q^{\bar{D}_G^Y(X)}. \tag{1.13}$$

[1]This is not true for all boundary conditions, but it holds true for the boundary conditions we will discuss.

For any (possibly infinite) graph $G \subset (\mathbb{Z}^d, B)$, we will write $\mathscr{F}_{B_G}$ for a σ-field of subsets of $\Omega = \Omega_{(\mathbb{Z}^d,B)}$ generated by finite subsets of B_G, so that $\mathscr{F} = \mathscr{F}_B$. For any finite graph G we will write $\mathscr{T}_{B_G} = \mathscr{F}_{B \setminus B_G}$ for the "external" σ-field of G, and

$$\mathscr{T} = \bigcap_G \mathscr{T}_{B_G} \tag{1.14}$$

for the tail σ-field.

There are, as usually, two natural candidates for a definition of random-cluster measure on an infinite graph. The first one is in terms of a "specification" and the second one is as a weak limit of measures defined on finite graphs.

Definition 1.1 *A probability measure μ on $(\Omega, \mathscr{F})$ is called a random-cluster measure on graph G with parameters p, q if*

$$\mu(A \mid \mathscr{T}_{B_F}) = \mu_F^{\cdot}(A), \quad \mu\text{-a.s., for all } A \in \mathscr{F}_{B_G} \text{ and any finite } F \subset G. \tag{1.15}$$

Moreover, μ is a random-cluster measure on G with a boundary condition Y if

$$\mu(X \in \Omega \mid X_{B \setminus B_G} = Y_{B \setminus B_G}) = 1. \tag{1.16}$$

Definition 1.2 *A probability measure μ on $(\Omega, \mathscr{F})$ is called a limit random-cluster measure on graph G with parameters p, q and with boundary condition Y,if there exists an increasing sequence $\{G_n\}_{n \geq 1}$ of finite graphs, such that $G_n \nearrow G$ as $n \to \infty$ and*

$$\mu_{G_n}^Y(\cdot) \overset{\text{weakly}}{\longrightarrow} \mu(\cdot). \tag{1.17}$$

We can see that the expression of partition function (1.13) is very similar to (1.9). Namely, setting

$$p = \frac{e^\beta - 1}{e^\beta} \tag{1.18}$$

and we multiplying[2] (1.13) by $e^{|B_G|}$ and taking in G the bonds from B_Λ, if $\lambda = 0$, and from $\mathbb{B}_\Lambda$, if $\lambda = 1$, we get (1.9). From this we can see that Potts model is in a sense a special case of the random-cluster model with $q \in \mathbb{N}$.

Further more, we will work directly with the random-cluster model, i.e. with the partition function (1.13). We will restrict ourselves only to $q \geq 1$. Only graphs $G(\Lambda)$ that "belong" to a set $\Lambda \subset \mathbb{Z}^d$ will be considered instead using of arbitrary subgraphs of $(\mathbb{Z}^d, B)$. Namely, we take only $G(\Lambda)$ such that

$$(\Lambda, B_\Lambda) \subset G(\Lambda) \subset (V(\mathbb{B}_\Lambda), \mathbb{B}_\Lambda). \tag{1.19}$$

In particular, we can consider for $G(\Lambda)$ either the graph (Λ, B_Λ) or the graph $(V(\mathbb{B}_\Lambda), \mathbb{B}_\Lambda)$. We will specify other options later in terms of boundary conditions. We will use the index Λ instead of index $G(\Lambda)$ whenever it cannot be misunderstood.

[2] Of course we must also multiply the right-hand side of (1.12) to assure that the probability measure stays the same.

Given $G(\Lambda)$ as above, we will use the notation introduced in the section about the Potts model, i.e. X ($\mathcal{X}$) is a set of ordered bonds from B_Λ ($\mathcal{B}_\Lambda$), $\mathbb{X} = X \cup \mathcal{X}$, $E_\Lambda(\mathbb{X})$ is the number of vertices of $G(\Lambda)$ not connected to any ordered bond in $\mathbb{X}$, and $D^Y_\Lambda(\mathbb{X})$ is the number of components of $(\mathbb{Z}^d, \mathbb{X} \circ Y)$ intersecting $G(\Lambda)$ and having at least one ordered edge. We use $\delta_r X$ to denote the set of bonds b from $B_\Lambda \setminus X$ for which $|V(b) \cap V(\mathbb{X})| = r$ ($r = 1, 2$) and $\delta_s \mathcal{X}$ the set of bonds $\bar{b}$ from $\mathcal{B}_\Lambda \setminus \mathcal{X}$ for which $|V(\bar{b}) \cap V(\mathbb{X})| = s$ ($s = 0, 1$). Now we can write some obvious equalities:

$$|E_\Lambda(\mathbb{X})| = |V(B_\Lambda)| - |V(\mathbb{X}) \cap \Lambda|, \tag{1.20}$$

$$2d|V(B_\Lambda)| = 2|B_\Lambda| + |\mathcal{B}_\Lambda|, \tag{1.21}$$

$$2d|V(\mathbb{X}) \cap \Lambda| = 2|X| + |\mathcal{X}| + 2|\delta_2 X| + |\delta_1 X| + |\delta_1 \mathcal{X}|. \tag{1.22}$$

Hence, we have

$$|E_\Lambda(\mathbb{X})| = \frac{|B_\Lambda \setminus X|}{d} + \frac{|\mathcal{B}_\Lambda \setminus \mathcal{X}|}{2d} - \frac{\|\delta\mathbb{X}\|}{2d} \tag{1.23}$$

with $\|\delta\mathbb{X}\| = 2|\delta_2 X| + |\delta_1 X| + |\delta_1 \mathcal{X}|$. Using this, we can rewrite the partition function (1.13) as follows

$$Z^Y_\Lambda = \sum_{\mathbb{X}} p^{|\mathbb{X}|}(1-p)^{|B_{G(\Lambda)} \setminus \mathbb{X}|} q^{\frac{|B_\Lambda \setminus X|}{d}} q^{\frac{|\mathcal{B}_\Lambda \setminus \mathcal{X}|}{2d}} q^{|D^Y_\Lambda(\mathbb{X})| - \frac{\|\delta\mathbb{X}\|}{2d}}. \tag{1.24}$$

1.3 Introduction of contours

The aim of this section and following ones is to convert the partition function (1.24) into partition functions of contour models more suitable for our work. First we state some general notations that do not depend on the type of the boundary condition.

For an arbitrary set $Y \subset B$ of ordered bonds we use $P(Y)$ to denote the set consisting of the union of all ordered bonds together with all unit squares whose all four edges are ordered, all unit cubes having all faces in $P(Y)$, etc. (according to dimension).

Under the term contour we will understand a connected component of boundary (in $\mathbb{R}^d$) of the closed set of all points having distance (in maximal norm) less or equal to 1/4 from $P(Y)$. We use $\partial(Y) = \{\gamma_i\}$ to denote the set of all contours of a configuration Y, $\mathbb{K}$ the set of all contours and $\mathcal{K}$ the set of all collections of contours from $\mathbb{K}$.

We call a collection $\partial \in \mathcal{K}$ compatible if there exists configuration Y such that ∂ is set of contours of Y, $\partial = \partial(Y)$, $\mathcal{K}^{co}$ will be set of all compatible collections. Since mapping between all configurations and all compatible collections of contours ∂ is one-to-one, we can use the notation $Y(\partial)$, $D_\Lambda(\partial)$, etc.

In addition, we will need something like a length of contour. It is suitable not to consider the physical length but to use the number of bonds intersected by γ. The length will be denoted by $\|\gamma\| = |B \cap \gamma|$, and $\|\partial\| = \sum_{\gamma \in \partial} \|\gamma\|$ if this sum exists.

The next notion we need to introduce is a "colour" of a contour. A contour of a finite length divides $\mathbb{R}^d$ into two components. The infinite one we denote $\mathrm{Ext}\,\gamma$ and the finite one $\mathrm{Int}\,\gamma$. For contours of infinite length we will not define these terms. Let us imagine a configuration Y such that γ is the only contour of Y. We call γ ordered (disordered) if all bonds laying entirely in $\mathrm{Ext}\,\gamma$, are ordered (disordered). For future reference we use $\mathbb{K}_d$ ($\mathbb{K}_o$) to denote the sets of all disordered (ordered) contours, $\mathcal{K}_d$, $\mathcal{K}_o$ the sets of collections from these sets and $\mathcal{K}_d^{co} \subset \mathcal{K}_d$, $\mathcal{K}_o^{co} \subset \mathcal{K}_o$ the sets of compatible (this means non intersecting here) collections.

The contour $\gamma \in \partial$ for which $\gamma \not\subset \mathrm{Int}\,\gamma'$ for all $\gamma' \in \partial$, $\gamma \neq \gamma'$ is called an external contour of the configuration ∂. We use $\mathcal{K}^e \subset \mathcal{K}^{co}$ (resp. $\mathcal{K}_o^e \subset \mathcal{K}_o^{co}$, $\mathcal{K}_d^e \subset \mathcal{K}_d^{co}$) to denote the set of all collections of mutually external (ordered, disordered) contours.

1.4 Ordered boundary condition

When all bonds outside $\mathbb{B}_\Lambda$ are ordered, we will talk about ordered boundary condition. The graph $G(\Lambda)$ "belonging" in this case to Λ is the graph $(V(\mathbb{B}_\Lambda), \mathbb{B}_\Lambda)$. From this we can see that all components of $\mathbb{X}$, that are connected to $B_{\Lambda^C} = (\mathbb{B}_\Lambda)^C$ must be counted as one component.

We use $\mathcal{K}_\Lambda^{co}(o)$ to denote the set of all collections ∂ of contours such that the configuration $Y(\partial)$ has all bonds outside $\mathbb{B}_\Lambda$ ordered. For these configurations we will write $X(\partial) = Y(\partial) \cap B_\Lambda$, $\mathcal{X}(\partial) = Y(\partial) \cap \mathcal{B}_\Lambda$.

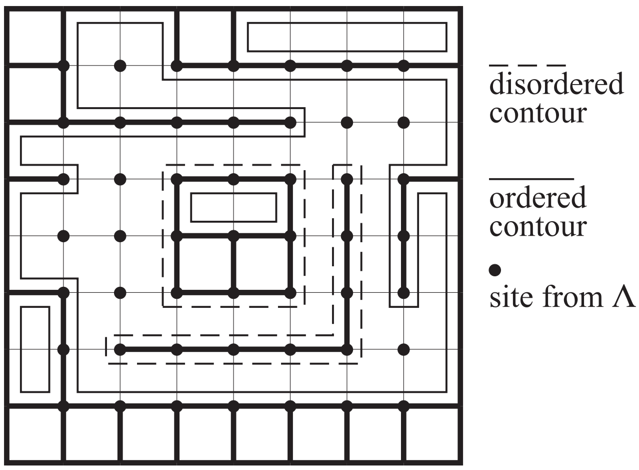

Figure 1.1: Ordered boundary condition

It is obvious for ordered boundary condition that $\|\partial\| = \|\delta\mathbb{X}(\partial)\| + |\mathcal{B} \setminus \mathcal{X}|$. Since all components of $\mathbb{X}$ not touching a boundary are separated from rest of the lattice by one disordered contour (see Figure 1.1, it is a pity it can be only two-dimensional), and the component which is connected to the boundary is always non empty (it contains at least all vertices from $V(\mathcal{B}_\Lambda) \setminus \Lambda$), we get

$$D_\Lambda(\partial) = \text{“number of disordered contours”} + 1. \tag{1.25}$$

Using this and setting

$$e_o = -\log p, \tag{1.26}$$

$$e_d = -\log(1-p) - \frac{1}{d}\log q \tag{1.27}$$

and

$$\rho(\gamma) = \begin{cases} q^{-\|\gamma\|/2d} & \text{for } \gamma \text{ ordered} \\ q\, q^{-\|\gamma\|/2d} & \text{for } \gamma \text{ disordered,} \end{cases} \tag{1.28}$$

we rewrite (1.24) as

$$Z^o_\Lambda = q \sum_{\partial \in \mathcal{K}^{co}_\Lambda(o)} e^{-e_o|\mathbb{X}(\partial)|} e^{-e_d|\mathbb{B}_\Lambda \setminus \mathbb{X}(\partial)|} \prod_{\gamma\in\partial} \rho(\gamma). \tag{1.29}$$

We can view (1.29) as the partition function of a model where each edge has the energy e_o or e_d and the “boundary” between edges with different “colours” costs us energy

$$E(\gamma_o) = (\|\gamma\|/2d)\log q \tag{1.30}$$

or

$$E(\gamma_d) = (-1 + \|\gamma\|/2d)\log q, \tag{1.31}$$

respectively.

On the first sight the very short and elegant expression of partition function (1.29) has one disadvantage. There is a long-range order interaction between contours. Namely, when we remove one, for example disordered, contour from ∂ we will possibly get the collection where one ordered contour is directly inside another ordered contour but this collection is not compatible. In some other models it is possible to re-colour the inner contours (for example as in Ising model). However, in this model it is not possible because the value of ρ depends on a contour colour and in addition we can see the contour colour from its shape.

In the following paragraphs we will remove this long-range interaction by the standard Pirogov-Sinai procedure [2, 12, 13]. For ordered boundary condition, all external contours are ordered. We use $\Theta(\partial)$ to denote the set of external contours of collection ∂. Let us introduce, for any $\Delta \subset \mathbb{R}^d$, two sets: $B_\Lambda(\Delta) = \{b \mid b \in B_\Lambda, c(b) \in \Delta\}$ and $\mathcal{B}_\Lambda(\Delta) = \{b \mid b \in \mathcal{B}_\Lambda, c(b) \in \Delta\}$, where we use $c(b)$ to denote the center of bond b. It is clear that

$$|X(\partial)| = |B_\Lambda(\operatorname{Ext}\partial)| + \sum_{\gamma\in\Theta(\partial)} |B_\Lambda(\operatorname{Int}\gamma) \cap \mathbb{X}(\partial'(\gamma))|, \tag{1.32}$$

$$|B_\Lambda \setminus X(\partial)| = \sum_{\gamma\in\Theta(\partial)} |B_\Lambda(\operatorname{Int}\gamma) \setminus \mathbb{X}(\partial'(\gamma))|, \tag{1.33}$$

$$|\mathcal{X}(\partial)| = |\mathcal{B}_\Lambda(\operatorname{Ext}\partial)| + \sum_{\gamma\in\Theta(\partial)} |\mathcal{B}_\Lambda(\operatorname{Int}\gamma) \cap \mathbb{X}(\partial'(\gamma))|, \tag{1.34}$$

$$|\mathcal{B}_\Lambda \setminus \mathcal{X}(\partial)| = \sum_{\gamma\in\Theta(\partial)} |\mathcal{B}_\Lambda(\operatorname{Int}\gamma) \setminus \mathbb{X}(\partial'(\gamma))|, \tag{1.35}$$

where $\operatorname{Ext}\partial = \bigcap_{\gamma\in\Theta(\partial)} \operatorname{Ext}\gamma$ and $\partial'(\gamma) = \{\gamma' \in \partial \mid \gamma' \subset \operatorname{Int}\gamma\}$.

Using this we can write

$$\begin{aligned} Z^o_\Lambda = q \sum_{\partial\in\mathcal{K}^{co}_\Lambda(o)} & e^{-e_o[|B_\Lambda(\operatorname{Ext}\partial)|+|\mathcal{B}_\Lambda(\operatorname{Ext}\partial)|]} \prod_{\gamma\in\Theta(\partial)} e^{-e_o|B_\Lambda(\operatorname{Int}\gamma)\cap\mathbb{X}(\partial'(\gamma))|} \times \\ & \times e^{-e_o|\mathcal{B}_\Lambda(\operatorname{Int}\gamma)\cap\mathbb{X}(\partial'(\gamma))|} e^{-e_d[|B_\Lambda(\operatorname{Int}\gamma)\setminus\mathbb{X}(\partial'(\gamma))|+|\mathcal{B}_\Lambda(\operatorname{Int}\gamma)\setminus\mathbb{X}(\partial'(\gamma))|]} \prod_{\gamma\in\partial}\rho(\gamma) \quad (1.36) \\ = q \sum_\theta & e^{-e_o[|B_\Lambda(\operatorname{Ext}\theta)|+|\mathcal{B}_\Lambda(\operatorname{Ext}\theta)|]} \prod_{\gamma\in\theta} \rho(\gamma) Z^o(\operatorname{Int}\gamma) \frac{Z^d(\operatorname{Int}\gamma)}{Z^o(\operatorname{Int}\gamma)}. \quad (1.37) \end{aligned}$$

Here, the sum runs over all collections θ from $\mathcal{K}^{co}_\Lambda(o)$ of mutually external contours and for finite $\Delta \subset \mathbb{R}^d$ we define

$$Z^d(\Delta) = \sum_{\substack{\partial_{(d)}\in\mathcal{K}^{co} \\ \partial_{(d)}\subset\Delta}} e^{-e_o|B(\Delta)\cap Y(\partial)|} e^{-e_d|B(\Delta)\setminus Y(\partial)|} \prod_{\gamma\in\partial}\rho(\gamma), \tag{1.38}$$

$$Z^o(\Delta) = \sum_{\substack{\partial_{(o)}\in\mathcal{K}^{co} \\ \partial_{(o)}\subset\Delta}} e^{-e_o|B(\Delta)\cap Y(\partial)|} e^{-e_d|B(\Delta)\setminus Y(\partial)|} \prod_{\gamma\in\partial}\rho(\gamma). \tag{1.39}$$

The index o (d) by ∂ refers to the condition that all external contours of ∂ are ordered (disordered).

We can iterate the step (1.37) by expanding $Z^o(\operatorname{Int}\gamma)$ until there are so small ordered contours that there cannot be other ordered contours inside. The number of iterations will necessarily be finite because we have a finite Λ.

After this we get

$$\begin{aligned} Z^o_\Lambda &= q e^{-e_o(|B_\Lambda|+|\mathcal{B}_\Lambda|)} \sum_{\partial\in\mathcal{K}^{co}_o(\Lambda)} \prod_{\gamma\in\partial} \rho_{(o)}(\gamma) \frac{Z^d(\operatorname{Int}\gamma)}{Z^o(\operatorname{Int}\gamma)} & (1.40) \\ &= q e^{-e_o(|B_\Lambda|+|\mathcal{B}_\Lambda|)} \sum_{\partial\in\mathcal{K}^{co}_o(\Lambda)} \prod_{\gamma\in\partial} \Phi_o(\gamma) & (1.41) \\ &= q e^{-e_o(|B_\Lambda|+|\mathcal{B}_\Lambda|)} \mathcal{Z}(\mathbb{K}_o(\Lambda), \Phi_o) & (1.42) \end{aligned}$$

with $\mathbb{K}_o(\Lambda)$ being the set of all ordered contours laying in 1-neighbourhood of Λ, $\mathcal{K}^{co}_o(\Lambda)$ the set of compatible collections from this set (i.e. non-intersecting) and using notation from Appendix A.

Hence, we get the expression of partition function for ordered boundary condition in terms of contour model with contour functional

$$\Phi_o(\gamma) = \rho(\gamma)\frac{Z^d(\mathrm{Int}\,\gamma)}{Z^o(\mathrm{Int}\,\gamma)} \tag{1.43}$$

and mainly without long-range order interaction.

1.5 Disordered boundary condition

Under this term we will understand such the case where all bonds outside B_Λ will be disordered (i.e. also all bonds in $\mathcal{B}$ are disordered, and so $\mathbb{X} = X$). A graph "belonging" to Λ will be the graph (Λ, B_Λ). Remembering the Potts model we find out that disordered boundary condition in random-cluster model corresponds to Potts model free boundary condition. The notation of contour, its length, colour, interior and exterior remains the same as in the previous section. It is obvious that $\|\partial\| = \|\delta\mathbb{X}(\partial)\|$ and

$$D_\Lambda(\partial) = \text{"number of disordered contours"}. \tag{1.44}$$

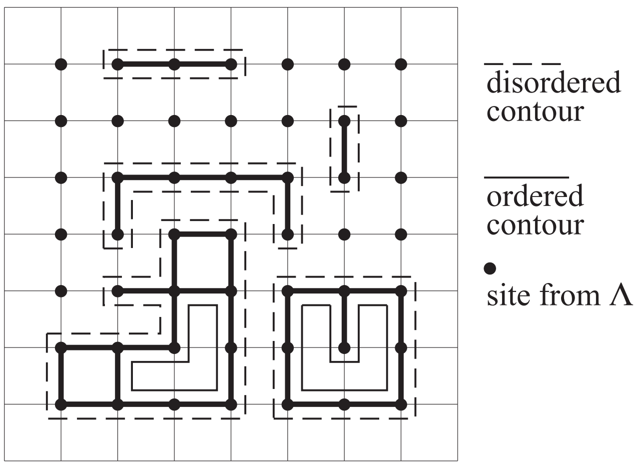

Figure 1.2: Disordered boundary condition

When we put these facts into (1.24) we get

$$Z^d_\Lambda = q^{|\mathcal{B}_\Lambda|/2d} \sum_{\partial\in\mathcal{K}^{co}_\Lambda(d)} e^{-e_o|X(\partial)|}e^{-e_d|B_\Lambda\setminus X(\partial)|}\prod_{\gamma\in\partial}\rho(\gamma), \tag{1.45}$$

where again

$$\rho(\gamma) = \begin{cases} q^{-\|\gamma\|/2d} & \text{for } \gamma \text{ ordered} \\ q\,q^{-\|\gamma\|/2d} & \text{for } \gamma \text{ disordered,} \end{cases} \tag{1.46}$$

and $\mathcal{K}^{co}_{\Lambda}(d)$ is the set of compatible collections of contours, such that their configurations are compatible with disordered boundary condition outside Λ. It is trivial that

$$|B_\Lambda \setminus X(\partial)| = |B_\Lambda(\mathrm{Ext}\,\partial)| + \sum_{\gamma\in\Theta(\partial)} |B_\Lambda(\mathrm{Int}\,\gamma) \setminus \mathbb{X}(\partial'(\gamma))|, \tag{1.47}$$

$$|X(\partial)| = \sum_{\gamma\in\Theta(\partial)} |B_\Lambda(\mathrm{Int}\,\gamma) \cap \mathbb{X}(\partial'(\gamma))|, \tag{1.48}$$

$$|\mathcal{B}_\Lambda \setminus \mathcal{X}(\partial)| = |\mathcal{B}_\Lambda(\mathrm{Ext}\,\partial)| + \sum_{\gamma\in\Theta(\partial)} |\mathcal{B}(\mathrm{Int}\,\gamma) \setminus \mathbb{X}(\partial'(\gamma))| = |\mathcal{B}_\Lambda|, \tag{1.49}$$

$$|\mathcal{X}(\partial)| = \sum_{\gamma\in\Theta(\partial)} |\mathcal{B}_\Lambda(\mathrm{Int}\,\gamma) \cap \mathbb{X}(\partial'(\gamma))| = 0. \tag{1.50}$$

For disordered boundary condition, all external contours are disordered. As in the previous section we extract from the partition function the dependence on the external configuration using $Z^d(\Delta)$ and $Z^o(\Delta)$. After the same iteration as in Section 1.4 it yields

$$Z^d_\Lambda = e^{-e_d|B_\Lambda|} q^{|\mathcal{B}_\Lambda|/2d} \sum_{\partial\in\mathcal{K}^{co}_d(\Lambda)} \prod_{\gamma\in\partial} \rho_{(d)}(\gamma) \frac{Z^o(\mathrm{Int}\,\gamma)}{Z^d(\mathrm{Int}\,\gamma)} \tag{1.51}$$

$$= e^{-e_d|B_\Lambda|} q^{|\mathcal{B}_\Lambda|/2d} \sum_{\partial\in\mathcal{K}^{co}_d(\Lambda)} \prod_{\gamma\in\partial} \Phi_d(\gamma) \tag{1.52}$$

$$= e^{-e_d|B_\Lambda|} q^{|\mathcal{B}_\Lambda|/2d} \mathcal{Z}(\mathbb{K}_d(\Lambda), \Phi_d) \tag{1.53}$$

with $\mathbb{K}_d(\Lambda)$ being the set of all disordered contours from 1-neighbourhood of Λ, and $\mathcal{K}^{co}_d(\Lambda)$ the compatible collection of contours from this set.

At the end of this section we note that the same expansion as for Z^o_Λ and Z^d_Λ is valid also for $Z^o(\Delta)$ and $Z^d(\Delta)$:

$$Z^o(\Delta) = e^{-e_o|B(\Delta)|} \mathcal{Z}(\mathbb{K}_o(\Delta), \Phi_o) \tag{1.54}$$

and

$$Z^d(\Delta) = e^{-e_d|B(\Delta)|} \mathcal{Z}(\mathbb{K}_d(\Delta), \Phi_d), \tag{1.55}$$

where this time $\mathbb{K}_o(\Delta)$ ($\mathbb{K}_d(\Delta)$) is the set of ordered (disordered) contours laying entirely in Δ.[3]

Note: We define configurations for disordered boundary condition only on the set B_Λ, unlike to ordered boundary condition. Due to this, the contours cannot overlap from 1-neighbourhood of Λ. This asymmetry does not have an influence when we make the limit $\Lambda \nearrow \mathbb{Z}^d$, but has a big influence on the simplicity of expressions.

[3] This is not in contradiction with the definition of $\mathbb{K}_o(\Lambda)$ and $\mathbb{K}_d(\Lambda)$ because $\Delta \subset \mathbb{R}^d$ and $\Lambda \subset \mathbb{Z}^d$.

1.6 Properties of contour functionals

In this section we will show, using the standard Pirogov-Sinai procedure, that there exist a transition point p_c such that both contour functionals that we got in previous sections fulfill the assumptions of Theorem A.2. After proving it, we will have tools from this theorem to make a very accurate description of behavior of models with these functionals also in an infinite volume.

To define p_c and to prove that Φ_o and Φ_d (we will use the symbol $\diamond$ for both o of d to save place) are τ-functionals for this value of p, we introduce a metastable free energy. First we suppress all contours whose weights are not dumped. Putting thus for $\tau \geq 1 + \log(2c)$

$$\bar{\Phi}_\diamond(\gamma) = \begin{cases} \Phi_\diamond(\gamma) & \text{if } |\Phi_\diamond(\gamma)| \leq e^{-\tau\|\gamma\|}, \\ 0 & \text{otherwise} \end{cases} \tag{1.56}$$

we define

$$\bar{Z}^\diamond(\Delta) = e^{-e_\diamond|B(\Delta)|}\mathcal{Z}(\mathbb{K}_\diamond(\Delta), \bar{\Phi}_\diamond) \tag{1.57}$$

and similarly

$$\bar{Z}^o_\Lambda = q\, e^{-e_o|\mathbb{B}_\Lambda|}\mathcal{Z}(\mathbb{K}_o(\Lambda), \bar{\Phi}_o), \tag{1.58}$$

$$\bar{Z}^d_\Lambda = q^{|\mathcal{B}_\Lambda|/2d}e^{-e_d|B_\Lambda|}\mathcal{Z}(\mathbb{K}_d(\Lambda), \bar{\Phi}_d). \tag{1.59}$$

The functionals $\bar{\Phi}_\diamond$ are constructed in such a way to be automatically dumped and translation invariant and so we can use the cluster expansion to have good control over limit

$$p(\bar{\Phi}_\diamond) = \lim_{\Lambda\nearrow\mathbb{Z}^d} |B_\Lambda|^{-1}\log\mathcal{Z}(\mathbb{K}_\diamond(\Lambda), \bar{\Phi}_\diamond). \tag{1.60}$$

Especially using (A.20) we can write

$$\bar{Z}^\diamond_\Lambda = \exp(-f_\diamond|\mathbb{B}_\Lambda| + \varepsilon|\partial\mathbb{B}_\Lambda|) \tag{1.61}$$

with

$$f_\diamond = e_\diamond - p(\bar{\Phi}_\diamond) \tag{1.62}$$

and with ε of order $e^{-\omega m_\diamond}$ (see Appendix A).

The metastable free energy defined by (1.62) plays an important role in determining which phase is stable. It turns out that the stable phase is characterized by having minimal metastable energy. We define

$$a_\diamond = f_\diamond - \min(f_o, f_d) \tag{1.63}$$

and we claim that Φ_o is dumped once $a_o = 0$ and similarly for Φ_d. Now we will prove it but in a very special situation that it will be sufficient for us. For a sketch of general proof see for example [6].

Proposition 1.3 *Let q such that*

$$\frac{\log q}{2d} \geq 1 + \log(2c) + 4d\varepsilon. \tag{1.64}$$

Then there exist $p_c \in [0,1]$ such that both a_o, $a_d = 0$, and both functionals Φ_o and Φ_d are τ-functionals, i.e. $\Phi_\diamond(\gamma) \leq e^{-\tau\|\gamma\|}$ for every $\gamma \in \mathbb{K}_\diamond$ with

$$\tau = \frac{\log q}{2d} - 4d\varepsilon \geq 1 + \log(2c). \tag{1.65}$$

Proof: Existence of a point p_c for which $a_d = a_o = 0$ follows once we observe that it is determined by the equation $f_o = f_d$ that is a well controlled disturbance of the equation $e_o = e_d$ and has a unique solution p_c close to $p_o = q^{1/d}/(1 + q^{1/d})$ solving $e_o = e_d$ (see [8]).

The τ-functionality of Φ_o and Φ_d we prove by induction in diameter of γ. By induction hypothesis we can replace $Z^\diamond(\operatorname{Int}\gamma)$ by $\bar{Z}^\diamond(\operatorname{Int}\gamma)$. Using equality (1.61) also holding for $Z^\diamond(\operatorname{Int}\gamma)$, we get

$$\frac{Z^o(\operatorname{Int}\gamma)}{Z^d(\operatorname{Int}\gamma)} = \frac{\bar{Z}^o(\operatorname{Int}\gamma)}{\bar{Z}^d(\operatorname{Int}\gamma)} \leq \exp[-(f_o - f_d)|B(\operatorname{Int}\gamma)| + 2\varepsilon|\partial B(\operatorname{Int}\gamma)|] \tag{1.66}$$

with $\partial B(\Delta)$ being the set of bonds from $B \setminus B(\Delta)$ which share vertex with some bond from $B(\Delta)$.

Since $a_o = a_d = 0$, i.e. we work at the transition point of random-cluster model (or at the critical temperature of Potts model), we have $f_o - f_d = 0$. The value of $|\partial\mathbb{B}(\operatorname{Int}\gamma)|$ we can bound by $2d\|\gamma\|$. Hence

$$\Phi_d(\gamma) \leq \rho_d(\gamma)\exp(4d\varepsilon\|\gamma\|) = \exp\left[\left(1 - \frac{\|\gamma\|}{2d}\right)\log q + 4d\varepsilon\|\gamma\|\right]. \tag{1.67}$$

Therefore, for q such that $(\log q)/(2d) \geq \tau + 4d\varepsilon$ we have $\Phi_d(\gamma) \leq e^{-\tau\|\gamma\|}$. In similar way we can prove it for $\Phi_o(\gamma)$. □

1.7 Uniqueness of limit random-cluster measure

In this section we will prove that in finite base cylinder it is only one limit random-cluster measure. We will not make a general proof but we will show it only for a special type of boundary conditions. First we will state a notation needed here.

We will use the symbol $\mu_{G,p,q}$ if we will need to denote the dependence on p and q explicitly. For two measures μ_G and μ'_G on Ω_G we will write $\mu_G \leq \mu'_G$ if for every increasing $\mathscr{F}_G$-measurable function f the inequality $\mu_G(f) \leq \mu'_G(f)$ holds. We say that the measure μ'_G FKG-dominates the measure μ_G. For every random-cluster measure μ with $q \geq 1$ the FKG inequality

$$\mu_G(fg) \geq \mu_G(f)\mu_G(g) \tag{1.68}$$

holds true with f, g being increasing measurable functions.

First we will state an auxiliary lemma which we will not prove. For proof you can see [4] and his references.

Lemma 1.4 *Let G be an arbitrary graph and $\mu_{G,p,q}$ and $\mu_{G,p',q'}$ are two random-cluster measures with different values of p and q. Then*

$$\mu_{G,p,q} \leq \mu_{G,p',q'} \qquad \text{if } q \geq q',\ q \geq 1 \text{ and } p \leq p' \tag{1.69}$$

Further, let V be a cylinder with a finite base $Q \subset \{\, i \in \mathbb{Z}^d \mid i_d = 0 \,\}$ and G(V) a graph belonging to V. We use $V_{m,n}$ to denote the set of points from V having value of d^{th} coordinate in the interval $[m, n]$, $V_n = V_{-n,n}$. We use D_m to denote the event realized if all vertical bonds $\langle i, j\rangle$ from $B_{G(V)}$ such that $i_d = m$, $j_d = m+1$ are disordered. The event $D_{m,n}$ realizes if at least one event D_i, $i = 0, \dots, n$ realizes. We will use $\bar{D}_m$ and $\bar{D}_{m,n}$ for negation of D_m and $D_{m,n}$.

Lemma 1.5 *Let V be a cylinder with finite base Q and $Y \subset B$. Then for every $\delta > 0$ and for every random-cluster measure on $G(V)$ with parameters p and $q \geq 1$ there exists n such that*

$$\mu_V^Y(D_{m,m+n}) > 1 - \delta. \tag{1.70}$$

Proof: Let us denote by $\mu_V^\star$ the measure $\mu_{V,P,1}$ with $P \geq p$. According to Lemma 1.4 the inequality $\mu_V^\star \geq \mu_V^Y$ holds true. Since D_m is decreasing event (i.e. $\bar{D}_m$ is increasing) we can write

$$\mu_V^Y(\bar{D}_m) \leq \mu_V^\star(\bar{D}_m) = P^{|Q|}. \tag{1.71}$$

Therefore,

$$\mu_V^Y(D_{m,m+n}) = 1 - \mu_V^Y(\bar{D}_{m,m+n}) \geq 1 - P^{(n+1)|Q|}. \tag{1.72}$$

□

Let us now consider an arbitrary boundary condition $Y \subset B$. We use Y_o (Y_d) to denote $Y \cup B_{G(V)}$ ($Y \setminus B_{G(V)}$). We say that Y is V-good if and only for every hight h there are not two mutually not connected ordered clusters in $B \setminus B_{G(V)}$ which are connected to B_V above and below the hight h.

Lemma 1.6 *Let $V \subset \mathbb{Z}^d$ be a cylinder with finite base, $G(V)$ let be arbitrary graph such that $(V, B_V) \subseteq G(V) \subseteq (V(\mathbb{B}(V)), \mathbb{B}(V))$, and let $Y \subset B$. Then the limit random-cluster measures*

$$\mu_V^{Y_o} = \lim_{n\to\infty} \mu_{V_n}^{Y_o} \tag{1.73}$$

and

$$\mu_V^{Y_d} = \lim_{n\to\infty} \mu_{V_n}^{Y_d} \tag{1.74}$$

exist. Moreover, if the measure

$$\mu_V^{Y} = \lim_{n\to\infty} \mu_{V_n}^{Y} \tag{1.75}$$

exist then

$$\mu_V^{Y_d} \le \mu_V^{Y} \le \mu_V^{Y_o}. \tag{1.76}$$

Proof: Using FKG inequality we can easily prove that

$$\mu_F^{Y_o} \ge \mu_{\bar F}^{Y_o} \tag{1.77}$$

$$\mu_F^{Y_d} \le \mu_{\bar F}^{Y_d} \tag{1.78}$$

with F being the subgraph of $\bar F$. Using this it is simple to prove the existence of $\mu_V^{Y_o}$ and $\mu_V^{Y_d}$. The inequality (1.76) is simple consequence of FKG inequality again. □

Proposition 1.7 *Let $V \subset \mathbb{Z}^d$ be a cylinder with finite base, $G(V)$ let be arbitrary graph such that $(V, B_V) \subseteq G(V) \subseteq (V(\mathbb{B}(V)), \mathbb{B}(V))$, and let $Y \subset B$ be V-good. Then there exists only one limit random-cluster measure on $G(V)$ with boundary condition Y.*

Proof: The only thing we need is to prove $\mu_V^{Y_o} = \mu_V^{Y_d}$. Consider now an increasing $\mathscr{F}_\Lambda$-measurable function f with $\Lambda \subset V_m$. Then

$$\begin{aligned}\mu_{V_n}^{Y_o}(f) &= \mu_{V_n}^{Y_o}(f \mid \bar D_{-n,-m-1} \cup \bar D_{m,n-1})\mu_{V_n}^{Y_o}(\bar D_{-n,-m-1} \cup \bar D_{m,n-1}) \\ &+ \sum_{i=-n}^{-m-1}\sum_{j=m}^{n-1} \mu_{V_n}^{Y_o}(f \mid E)\mu_{V_n}^{Y_o}(E).\end{aligned} \tag{1.79}$$

with E being the event which realizes if all D_i, D_j, $\bar D_{i+1,-m-1}$ and $\bar D_{m,j-1}$ realize. We use $\varepsilon(n)$ to denote $\mu_{V_n}^{Y_o}(\bar D_{-n,-m-1} \cup \bar D_{m,n-1})$. Since Y is V-good

$$\mu_{V_n}^{Y_o}(f) \le \varepsilon(n)\|f\| + \sum_{i=-n}^{-m-1}\sum_{j=m}^{n-1} \mu_{V_{i+1,j}}^{Y_d}(f)\mu_{V_n}^{Y_o}(E) \tag{1.80}$$

$$\le \varepsilon(n)\|f\| + (1-\varepsilon(n))\mu_{V_n}^{Y_d}(f) \le \varepsilon(n)\|f\| + \mu_{V_n}^{Y_d}(f). \tag{1.81}$$

We used the inequality (1.78) when we have changed $\mu_{V_{i,j}}^{Y_d}$ to $\mu_{V_n}^{Y_d}$. $\varepsilon(n)$ is the probability that at least one from events $D_{-n,-m-1}$ and $D_{m,n-1}$ realizes. These probabilities can be bound using Lemma 1.5 by $P^{(n-m)|Q|}$. Therefore, when we take the limit $n \to \infty$ we get

$$\mu_V^{Y_o}(f) \le \mu_V^{Y_d}(f). \tag{1.82}$$

Using this inequality, (1.76) and the fact that every function can be approximate by linear combination of local increasing functions we get $\mu_V^{Y_o} = \mu_V^{Y_d}$. □

1.8 Description of stable phase

We describe now more accurate the behavior of stable phases, i.e. translation invariant limit random-cluster measures with minimal metastable free energy, in infinite volume. Furthermore, we will always consider that $a_d = a_o = 0$ (i.e. $p = p_c$).

First, let us define a probability measure inside an ordered contour γ_o. We attach to each configuration Y, such that γ_o is the only external contour of Y, its probability

$$\mu(Y \mid \gamma_o) = \frac{1}{Z^d(\operatorname{Int}\gamma_o)} e^{-e_o|Y\cap B(\operatorname{Int}\gamma_o)|-e_d|B(\operatorname{Int}\gamma_o)\setminus Y|} \prod_{\gamma'\in(\partial(Y)\setminus\gamma_o)} \rho(\gamma') \tag{1.83}$$

and similarly for the interior of some disordered contour γ_d

$$\mu(Y \mid \gamma_d) = \frac{1}{Z^o(\operatorname{Int}\gamma_d)} e^{-e_o|Y\cap B(\operatorname{Int}\gamma_d)|-e_d|B(\operatorname{Int}\gamma_d)\setminus Y|} \prod_{\gamma'\in(\partial(Y)\setminus\gamma_d)} \rho(\gamma'). \tag{1.84}$$

For a bounded $\mathscr{F}$-measurable function φ we define

$$\mu(\varphi \mid \gamma) = \sum_{Y;\Theta(\partial(Y))=\gamma} \varphi(Y)\mu(Y \mid \gamma). \tag{1.85}$$

For a collection of mutually external contours θ we denote $\mu(\varphi \mid \theta) = \prod_{\gamma\in\theta} \mu(\varphi \mid \gamma)$.

In addition, we use $U \div V$ to denote the symetrical difference of U and V, and for two sets Δ and $\Delta' \subset \mathbb{R}^d$ we write

$$d(\Delta, \Delta') = \inf_{x\in\Delta, y\in\Delta'} d(x, y), \tag{1.86}$$

with $d(x, y)$ being the distance of two points x, $y \in \mathbb{R}^d$ in maximal norm.

Since $\operatorname{Int}\theta$ consists only from finite components there exists a unique measure $\mu(. \mid \theta)$ and due to this the following proposition can be made.

Proposition 1.8 *Let*

$$\frac{\log q}{2d} \geq 1 + 4d\varepsilon + \log(2c) + 2\frac{\log m_\diamond}{m_\diamond}. \tag{1.87}$$

Then for $\diamond = o$, d there exists limit random-cluster measure $\mu^\diamond_V(.)$ in an arbitrary volume V and a probability measure $P^e_{\diamond,V}$ on $\mathcal{K}^e_\diamond(V)$ such that for every bounded measurable φ one has

$$\mu^\diamond_V(\varphi) = \int_{\mathcal{K}^e_\diamond(V)} \mu(\varphi \mid \theta) P^e_{\diamond,V}(d\theta). \tag{1.88}$$

Moreover:

(i) $\mu^\diamond_V$ is a weak limit of $\mu^\diamond_U$ over finite $U \subset V$, ordered by inclusion

(ii) Denoting $\mathcal{K}^e_\diamond(\theta, V) = \{\, \bar\theta \in \mathcal{K}^e_\diamond(V) \mid \bar\theta \supset \theta \,\}$, $\rho^e_{\diamond,V}(\theta) = P^e_{\diamond,V}(\mathcal{K}^e_\diamond(\theta, V))$ *whenever* $\theta \in \mathcal{K}^e_\diamond(V)$, *and taking*

$$\omega = \tau - 1 - \log(2c) - 2\frac{\log m_\diamond}{m_\diamond} \tag{1.89}$$

we have:

(a) $\rho^e_{\diamond,V}(\theta) \le e^{-\tau\|\theta\|+1}$
for every $\theta \in \mathcal{K}^e_\diamond(V)$.

(b) $|\rho^e_{\diamond,V_1}(\theta) - \rho^e_{\diamond,V_2}(\theta)| \le \|\theta\| \exp[-\tau\|\theta\| + 1 - \omega d(\theta, V_1 \div V_2)]$
for every V_1, $V_2 \subset \mathbb{Z}^d$ *and* $\theta \in \mathcal{K}^e_\diamond(V_1 \cap V_2)$.

(c) $$|\rho^e_{\diamond,V}(\theta_1 \cup \theta_2) - \rho^e_{\diamond,V}(\theta_1)\rho^e_{\diamond,V}(\theta_2)| \le \|\theta_1 \cup \theta_2\| \exp[-\tau\|\theta_1 \cup \theta_2\| + 1 - \omega d(\theta_1, \theta_2)]$$
whenever $\theta_1 \cup \theta_2 \in \mathcal{K}^e_\diamond(V)$.

Proof: Since $\Phi_\diamond$ are τ-functionals according to Proposition 1.3, we can use Theorem A.2. There is a probability measure $P_{\diamond,V}$ on $\mathcal{K}^a_\diamond(V)$ that recovers its correlation functions. Introducing a map $\mathcal{K}^a_\diamond(V) \mapsto \mathcal{K}^e_\diamond(V)$ by attributing to each $\partial \in \mathcal{K}^a_\diamond(V)$ the set of its external contours $\Theta(\partial) \in \mathcal{K}^e_\diamond(V)$, we can define the measure $P^e_{\diamond,V}$ on $\mathcal{K}^e_\diamond(V)$ as the image of $P_{\diamond,V}$ under this map. Let us observe that for a finite Λ one has

$$\rho^e_{\diamond,V}(\theta) = \Phi_\diamond(\theta)\frac{\mathcal{Z}(\mathbb{K}_\diamond(\Lambda) \setminus [[\theta]])}{\mathcal{Z}(\mathbb{K}_\diamond(\Lambda))} = \Phi_\diamond(\theta) \exp\Big[-\sum_{\substack{\mathbb{C}\in\mathcal{K}^{cl}_\diamond(\Lambda)\\ \mathbb{C}\cap[[\theta]]\neq\emptyset}} \Phi^T_\diamond(\mathbb{C})\Big], \tag{1.90}$$

where $[[\theta]] = \{\, \gamma \in \mathbb{K}_\diamond \mid \text{either } \gamma\iota\theta \text{ or there exists } \bar\gamma \in \theta \text{ such that } \bar\gamma \subset \operatorname{Int}\gamma \,\}$.

To compute the sum in the previous expression we will bound first

$$\sum_{\mathbb{C};\bigcup_{\gamma\in\mathbb{C}}(\gamma\cup\operatorname{Int}\gamma)\ni i} |\Phi^T_\diamond(\mathbb{C})|e^{\omega\|\mathbb{C}\|} \le \sum_{\mathbb{C}\ni i} |\Phi^T_\diamond(\mathbb{C})|\, \|\mathbb{C}\| e^{\omega\|\mathbb{C}\|}. \tag{1.91}$$

To prove this let us consider a sum over clusters with fixed length $\|\mathbb{C}\| = n$. Considering the half-line starting in i that is parallel with a fixed coordinate axis, there are fewer than n possibilities for the first intersection with such cluster. Therefore,

$$\sum_{\substack{\mathbb{C};\ \|\mathbb{C}\|=n\\ \bigcup_{\gamma\in\mathbb{C}}(\gamma\cup\operatorname{Int}\gamma)\ni i}} |\Phi^T_\diamond(\mathbb{C})|e^{\omega\|\mathbb{C}\|} \le n \sum_{\substack{\mathbb{C};\ \mathbb{C}\ni i\\ \|\mathbb{C}\|=n}} |\Phi^T_\diamond(\mathbb{C})|e^{\omega\|\mathbb{C}\|}. \tag{1.92}$$

Using the bounds (1.91),

$$\|\mathbb{C}\| \le \exp\left[\frac{\log m_\diamond}{m_\diamond}\|\mathbb{C}\|\right] \tag{1.93}$$

and the assumptions on ω we easily prove (ii)(a). The proof of (ii)(b,c) is an easy application of the previous procedure and the bound $|e^u - e^v| \le \max(e^u, e^v)|u - v|$.

To prove (i) and (1.88), let us consider a cylinder function φ living in $\Lambda \subset \mathbb{Z}^d$ and choose $\varepsilon > 0$. We shall prove that for $U \subset V$ finite and large enough

$$\left| \mu_U^{\diamond}(\varphi) - \int_{\mathcal{K}^e_{\diamond,V}} \mu(\varphi \mid \theta) P^e_{\diamond,V}(d\theta) \right| \leq \varepsilon \|\varphi\|. \tag{1.94}$$

One can easily verify that

$$\mu_U^{\diamond}(\varphi) = \sum_{\theta \in \mathcal{K}^e_{\diamond}(U)} \mu(\varphi \mid \theta) \frac{\prod_{\gamma \in \theta} \rho(\gamma) \mathcal{Z}(\mathbb{K}_{\diamond}(\operatorname{Int} \gamma), \Phi_{\diamond})}{\mathcal{Z}(\mathbb{K}_{\diamond}(U), \Phi_{\diamond})} \tag{1.95}$$

$$\equiv \int_{\mathcal{K}^e_{\diamond}(U)} \mu(\varphi \mid \theta) P^e_{\diamond,U}(d\theta). \tag{1.96}$$

Hence, to prove (1.94) we must verify

$$\left| \int_{\mathcal{K}^e_{\diamond}(U)} \mu(\varphi \mid \theta) P^e_{\diamond,U}(d\theta) - \int_{\mathcal{K}^e_{\diamond}(V)} \mu(\varphi \mid \theta) P^e_{\diamond,V}(d\theta) \right| \leq \varepsilon \|\varphi\| \tag{1.97}$$

for U large enough. For each $\theta \in \mathcal{K}^e_{\diamond}$ we will consider a subset $\theta^{(k)} \subset \theta$ of those $\gamma \in \theta$ for which $\|\gamma\| < k$. Denoting by $\mathcal{K}^e_{\diamond}(\Lambda, k)$ the set $\{\, \theta \in \mathcal{K}^e_{\diamond} \mid \text{there exists } \gamma \in \theta$ such that $(\gamma \cup \operatorname{Int}\gamma) \cap \mathbb{B}_{\Lambda} \neq \emptyset$ and $\|\gamma\| \geq k \,\}$, we get using (ii)(a) and the similar reasoning as in the proof of (1.91) the estimate

$$\int [\mu(\varphi \mid \theta) - \mu(\varphi \mid \theta^{(k)})] P^e_{\diamond,U}(d\theta) \leq 2\|\varphi\| P^e_{\diamond,U}(\mathcal{K}^e_{\diamond}(\Lambda, k)) \tag{1.98}$$

$$\leq 4\|\varphi\| \, |\mathbb{B}_{\Lambda}| e^{(-\tau+1+\log c)k+1} \tag{1.99}$$

$$\leq \frac{1}{4} \varepsilon \|\varphi\|, \tag{1.100}$$

whenever $U \subset \mathbb{Z}^d$ and k large enough. Having chosen such k, the estimate (1.97) will be verified if we show that

$$\left| \int_{\mathcal{K}^e_{\diamond}(U)} \mu(\varphi \mid \theta^{(k)}) P^e_{\diamond,U}(d\theta) - \int_{\mathcal{K}^e_{\diamond}(V)} \mu(\varphi \mid \theta^{(k)}) P^e_{\diamond,V}(d\theta) \right| \leq \frac{1}{2} \varepsilon \|\varphi\| \tag{1.101}$$

for U large enough. Observing that $\mu(\varphi \mid \theta^{(k)})$ is a cylindrical function living in k-neighbourhood of the set Λ the estimate (1.101) follows from the weak convergence $\lim_{U \nearrow V} P^e_{\diamond,U} = P^e_{\diamond,V}$ (see Theorem A.2(iii)). Thus, we prove that the limit $\lim_{U \nearrow V} \mu_U^{\diamond}(\varphi)$ exists and is equal to

$$\int_{\mathcal{K}^e_{\diamond}(V)} \mu(\varphi \mid \theta) P^e_{\diamond,V}(d\theta) \tag{1.102}$$

for all cylindrical φ. Since both sides of (1.88) have the unique extensions from bounded cylindrical function to bounded measurable functions we have finished the proof. $\square$

Chapter 2

Translation non-invariant state

2.1 Interfaces and walls

To work with a translation non-invariant limit measure for random-cluster we need to introduce a configuration that will serve us similarly as pure ordered or pure disordered states did while we were exploring translation invariant states. As this configuration we will use the configuration ξ which each bond having at least one vertex in a lower subspace[1] is disordered and the rest of bonds from B is ordered. This configuration has a simple infinite contour which will be called an interface of configuration ξ and denoted by I_0.

We use $B^o \subset B$ to denote the set of all bonds having both vertices with non-negative d^{th} coordinate and $B^d = B \setminus B^o$. For an arbitrary $\Lambda \subset \mathbb{Z}^d$ we will write $B^o_\Lambda = B_\Lambda \cap B^o$, $\mathcal{B}^o_\Lambda = \mathcal{B}_\Lambda \cap B^o$, $\mathbb{B}^o_\Lambda = \mathbb{B}_\Lambda \cap B^o$ and analogically $B^d_\Lambda = B_\Lambda \cap B^d$, $\mathcal{B}^d_\Lambda = \mathcal{B}_\Lambda \cap B^d$ and $\mathbb{B}^d_\Lambda = \mathbb{B}_\Lambda \cap B^d$.

Our task in this section is to write the partition function for a subgraph $G(\Lambda)$ of $(\mathbb{Z}^d, B)$ that belongs to some finite set Λ under the boundary condition ξ in terms of a contour model. In this chapter we will take for Λ always a box in $\mathbb{Z}^d$, i.e. the subset of $\mathbb{Z}^d$ of the form

$$\Lambda = \prod_{i=1}^{d} \langle x_i, y_i \rangle \tag{2.1}$$

with x_i, $y_i \in \mathbb{R}$ and $\langle x_i, y_i \rangle = [x_i, y_i] \cap \mathbb{Z}$.

In analogy with Sections 1.4 and 1.5 we say that a configuration Y on B is compatible with the boundary condition ξ outside $G(\Lambda)$ if and only if for Y the following is true: all bonds from $(\mathbb{B}_\Lambda)^C \cap B^o$ are ordered and all bonds from $(B_\Lambda)^C \cap B^d$ are disordered in Y, i.e. a graph $G(\Lambda)$ we talk about is the graph $(\Lambda \cup V(\mathcal{B}^o_\Lambda), \mathbb{B}^o_\Lambda \cup B^d_\Lambda)$. As in the previous chapter we will write $X = Y \cap B_\Lambda$, $\mathcal{X} = Y \cap \mathcal{B}_\Lambda$.

If Y is compatible with ξ outside a finite Λ, it is trivial to prove that there is just one infinite contour in $\partial(Y)$. We will call it an interface of the configuration Y and denote it by $I(Y)$. $I(Y)$ can differ from I_0 only in 1-neighbourhood of Λ. We use

[1] Under this term we understand the set of points from $\mathbb{R}^d$ having the d^{th} coordinate negative

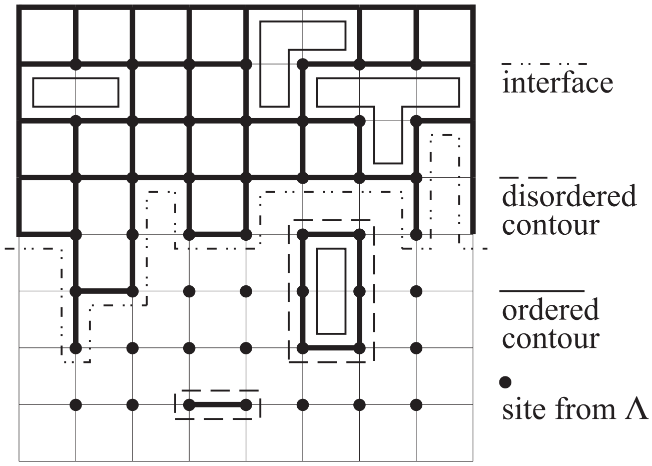

Figure 2.1: Configuration with an interface

$\mathcal{I}(\Lambda)$ to denote the set of such interfaces. For each interface from $\mathcal{I}(\Lambda)$ there exists at least one configuration Y compatible with boundary condition ξ outside Λ.

On the other side, let us have an infinite contour γ. We say that γ is an interface if and only if $\gamma \setminus I_0$ has finite components.

An interface divides $\mathbb{R}^d$ into two connected components. We use $\mathbb{R}^d_o(I)$ to denote the upper one and $\mathbb{R}^d_d(I)$ the lower one. We will write $B^o_\Lambda(I)$ for $B_\Lambda(\mathbb{R}^d_o(I))$, $B^d_\Lambda(I)$ for $B_\Lambda(\mathbb{R}^d_d(I))$, $\mathcal{B}^o_\Lambda(I)$ for $\mathcal{B}_\Lambda(\mathbb{R}^d_o(I))$ and $\mathcal{B}^d_\Lambda(I)$ for $\mathcal{B}_\Lambda(\mathbb{R}^d_d(I))$. We also denote $\Lambda^o(I) = \mathbb{R}^d_o(I) \cap \Lambda$ and $\Lambda^d(I) = \mathbb{R}^d_d(I) \cap \Lambda$.

The length of an interface is according to the previous definition infinity. Thus, we define $\|I\|_\Lambda = |\mathbb{B}_\Lambda \cap I|$. We denote $\rho_\Lambda(I) = q^{-\|I\|_\Lambda/2d}$. We use $\mathcal{K}^{co}_\Lambda(\xi)$ to denote the set of such collections ∂ from $\mathcal{K}$ such that $Y(\partial)$ is compatible with ξ outside Λ and $I(\partial)$ for the interface of ∂.

Now we rewrite the partition function (1.24). We from the Figure 2.1 we can see that

$$\|\partial\| = \sum_{\substack{\gamma\in\partial \\ \gamma\neq I(\partial)}} \|\gamma\| + \|I(\partial)\|_\Lambda = \|\delta\mathbb{X}\| + |\mathcal{B}^o_\Lambda \setminus \mathcal{X}(\partial)|. \tag{2.2}$$

Similarly to Section 1.4, the following equality holds true:

$$D_\Lambda(\partial) = \text{“number of disordered contours”} + 1. \tag{2.3}$$

It comes from the fact that the infinite component of ξ always intersects $G(\Lambda)$ and every finite component of $\mathbb{X}$ is divided from the rest of the lattice by one disordered contour.

Using this we can write

$$Z^\xi_\Lambda = \sum_{I\in\mathcal{I}(\Lambda)} Z^\xi_\Lambda(I) \tag{2.4}$$

with

$$Z_\Lambda^\xi(I) = qq^{|\mathcal{B}_\Lambda^d|/2d} \sum_{\substack{\partial\in\mathcal{K}_\Lambda^{co}(\xi)\\ I(\partial)=I}} e^{-e_o|\mathbb{X}(\partial)|} e^{-e_d|(\mathbb{B}_\Lambda^o\setminus\mathbb{X}(\partial))\cup(B_\Lambda^d\setminus\mathbb{X}(\partial))|} \rho_\Lambda(I) \prod_{\substack{\gamma\in\partial\\ \gamma\neq I}} \rho(\gamma) \tag{2.5}$$

$$= qq^{-|\mathcal{B}_\Lambda^d|/2d}(1-p)^{-|\mathcal{B}_\Lambda^d|} \sum_{\substack{\partial\in\mathcal{K}_\Lambda^{co}(\xi)\\ I(\partial)=I}} e^{-e_o|\mathbb{X}(\partial)|} e^{-e_d|\mathbb{B}_\Lambda\setminus\mathbb{X}(\partial)|} \rho_\Lambda(I) \prod_{\substack{\gamma\in\partial\\ \gamma\neq I}} \rho(\gamma). \tag{2.6}$$

There are all external contours ordered in $\mathbb{R}_o^d(I)$ and disordered in $\mathbb{R}_d^d(I)$. We will use ∂^o to denote $\partial\cap\mathbb{R}_o^d(I)$, ∂^d to denote $\partial\cap\mathbb{R}_d^d(I)$ and for $\Delta\in\mathbb{R}^d$ we will write $B_\Lambda^\diamond(I)(\Delta) = B_\Lambda^\diamond(I)\cap B(\Delta)$ and similarly for $\mathcal{B}$ and $\mathbb{B}$. Thus,

$$\begin{aligned} Z_\Lambda^\xi(I) = q^{1-|\mathcal{B}_\Lambda^d|/2d}(1-p)^{-|\mathcal{B}_\Lambda^d|} \sum_{\theta^o,\theta^d} e^{-e_o|\mathbb{B}_\Lambda^o(I)(\operatorname{Ext}\theta^o)|} e^{-e_d|\mathbb{B}_\Lambda^d(I)(\operatorname{Ext}\theta^d)|} \\ \rho(I) \prod_{\gamma\in\theta^o} \rho(\gamma) Z_\Lambda^d(\operatorname{Int}\gamma) \prod_{\gamma\in\theta^d} \rho(\gamma) Z_\Lambda^o(\operatorname{Int}\gamma). \end{aligned} \tag{2.7}$$

Here the sum runs over all configurations of mutually external contours such that all contours from θ^o lay in $\mathbb{R}_o^d(I)$, from θ^d in $\mathbb{R}_d^d(I)$ and both from θ^o and θ^d lay in 1-neighbourhood of Λ.

After the procedure described in Section 1.4 we get:

$$\begin{aligned} Z_\Lambda^\xi(I) &= q^{1-|\mathcal{B}_\Lambda^d|/2d}(1-p)^{-|\mathcal{B}_\Lambda^d|} e^{-e_o|\mathbb{B}_\Lambda^o(I)|} e^{-e_d|\mathbb{B}_\Lambda^d(I)|} \rho_\Lambda(I) \\ &\quad\times \sum_{\substack{\partial^o\in\mathcal{K}_o^{co}(\Lambda,I)\\ \partial^d\in\mathcal{K}_d^{co}(\Lambda,I)}} \prod_{\gamma\in\partial^o} \Phi^o(\gamma) \prod_{\gamma\in\partial^d} \Phi^d(\gamma) \qquad (2.8) \\ &= q^{1-|\mathcal{B}_\Lambda^d|/2d}(1-p)^{-|\mathcal{B}_\Lambda^d|} e^{-e_o|\mathbb{B}_\Lambda^o(I)|} e^{-e_d|\mathbb{B}_\Lambda^d(I)|} \rho_\Lambda(I) \\ &\quad\times \mathcal{Z}(\mathbb{K}_o(\Lambda^o, I), \Phi^o)\mathcal{Z}(\mathbb{K}_d(\Lambda^d, I), \Phi^d). \qquad (2.9) \end{aligned}$$

Here $\mathbb{K}_o(\Lambda^o, I)$ is the set of ordered contours being in 1-neighbourhood of $\Lambda^o(I)$ but not intersecting I and similarly $\mathbb{K}_d(\Lambda^d, I)$ is the set of disordered contours laying in 1-neighbourhood of $\Lambda^d(I)$ and not intersecting I. $\mathcal{K}_o^{co}(\Lambda, I)$, $\mathcal{K}_d^{co}(\Lambda, I)$ are non-intersecting collections of contours from these sets.

In addition, we will need another object called a wall. It will be a decoration of the flat interface I_0. We will express the partition function in terms of these decorations.

To define a wall we first introduce the shift $T_h : \mathbb{R}^d \mapsto \mathbb{R}^d, (x_1, x_2, \ldots, x_d) \mapsto (x_1, x_2, \ldots, x_d + h)$ for every $h\in\mathbb{R}$. We also introduce two mappings on $\mathbb{R}^d$: $\tilde\pi(\Delta) = \bigcup_{h\in\mathbb{R}} T_h(\Delta)$ for $\Delta\subset\mathbb{R}^d$ and $\pi(\Delta) = \tilde\pi(\Delta)\cap I_0$.

Every interface I we divide into two sets

$$\operatorname{good}(I) = \{\, x\in\mathbb{R}^d \mid x\in I, \tilde\pi(x)\cup I = x \,\} \tag{2.10}$$

and $\mathrm{bad}(I) = I \setminus \mathrm{good}(I)$. We use $\widetilde{\mathbb{W}}(I)$ to denote the intersection of interface I with the set of all points having distance less or equal to $1/2$ from $\mathrm{bad}(I)$. Under the term wall we understand a connected component of $\widetilde{\mathbb{W}}(I)$.

The wall w is called a standard wall if there exists an interface I_w such that w is the only wall of I_w. We use $\mathbb{W}$ to denote the set of all standard walls. We call the collection of standard walls $\mathbb{V} \subset \mathbb{W}$ compatible if $\pi(w_1)$ and $\pi(w_2)$ do not touch or intersect whenever $w_1, w_2 \in \mathbb{V}$. The set of all compatible collections of standard walls we denote by $\mathcal{W}^{co}$.

If w is a wall we use $\mathrm{Ext}_{I_0} w$ to denote the infinite component of $I_0 \setminus \pi(w)$ and $\mathrm{Int}_{I_0} w = I_0 \setminus (\pi(w) \cup \mathrm{Ext}_{I_0} w)$. If w_1 and w_2 are compatible walls we say that w_2 is inside of w_1 if $\pi(w_2) \subset \mathrm{Int}_{I_0} w_1$. We use $\mathbb{E}(\mathbb{V})$ to denote the set of all external walls of compatible collection $\mathbb{V}$, i.e. such walls w for those $w' \in \mathbb{V}$, $w \neq w'$ implies that $\pi(w) \subset \mathrm{Ext}_{I_0} w'$. The set of all collections for that $\mathbb{V} = \mathbb{E}(\mathbb{V})$ we denote by $\mathcal{W}^e$.

The compatible collection of walls $\mathbb{V}$ is called admissible if every wall from $\mathbb{V}$ is inside a finite number of walls from $\mathbb{V}$. We use $\mathcal{W}^a$ to denote the set of all admissible collections. We also introduce the sets $\mathbb{W}(V)$ of walls laying in 1-neighbourhood of V and $\mathcal{W}^a(V)$ of admissible collections from $\mathbb{W}(V)$.

The following lemma describes the mutual relation between walls and standard walls and between collections of standard walls and the set of all interfaces.

Lemma 2.1 *(a) For every wall w there exists one and only one $h = h(w)$ such that $T_h(w)$ is in $\mathbb{W}$. We call the shift $T_{h(w)}(w)$ the standard position of w.*

(b) The mapping $\mathbb{W}(\cdot)$ that ascribes to an interface I the collection of its walls in standard positions maps $\mathcal{I}$, the set of all interfaces, into $\mathcal{W}^{co}$ and is one-to-one from $\mathcal{I}^a = \mathbb{W}^{-1}(\mathcal{W}^a)$.

For proof see for example Appendix A of [5]

2.2 Normalization of partition function

The task of this section is to extract from $Z^{\xi}_{\Lambda}(I)$ the terms not depending on I. We will find the new, normalized partition functions $\tilde{Z}^{\xi}_{\Lambda}(I)$. The advantage of these new partition functions is a possibility of rewriting them in a form which is very close to the one used in contour models. Moreover, these "contours" live near I_0. We will obtain a $(d-1)$-dimensional contour model and we will explore it by methods from Theorem A.1.

Before continuing extraction, let us define the function $\chi_\diamond(\mathbb{C})$ on the set $\mathcal{K}^{cl}_\diamond$. $\chi_\diamond(\mathbb{C}) = 1$ if there exists bond b from $\mathbb{B}^\diamond_\Lambda$ and bond b' from $B^\diamond \setminus \mathbb{B}^\diamond_\Lambda$ such that $b \cap \mathbb{C} \neq \emptyset$ and $b' \cap \mathbb{C} \neq \emptyset$ and moreover both b and b' are edges of the same hypercube in $\mathbb{Z}^d$, otherwise $\chi_\diamond(\mathbb{C}) = 0$.

Since we will use the results of Proposition 1.3 we will always suppose that the assumptions of this proposition are fulfilled. In particular, we suppose that

$$\frac{\log q}{2c} \geq 4d\varepsilon + 1 + \log(2c). \tag{2.11}$$

Proposition 2.2 *If $I \in \mathcal{I}(\Lambda)$ we have*

$$\begin{aligned}\tilde{Z}^{\xi}_{\Lambda}(I) &= \frac{Z^{\xi}_{\Lambda}(I)}{N^{\xi}_{\Lambda}} \\ &= \frac{\rho_{\Lambda}(I)}{\rho_{\Lambda}(I_0)} \exp\left\{ - \sum_{\diamond} \sum_{\substack{\mathbb{C}\in\mathcal{K}^{cl}_{\diamond} \\ \mathbb{C}\cap I\neq\emptyset}} \Phi^T_{\diamond}(\mathbb{C}) \left[\frac{|\mathbb{C}\cap\mathbb{B}^{\diamond}_{\Lambda}(I)|}{|\mathbb{C}\cap B|} - \chi_{\diamond}(\mathbb{C}) \frac{|\mathbb{C}\cap\mathbb{B}_{\Lambda}|}{|\mathbb{C}\cap B|} \right] \right\} \end{aligned} \tag{2.12}$$

with

$$N^{\xi}_{\Lambda} = \rho(I_0) q^{1-|\mathcal{B}^{\Lambda}_d|/2d} (1-p)^{-|\mathcal{B}^{\Lambda}_d|} \exp\left\{ - f_o|\mathbb{B}_{\Lambda}| - \sum_{\diamond} \sum_{\substack{\mathbb{C}\in\mathcal{K}^{cl}_{\diamond} \\ \chi(\mathbb{C})=1}} \Phi^T_{\diamond}(\mathbb{C}) \frac{|\mathbb{C}\cap\mathbb{B}_{\Lambda}|}{|\mathbb{C}\cap B|} \right\}. \tag{2.13}$$

Proof: One substitutes $\sum_{\mathbb{C}\in\mathcal{K}^{cl}_{\diamond}(\Lambda,I)} \Phi^T_{\diamond}(\mathbb{C})$ for $\log \mathcal{Z}(\mathbb{K}_{\diamond}(\Lambda^{\diamond}, I))$ according to Theorem A.2, and

$$|\mathbb{B}^{\diamond}_{\Lambda}(I)|(f_{\diamond} + p(\Phi_{\diamond})) = f_{\diamond}|\mathbb{B}^{\diamond}_{\Lambda}(I)| + \sum_{\mathbb{C}:\mathbb{C}\cap\mathbb{B}^{\diamond}_{\Lambda}(I)\neq\emptyset} \Phi^T_{\diamond}(\mathbb{C}) \frac{|\mathbb{C}\cap\mathbb{B}^{\diamond}_{\Lambda}(I)|}{|\mathbb{C}\cap B|} \tag{2.14}$$

for $e_{\diamond}|\mathbb{B}^{\diamond}_{\Lambda}(I)|$ according to (1.62) and (A.18) in (2.9). Using the fact that $f_o = f_d$ we can easily prove this proposition. □

According to the definition of $\rho_{\Lambda}(I)$ and $\|I\|_{\Lambda}$ holds true

$$\log \frac{\rho_{\Lambda}(I)}{\rho_{\Lambda}(I_0)} = -(\|I\|_{\Lambda} - \|I_0\|_{\Lambda}) \frac{\log q}{2d}. \tag{2.15}$$

Denoting $\|w\| = |w \cap B|$ and $\|\pi(w)\| = |\pi(w) \cap B|$ for every $w \in \mathbb{W}$ we can write

$$\|I\|_{\Lambda} = \|I_0\|_{\Lambda} + \sum_{w\in\mathbb{W}(I)} (\|w\| - \|\pi(w)\|), \tag{2.16}$$

$$\log \frac{\rho_{\Lambda}(I)}{\rho_{\Lambda}(I_0)} = -\frac{\log q}{2d} \sum_{w\in\mathbb{W}(I)} (\|w\| - \|\pi(w)\|) = -\sum_{w\in\mathbb{W}(I)} E(w), \tag{2.17}$$

where

$$E(w) = \frac{\log q}{2d} (\|w\| - \|\pi(w)\|) \tag{2.18}$$

and thus

$$\begin{aligned}\tilde{Z}^{\xi}_{\Lambda}(I) &= \prod_{w\in\mathbb{W}(I)} \exp[-E(w)] \\ &\quad \times \exp\left\{ - \sum_{\diamond} \sum_{\substack{\mathbb{C}\in\mathcal{K}^{cl} \\ \mathbb{C}\cap I\neq\emptyset}} \Phi^T_{\diamond}(\mathbb{C}) \left[\frac{|\mathbb{C}\cap\mathbb{B}^{\diamond}_{\Lambda}(I)|}{|\mathbb{C}\cap B|} - \chi_{\diamond}(\mathbb{C}) \frac{|\mathbb{C}\cap\mathbb{B}_{\Lambda}|}{|\mathbb{C}\cap B|} \right] \right\}. \end{aligned} \tag{2.19}$$

2.3 Expression in terms of aggregates

Now, we would like to investigate a cylindrical volume V with a finite base $Q \subset \mathbb{Z}^{d-1}$. We note that according to Lemma 1.7 there is the unique limit random-cluster measure and we have

$$\lim_{U \nearrow V} \mu_U^\xi(\cdot) = \mu_V^\xi(\cdot), \tag{2.20}$$

where we consider a weak limit over the directed set of finite volumes $U \subset V$.

Lemma 2.3 *Let the assumptions of Proposition 1.3 be fulfilled, i.e.*

$$\frac{\log q}{2d} \geq 1 + 4d\varepsilon + \log(2c) \tag{2.21}$$

and

$$\frac{\log q}{2d} \geq 4 + \log(2c). \tag{2.22}$$

Then:

(a) There exists $K_Q : \mathcal{I}(V) \mapsto \mathbb{R}$ *such that* $\tilde{Z}_U^\xi(I) \leq K_Q(I)$ *for any* $\mathbb{I} \in \mathcal{I}(U)$ *with* $\sum_{I \in \mathcal{I}(V)} K_Q(I) < \infty$.

(b) There exists a finite limit $\lim_{U \nearrow V} \tilde{Z}_U^\xi(I)$ *and it equals*

$$\begin{aligned} \tilde{Z}_V^\xi(I) = & \prod_{w \in \mathbb{W}(I)} \exp[-E(w)] \\ & \times \exp\left\{ -\sum_{\diamond} \sum_{\substack{\mathbb{C} \in \mathcal{K}^{cl} \\ \mathbb{C} \cap I \neq \emptyset}} \Phi_\diamond^T(\mathbb{C}) \left[\frac{|\mathbb{C} \cap \mathbb{B}_V^\diamond(I)|}{|\mathbb{C} \cap B|} - \chi_\diamond(\mathbb{C}) \frac{|\mathbb{C} \cap \mathbb{B}_V|}{|\mathbb{C} \cap B|} \right] \right\} \end{aligned} \tag{2.23}$$

(c) The probabilities $P_U^{\mathcal{I}}$ *of interfaces from* $\mathcal{I}(U)$ *defined by*

$$\mu_U^\xi(\{ Y \mid I(Y) = I \}) = P_U^{\mathcal{I}}(I) = \frac{\tilde{Z}_U^\xi(I)}{\sum_{I' \in \mathcal{I}(U)} \tilde{Z}_U^\xi(I')} \tag{2.24}$$

converge to a probability on $\mathcal{I}(V)$ *which is defined by* $\tilde{Z}_V^\xi(I)$, *i.e.*

$$P_V^{\mathcal{I}}(I) = \frac{\tilde{Z}_V^\xi(I)}{\sum_{I' \in \mathcal{I}(V)} \tilde{Z}_V^\xi(I)}. \tag{2.25}$$

Proof: According to the definition of $E(w)$ and (A.12) we can write

$$\tilde{Z}_U^\xi(I) \leq \exp\Big\{ -\sum_{w \in \mathbb{W}(I)} \frac{\log q}{2d}(\|w\| - \|\pi(w)\|) + 4\|I\|_U \Big\} \tag{2.26}$$

$$= \exp\Big\{ -\sum_{w \in \mathbb{W}(I)} \left(\frac{\log q}{2d} - 4 \right) (\|w\| - \|\pi(w)\|) + 4\|I_0\|_U \Big\}. \tag{2.27}$$

Since both inequalities $\|I\|_U \le \|I\|_V$ and $\|\pi(w)\| \le \|w\|/2$ are obvious for every interface $I \in \mathcal{I}(U)$ and every wall we get

$$\tilde{Z}_U(\beta \mid I, \xi) \le \exp\Big\{ -\frac{1}{2} \sum_{w \in \mathbb{W}(I)} \left(\frac{\log q}{2d} - 4 \right) \|w\| + 4\|I_0\|_V \Big\} \tag{2.28}$$

$$\equiv K_Q(I). \tag{2.29}$$

One can see from this expression that $K_Q(I)$ does not depend on the volume U but only on the base Q and on the interface I. The sum can be bound by the following way

$$\sum_{I \in \mathcal{I}(V)} K_Q(I) \le \exp(4\|I_0\|_V) \prod_{i \in I_0 \cap \mathbb{B}_V} \sum_{\substack{w \in \mathbb{W} \\ i \in w}} \exp\left[-\left(\frac{\log q}{2d} - 4 \right) \|w\| \right]. \tag{2.30}$$

The number of walls that contains an arbitrary site $i \in \mathbb{Z}^d_\star$ can be bound in the same way as the number of such contours (see (A.11)) by

$$|\{\, w \mid w \ni i, \|w\| = n \,\}| \le c^n. \tag{2.31}$$

Hence

$$\sum_{\substack{w \in \mathbb{W} \\ i \in w}} \exp\left[-(\log q/2d - 4)\, \|w\| \right]$$

$$\le \sum_{n=m_w}^{\infty} c^n \exp[-(\log q/2d - 4)\, n] \tag{2.32}$$

$$\le \frac{\{c \exp\left[-(\log q/2d - 4) \right]\}^{m_w}}{1 - c \exp[-(\log q/2d - 4)\, n]} \le 1 \tag{2.33}$$

if

$$\frac{\log q}{2d} \ge 4 + \log(2c) \tag{2.34}$$

and where we use m_w to denote the minimal length of wall. □

In the next lemma we will rewrite the partition function $\tilde{Z}^\xi_U(I)$ as a sum over triplets $T = (T_w, T_o, T_d) \in \mathcal{T}(V)$ defined so that $T_w \in \mathcal{W}^a(V)$ and T_o, T_d are finite subsets of $\mathcal{K}^{cl}_o$ or $\mathcal{K}^{cl}_d$ and the elements of T_o and T_d intersect $\mathbb{B}_V$ and $I(T_w)$. We will use $\mathcal{T} = \mathcal{T}(\mathbb{Z}^d)$.

Lemma 2.4 *Under the assumption of Lemma 2.3, one gets for $I \in \mathcal{I}(V)$ that*

$$\tilde{Z}^\xi_U(I) = \prod_{w \in T_w = \mathbb{W}(I)} e^{-E(w)} \sum_{T \in \mathcal{T}(V), I(T_w) = I} \prod_{\diamond} \prod_{\mathbb{C} \in T_\diamond} f^\diamond_{V,I}(\mathbb{C}), \tag{2.35}$$

where

$$f^\diamond_{V,I}(\mathbb{C}) = \exp\Big\{ -\Phi^T_\diamond(\mathbb{C}) \Big[\frac{|\mathbb{C} \cap \mathbb{B}^\diamond_V(I)|}{|\mathbb{C} \cap B|} - \chi_\diamond(\mathbb{C}) \frac{|\mathbb{C} \cap \mathbb{B}_V|}{|\mathbb{C} \cap B|} \Big] \Big\} - 1 \tag{2.36}$$

whenever $\mathbb{C} \in \mathcal{K}^{cl}$.

Proof: According to Lemma 2.1 we know that $\mathbb{W}(\mathcal{I}(V)) = \mathbb{W}(V)$. When we use this fact and the part (b) from the previous lemma we must only prove the equality

$$\exp\left\{-\sum_{\diamond}\sum_{\substack{\mathbb{C}\in\mathcal{K}^{cl}_{\diamond}\\ \mathbb{C}\cap I\neq\emptyset}} \Phi^T_{\diamond}(\mathbb{C})\left[\frac{|\mathbb{C}\cap\mathbb{B}^{\diamond}_V(I)|}{|\mathbb{C}\cap B|} - \chi_{\diamond}(\mathbb{C})\frac{|\mathbb{C}\cap\mathbb{B}_V|}{|\mathbb{C}\cap B|}\right]\right\}$$

$$= \sum_{\substack{T\in\mathcal{T}(V)\\ I(T_w)=I}}\prod_{\diamond}\prod_{\mathbb{C}\in T_{\diamond}} f^{\diamond}_{V,I}(\mathbb{C}). \tag{2.37}$$

This equality comes from the observation

$$\exp\Big(\sum_{n\in N} a_n\Big) = \prod_{n\in N}[(\exp a_n - 1) + 1] = \sum_{\substack{K\subset N\\ \text{finite}}}\prod_{n\in K}(\exp a_n - 1) \tag{2.38}$$

for countable N if $\sum|a_n| < \infty$. This inequality holds true because

$$\sum_{\diamond}\sum_{\substack{\mathbb{C}\in\mathcal{K}^{cl}_{\diamond}\\ \mathbb{C}\cap I\neq\emptyset}} |\Phi^T_{\diamond}(\mathbb{C})|\left|\left[\frac{|\mathbb{C}\cap\mathbb{B}^{\diamond}_V(I)|}{|\mathbb{C}\cap B|} - \chi_{\diamond}(\mathbb{C})\frac{|\mathbb{C}\cap\mathbb{B}_V|}{|\mathbb{C}\cap B|}\right]\right| \leq 4\|I\|_V. \tag{2.39}$$

□

Now we introduce the notation of an aggregate. Let $T \in \mathcal{T}(V)$ and $a = (a_w, a_o, a_d)$, where $a_w \subset T_w$, $a_o \subset T_o$, $a_d \subset T_d$ are such that

$$\pi(a) = \pi\left(\bigcup_{w\in a_w} w \cup \bigcup_{\mathbb{C}\in a_d}\mathbb{C} \cup \bigcup_{\mathbb{C}\in a_o}\mathbb{C}\right) \tag{2.40}$$

is connected component of $\pi(T)$. Then we say that a is the aggregate of T. On the other side the triplet (a_w, a_o, a_d) is called an aggregate (in V) if it is the aggregate of the triplet from $\mathcal{T}$ ($\mathcal{T}(V)$).

If a is the only aggregate of T then it is called a standard aggregate. The set of all standard aggregates of triplets from $\mathcal{T}(V)$ we denote by $\mathbb{A}(V)$. Further we use $\mathcal{A}^{co}(V)$ to denote the set of all finite subsets from $\mathbb{A}(V)$ consisting of standard aggregates such that for two of them, say a, a', the set $\pi(a)\cup\pi(a')$ is disconnected.

It can be proven that for any aggregate a of $T \in \mathcal{T}(V)$ there exists one and only one $h = h(a)$ such that the shift $T_h a$ is in $\mathbb{A}(V)$. The shift $T_h a$ we call the aggregate a in a standard position. The mapping $\mathbb{A}(\cdot)$ that ascribes to a triplet $T \in \mathcal{T}(V)$ the set of its aggregates in standard positions is one-to-one mapping from $\mathcal{T}(V)$ onto $\mathcal{A}^{co}(V)$.

Now we have all tools to express the partition function $\tilde{Z}^{\xi}_V(I)$ in terms of an abstract contour model. The following lemma is a simple consequence of Lemma 2.3 and 2.4. We must only realize that the value of $f^{\diamond}_{V,I}(\mathbb{C})$ for a cluster $\mathbb{C}$ from aggregate a depends only on the part of the interface which is directly influenced by a_w. Thus, we can write $f^{\diamond}_{V,I(a_w)}(\mathbb{C})$ instead of $f^{\diamond}_{V,I(T_w)}(\mathbb{C})$ if $a_w \in T_w$.

Lemma 2.5 *Let us denote for every cylindrical set V with finite base and for every standard aggregate $a \in \mathbb{A}(V)$*

$$\Psi^V(a) = \prod_{w \in a_w} e^{-E(w)} \prod_{\diamond} \prod_{\mathbb{C} \in a_\diamond} f^{\diamond}_{V,I(a_w)}(\mathbb{C}). \tag{2.41}$$

Then under assumptions of Lemma 2.3, one has

(a)
$$\tilde{Z}^{\xi}_V = \mathcal{Z}(\mathbb{A}(V), \Psi_V),$$

where

$$\tilde{Z}^{\xi}_V = \frac{Z^{\xi}_V}{N^{\xi}_V} \tag{2.42}$$

and

$$\mathcal{Z}(\mathbb{A}(V), \Psi_V) = \sum_{\mathbb{R} \in \mathcal{A}(V)} \prod_{a \in \mathbb{R}} \Psi^V(a). \tag{2.43}$$

(b)
$$P^{\mathcal{I}}_V(I(\mathbb{V})) = \sum_{\substack{\mathbb{S} \in \mathcal{A}(V) \\ \mathbb{W}(\mathbb{S}) = \mathbb{V}}} \rho_{\mathbb{A}(V)}(\mathbb{S}, \Psi^V),$$

where

$$P^{\mathcal{I}}_V(I) = \frac{\tilde{Z}^{\xi}_V(I)}{\sum_{I' \in \mathcal{I}(V)} \tilde{Z}^{\xi}_V(I')} \tag{2.44}$$

and

$$\rho_{\mathbb{A}(V)}(\mathbb{S}, \Psi^V) = \Big[\prod_{a \in \mathbb{S}} \Psi_V(a)\Big] \Big/ \mathcal{Z}(\mathbb{A}(V), \Psi_V) \tag{2.45}$$

for $\mathbb{S} \in \mathcal{A}^{co}(V)$.

2.4 Properties of aggregate contour model

The main purpose of this section is to prove the assumptions of Theorem A.1 for a contour model in which aggregates play the role of contours and which has the contour functional Ψ^V . It will be proven in Lemma 2.7

Before proving this, we must note that although in the previous section we rewrite $\tilde{Z}^{\xi}_V(I)$ only for cylinders with a finite base the definitions of $f^{\diamond}_{V,I}$ and Ψ^V are all right for arbitrary cylindrical volume. Thus, we will use these two terms also for a cylindrical volume V with an infinite base.

For proof of Lemma 2.7 we need the following estimate.

Lemma 2.6 *Let the assumptions of Lemma 2.3 be fulfilled. Then*

$$\sum_{\substack{\mathbb{C} \in \mathcal{K}^{cl}_{\diamond} \\ i \in \mathbb{C}}} |f^{\diamond}_{V,I}(\mathbb{C})| \exp(\omega \|C\|) \leq \kappa \tag{2.46}$$

whenever

$$\omega \leq -\frac{2+\log 2}{m_\diamond} + \frac{\log \kappa}{m_\diamond} + \tau - \left[1 + \log(2c) + \frac{\log m_\diamond}{m_\diamond}\right] \tag{2.47}$$

and $\kappa \leq 2e^2$, V *cylindrical volume in* $\mathbb{Z}^d$, I *is an interface from* $\mathcal{I}(V)$ *and* $i \in \mathbb{Z}^d_\star$.

Proof: According to Proposition 1.3 and Theorem A.2 we have

$$\sum_{\substack{\mathbb{C}\in\mathcal{K}_q^{cl}\\ \mathbb{C}\ni i}} |\Phi_\diamond^T(\mathbb{C})| e^{\tilde{\omega}\|C\|} \leq 1 \tag{2.48}$$

with

$$\tilde{\omega} = \tau - \left[1 + \log(2c) + \frac{\log m_\diamond}{m_\diamond}\right]. \tag{2.49}$$

Using the facts that $|e^u - 1| \leq e^v|u|$ if $|u| \leq v$ and that $|\mathbb{C} \cap B| \geq m_\diamond$ for every cluster from $\mathcal{K}_\diamond^{cl}$ we have

$$\begin{aligned}
&\sum_{\substack{\mathbb{C}\in\mathcal{K}_\diamond^{cl}\\ \mathbb{C}\ni i}} |f^\diamond_{V,I}(\mathbb{C})| \exp(\omega\|C\|)\\
&\quad= \sum_{\substack{\mathbb{C}\in\mathcal{K}_\diamond^{cl}\\ \mathbb{C}\ni i}} \left|\exp\left\{-\Phi_\diamond^T(\mathbb{C})\left[\frac{|\mathbb{C}\cap\mathbb{B}_V^\diamond(I)|}{|\mathbb{C}\cap B|} - \chi_\diamond(\mathbb{C})\frac{|\mathbb{C}\cap\mathbb{B}_V|}{|\mathbb{C}\cap B|}\right]\right\} - 1\right|\\
&\qquad \times \exp(\omega\|C\|) && (2.50)\\
&\quad\leq \sum_{\substack{\mathbb{C}\in\mathcal{K}_\diamond^{cl}\\ \mathbb{C}\ni i}} 2e^2|\Phi_\diamond^T(\mathbb{C})|\exp(\omega\|C\|) && (2.51)\\
&\quad\leq 2e^2\exp[(\omega - \tilde{\omega})m_\diamond] \leq \kappa. && (2.52)
\end{aligned}$$

Lemma 2.7 *Let us define*

$$\zeta = \log(4c) + 1 + \log(3c_{d-1}) \tag{2.53}$$

and let the assumptions of Lemma 2.6 be fulfilled. Then

$$\sum_{\substack{a\in\mathbb{A}(V)\\ \pi(A)\ni i}} \exp(\|\pi(a)\| + \omega\|a\|)\Psi^V(a) \leq 1, \tag{2.54}$$

where

$$\|a\| = \sum_{w\in a_w} \|w\| + \sum_{\mathbb{C}\in a_d\cup a_o} \|\mathbb{C}\| \tag{2.55}$$

and $i \in I_0 \cap B$, *whenever*

$$\omega \leq \min\left[\tau - 1 - \log(2c) - \max_\diamond\left(\frac{\log m_\diamond}{m_\diamond} - \frac{2+\log 4}{m_\diamond}\right) - \zeta, \frac{\log q}{2d} - \zeta\right]. \tag{2.56}$$

Proof: We use $I_0^\star$ to denote the set $\{\, i \mid i \in I_0$, the distance of i from all lines (planes, etc.) composing $\pi(B)$ is odd multiple of $1/4\,\}$. This is the set of points where projections of contours and walls have their corners. Let us now consider the set $P \subset I_0^\star$. We say that this set is connected if and only if the set $\bar{P}$ that contains all points from P, all abscissas of length $1/2$ with both vertices in P, all squares with all sides in P, etc., is connected. We use $\mathcal{P}$ to denote the set of all finite connected subsets of $I_0^\star$, and $\|P\| = |\bar{P} \cap B|$.

If a_w is an admissible collection of walls from $\mathbb{W}(V)$ and $P \in \mathcal{P}$ we define $I(a_w, P) = I(a_w) \cap \tilde{\pi}(\bar{P})$. Then

$$\begin{aligned}
&\sum_{\substack{a\in\mathbb{A}(V)\\ \pi(A)\ni i}} \exp(\|\pi(a)\| + \omega\|a\|)\Psi^V(a) \\
&\qquad \le \sum_{\substack{P\in\mathcal{P}\\ P\ni i}} (\exp\|P\|) \sum_{\substack{a\in\mathbb{A}(V)\\ \pi(a)=\bar{P}}} \exp(\omega\|a\|) \exp\left\{-\sum_{w\in a_w} E(w)\right\} \\
&\qquad\quad \times \prod_{\diamond} \prod_{\mathbb{C}\in a_\diamond} f^\diamond_{V,I(a_w)}(\mathbb{C}) \qquad (2.57)\\
&\qquad \le \sum_{\substack{P\in\mathcal{P}\\ P\ni i}} (\exp\|P\|) \sum_{\substack{\pi(a_w)\subset\bar{P}\\ a_w\in\mathcal{W}(V)}} \exp\left[\left(\omega - \frac{\log q}{2d}\right)\|a_w\|\right] \\
&\qquad\quad \times \sum_{(a_o,a_d)\in\mathbb{F}(a_w,P)} \prod_{\diamond} \exp(\omega\|a_\diamond\|) \prod_{\mathbb{C}\in a_\diamond} |f^\diamond_{V,I(a_w)}(\mathbb{C})| = (1), \qquad (2.58)
\end{aligned}$$

where we used the definition of $E(w)$, the fact that $\|\pi(w)\| \ge 0$, and where we denoted by $\mathbb{F}(a_w, P)$ the set $\{\, (a_o, a_d) \mid (a_w, a_o, a_d) \in \mathbb{A}(V), \pi(a) = \bar{P}\,\}$. Using the fact that $\|a\| \ge \|I(a, P)\|$ whenever $\pi(a) = \bar{P}$ we get

$$\begin{aligned}
(1) &\le \sum_{\substack{P\in\mathcal{P}\\ P\ni i}} (\exp\|P\|) \sum_{\substack{\pi(a_w)\subset P\\ a_w\in\mathcal{W}(V)}} \exp\left[\left(\omega - \frac{\log q}{2d} + \zeta\right)\|a_w\| - \zeta\|I(a_w,P)\|\right] \\
&\quad \times \sum_{(a_o,a_d)\in\mathbb{F}(a_w,P)} \prod_{\diamond} \exp[(\omega+\zeta)\|a_\diamond\|] \prod_{\mathbb{C}\in a_\diamond} |f^\diamond_{V,I(a_w)}(\mathbb{C})| = (2). \qquad (2.59)
\end{aligned}$$

Since $\omega - (\log q)/(2d) + \zeta \le 0$, and according to the previous lemma, which we can use because $\omega + \zeta \le \tau - \left[1 + \log(2c) + \frac{\log m_\diamond}{m_\diamond}\right] - \frac{2+\log 4}{m_\diamond}$, i.e. we take $\kappa = \frac{1}{2}$, we have

$$\begin{aligned}
(2) &\le \sum_{\substack{P\in\mathcal{P}\\ P\ni i}} (\exp\|P\|) \sum_{\substack{\pi(a_w)\subset P\\ a_w\in\mathcal{W}(V)}} \exp[-\zeta\|I(a_w,P)\|] \\
&\quad \times \prod_{\diamond} \prod_{j\in I(a_w,P)\cap\mathbb{B}} \sum_{k=0}^{\infty} \Bigg(\sum_{\substack{\mathbb{C}\in\mathcal{K}^{cl}_\diamond\\ \mathbb{C}\ni j}} \{\exp[(\omega+\zeta)\|\mathbb{C}\|]\} |f^\diamond_{V,I(a_w)}(\mathbb{C})| \Bigg)^k \qquad (2.60)
\end{aligned}$$

$$\leq \sum_{\substack{P\in\mathcal{P}\\ P\ni i}} (\exp\|P\|) \sum_{\substack{\pi(a_w)\subset P\\ a_w\in\mathcal{W}(V)}} \exp[-(\zeta-\log 4)\|I(a_w,P)\|] = (3). \tag{2.61}$$

Now we will estimate the number of admissible families a_w of walls, such that $I(a_w, P)$ is connected, $\|I(a_w, P)\| = n$ and it contains a point from boundary of $\bar{P}$. We will bound it in similar way as a number of contours with length n by c^n. We will also use the fact that the "shortest" interface with these properties has the length $\|P\|$.

$$(3) \leq \sum_{\substack{P\in\mathcal{P}\\ P\ni i}} (\exp\|P\|) \sum_{n=\|P\|}^{\infty} \exp[-(\zeta-\log 4)n]c^n \tag{2.62}$$

$$= \sum_{\substack{P\in\mathcal{P}\\ P\ni i}} (\exp\|P\|) \frac{\exp[(\log(4c)-\zeta)\|P\|]}{1-\exp[\log(4c)-\zeta]} \tag{2.63}$$

$$\leq 2\sum_{\substack{P\in\mathcal{P}\\ P\ni i}} \exp[(1+\log(4c)-\zeta)\|P\|] = (4). \tag{2.64}$$

To bound the last expression we will use the estimate

$$|\{\, P\in\mathcal{P} \mid P\ni i, \|P\| = n \,\}| \leq c_{d-1}^n. \tag{2.65}$$

And thus

$$(4) \leq 2\sum_{n=1}^{\infty} c_{d-1}^n \exp[(1+\log(4c)-\zeta)n] = (5). \tag{2.66}$$

With respect to the definition of ζ we have

$$(5) = 2\frac{\exp[1+\log(4c)-\zeta+\log c_{d-1}]}{1-\exp[1+\log(4c)-\zeta+\log c_{d-1}]} \leq 1. \tag{2.67}$$

$\square$

In the previous two lemmas we verified the assumptions of Theorem A.2 for every contour model with contour functional Ψ^V. The bounds we found are even independent on volume V, so we can work without fear with these contour models also for infinite-base cylinders.

The following proposition is a version of Theorem A.2 for the aggregate contour model. Before stating it we recall that

$$\tau = \frac{\log q}{2d} - 4d\varepsilon. \tag{2.68}$$

In order to make the notation shorter, we define a constant

$$c_1 = 1 + 4d\varepsilon + \log(2c) + \max_{\diamond}\left[\frac{\log m_\diamond}{m_\diamond} + \frac{2+\log 4}{m_\diamond}\right] + \zeta. \tag{2.69}$$

To have all things properly prepared, we have to define an incompatibility of aggregates. We say that two aggregates a_1 and a_2 are incompatible if and only if $\pi(a_1) \cup \pi(a_2)$ is connected. We use $\mathcal{A}^{cl}$ to denote the set of all collections of aggregates, such collections which are not decomposable into two compatible sets.

Proposition 2.8 *Let*

$$\frac{\log q}{2d} \geq c_1 \tag{2.70}$$

and let $V \subset \mathbb{Z}^d$ be a cylindrical set. Then there exists the unique function $\Psi^{V,T} : \mathcal{A} \mapsto \mathbb{R}$ such that

$$\log \mathcal{Z}(\mathbb{D}, \Psi^V) = \sum_{\mathbb{C} \in \mathcal{A}^{cl}(\mathbb{D})} \Psi^{V,T}(\mathbb{C}) \tag{2.71}$$

for every $\mathbb{D} \subset \mathbb{A}(U)$, where U is a cylinder with finite base. For every $i \in I_0 \cap \mathbb{Z}^d_\star$ we also have

$$\sum_{\substack{\pi(\mathbb{C}) \ni i \\ \mathbb{C} \in \mathcal{A}^{cl}}} |\Psi^{V,T}(\mathbb{C})| e^{\omega \|C\|} \leq 1 \tag{2.72}$$

with

$$\omega \leq \frac{\log q}{2d} - c_1. \tag{2.73}$$

Moreover,

$$\Psi^{V,T}(\mathbb{C}) = \sum_{\mathbb{D} \subset \mathbb{C}} (-1)^{\|\mathbb{D}\| - \|\mathbb{C}\|} \log \mathcal{Z}(\mathbb{D}, \Psi^V) \tag{2.74}$$

for every $\mathbb{C} \in \mathcal{A}^{cl}$.

For every $\mathbb{S} \in \mathcal{A}^{cl,f}$ there exists the unique function $\Delta^V_{\mathbb{S}} : \mathcal{A}^f \mapsto \mathbb{C}$ such that

$$\rho_{\mathbb{D}}(\mathbb{S}, \Psi^V) = \sum_{\mathbb{C} \in \mathcal{A}^{cl}(\mathbb{D})} \Delta^V_{\mathbb{S}}(\mathbb{C}) \tag{2.75}$$

for every $\mathbb{C} \subset \mathbb{A}$ and that

$$\sum_{\mathbb{C} \supset \mathbb{S}} |\Delta^V_{\mathbb{S}}(\mathbb{C})| e^{\omega \|C\|} \leq e^{\|\pi(\mathbb{S})\|} \Big| \prod_{a \in \mathbb{S}} \Psi^V(a) \Big|. \tag{2.76}$$

The function $\Delta^V_{\mathbb{S}}$ is given by

$$\Delta^V_{\mathbb{S}}(\mathbb{C}) = \sum_{\mathbb{G} \subset \mathbb{C}} (-1)^{\|\mathbb{G}\| - \|\mathbb{C}\|} \rho_{\mathbb{G}}(\mathbb{S}, \Psi^V) \tag{2.77}$$

and

$$\Delta^V_{\mathbb{S}_1 \cup \mathbb{S}_2}(\mathbb{C}_1 \cup \mathbb{C}_2) = \Delta^V_{\mathbb{S}_1}(\mathbb{C}_1) \Delta^V_{\mathbb{S}_2}(\mathbb{C}_2), \tag{2.78}$$

whenever every $a_1 \in \mathbb{S}_1 \cup \mathbb{C}_1$ is compatible with every $a_2 \in \mathbb{S}_2 \cup \mathbb{C}_2$.

Proof: According to assumptions of this proposition and according to the previous lemma, the assumptions of Theorem A.1 are fulfilled when we take $a(a) = \|\pi(a)\|$, $l(a) = \|a\|$ and $\|C\| = \sum_{a \in \mathbb{C}} \|a\|$. Then all statements of this proposition come from Theorem A.1. □

At the end of this section we notice that in respect to (2.77) and the definition of $\rho_{\mathbb{D}}(\mathbb{S}, \Psi^V)$ in Lemma 2.5, the sum in (2.75) can run only over such $\mathbb{C}$ for that $\mathbb{C} \supset \mathbb{S}$, because for other $\mathbb{C}$ is $\Delta^V_{\mathbb{S}}(\mathbb{C}) = 0$.

2.5 The existence of limit measure

In the previous section we found out that the aggregate contour model is a well behaving contour model. However, our main aim is to describe a limit random-cluster measure, primarily to describe the behaving of walls on the interface between ordered and disordered phases because we control the behaviour of these phases due to Proposition 1.8.

First we state an auxiliary lemma needed for following proofs.

Lemma 2.9 *Let*

$$\frac{\log q}{2d} \geq c_1 \tag{2.79}$$

and

$$\omega \leq \frac{\log q}{2d} - c_1 \tag{2.80}$$

and let $\mathcal{S} \subset \mathcal{A}^{co}$, $\mathcal{C} \subset \mathcal{A}^{cl}$, and a finite $M \subset I_0 \cap \mathbb{Z}^d_\star$. Let $a \in \mathbb{S}$ implies $\pi(a) \cap M \neq \emptyset$ whenever $\mathbb{S} \in \mathcal{S}$. Then

$$\sum_{\mathbb{S} \in \mathcal{S}} \sum_{\mathbb{C} \in \mathcal{C}} |\Delta^V_{\mathbb{S}}(\mathbb{C})| \leq 2^{|M|} \exp[-\omega(\inf_{\mathcal{C}} \|\mathbb{C}\| + \inf_{\mathcal{S}} \|\mathbb{S}\|)]. \tag{2.81}$$

Proof:

$$\sum\sum |\Delta^V_{\mathbb{S}}(\mathbb{C})| \leq \sum\sum [\exp(-\omega \inf_{\mathcal{C}} \|\mathbb{C}\|) \exp(\omega \|\mathbb{C}\|)] |\Delta^V_{\mathbb{S}}(\mathbb{C})| \tag{2.82}$$

$$\leq [\exp(-\omega \inf_{\mathcal{C}} \|\mathbb{C}\|)] \Big\{ \sum_{\mathbb{S}} [\exp \pi(\mathbb{S})] \prod_{a \in \mathbb{S}} |\Psi^U(a)| \Big\} \tag{2.83}$$

according to Proposition 2.8,

$$\leq \Big\{ \sum_{\mathbb{S}} [\exp \pi(\mathbb{S}) \exp(\omega \|\mathbb{S}\|)] \prod_{a \in \mathbb{S}} |\Psi^U(a)| \Big\}$$
$$\times \exp[-\omega(\inf_{\mathcal{C}} \|\mathbb{C}\| + \inf_{\mathcal{S}} \|\mathbb{S}\|)] \tag{2.84}$$

$$\leq \prod_{i \in M} \Big\{ 1 + \sum_{a : i \in \pi(a)} \exp[\|\pi(a)\| + \omega \|a\|] \, |\Psi^U(a)| \Big\}$$
$$\times \exp[-\omega(\inf_{\mathcal{C}} \|\mathbb{C}\| + \inf_{\mathcal{S}} \|\mathbb{S}\|)] \tag{2.85}$$

$$\leq 2^{|M|} \exp[-\omega(\inf_{\mathcal{C}} \|\mathbb{C}\| + \inf_{\mathcal{S}} \|\mathbb{S}\|)] \tag{2.86}$$

according to Proposition 2.8 again. □

In Lemma 2.5 we found the probability

$$P_U^{\mathcal{I}} = \sum_{\substack{\mathbb{S} \in \mathcal{A}(U) \\ \mathbb{W}(\mathbb{S}) = \mathbb{V}}} \rho_{\mathbb{A}(U)}(\mathbb{S}, \Psi^U) \tag{2.87}$$

on $\mathcal{I}(U)$, where U was a cylinder with finite base. Using this probability we now define correlations $\rho_U^{\mathcal{I}}(\mathbb{V})$ for $\mathbb{V} \in \mathcal{W}^{f,co}(U)$ by

$$\rho_U^{\mathcal{I}}(\mathbb{V}) = P_U^{\mathcal{I}}(\{ I \in \mathcal{I}(U) \mid \mathbb{W}(I) \supset \mathbb{V} \}). \tag{2.88}$$

In the following lemma we will show some properties of these correlations.

Lemma 2.10 *Let*

$$\frac{\log q}{2d} \geq c_1 \tag{2.89}$$

and

$$\omega \leq \frac{\log q}{2d} - c_1 \tag{2.90}$$

then we have

(a) $\rho_U^{\mathcal{I}}(\mathbb{V}) \leq \exp[-(2\omega - \log 2)\|\mathbb{V}\|]$

(b) $|\rho_{U_1}^{\mathcal{I}}(\mathbb{V}) - \rho_{U_2}^{\mathcal{I}}(\mathbb{V})| \leq 4 \exp[-(\omega - \log 2)\|\mathbb{V}\| - \omega d(\mathbb{V}, U_1 \div U_2)]$

whenever U_1, U_2 are cylinders with finite base, $U_1 \div U_2$ is their symmetrical difference and $\mathbb{V} \in \mathcal{W}^{co,f}(U_1 \cap U_2)$.

Proof: Let us denote the set

$$\{ \mathbb{S} \in \mathcal{A}^{co}(U) \mid \mathbb{W}(\mathbb{S}) \supset \mathbb{V}, a \in \mathbb{S} \Rightarrow \mathbb{V} \cap \mathbb{W}(a) \neq \emptyset \} \tag{2.91}$$

by $\mathcal{A}^{co}(U, \mathbb{V})$. Then, due to Lemma 2.5,

$$\rho_U^{\mathcal{I}}(\mathbb{V}) = \sum_{\mathbb{S} \in \mathcal{A}^{co}(U,\mathbb{V})} \rho_{\mathbb{A}(U)}(\mathbb{S}, \Psi^U). \tag{2.92}$$

(a) According to Proposition 2.8 and Lemma 2.9 we can write

$$\rho_U^{\mathcal{I}}(\mathbb{V}) = \sum_{\mathbb{S} \in \mathcal{A}^{co}(U,\mathbb{V})} \sum_{\substack{\mathbb{C} \in \mathcal{A}^{cl} \\ \mathbb{C} \subset \mathbb{A}(U)}} \Delta_{\mathbb{S}}^U(\mathbb{C}) \leq \exp[-(2\omega - \log 2)\|\mathbb{V}\|], \tag{2.93}$$

because $\mathbb{S} \in \mathcal{A}^{co}(U, \mathbb{V})$ implies $[a \in \mathbb{S} \Rightarrow \pi(a) \cap \pi(\mathbb{V}) \neq \emptyset]$, $\Delta_{\mathbb{S}}^U(\mathbb{C}) \neq 0 \Rightarrow \mathbb{C} \supset \mathbb{S}$, $\|\mathbb{S}\| \geq \|\mathbb{V}\|$ for $\mathbb{S} \in \mathcal{A}^{co}(U, \mathbb{V})$, and $\|\pi(\mathbb{V})\| \leq \|\mathbb{V}\|$.

(b) We first express the difference $\rho^{\mathcal{I}}_{U_1}(\mathbb{V}) - \rho^{\mathcal{I}}_{U_2}(\mathbb{V})$ using $\mathcal{A}^{co}(U,\mathbb{V})$:

$$\rho^{\mathcal{I}}_{U_1}(\mathbb{V}) - \rho^{\mathcal{I}}_{U_2}(\mathbb{V}) = \sum_{\mathbb{S}\in\mathcal{A}^{co}(U_1,\mathbb{V})} \rho_{\mathbb{A}(U_1)}(\mathbb{S},\Psi^U_1) - \sum_{\mathbb{S}\in\mathcal{A}^{co}(U_2,\mathbb{V})} \rho_{\mathbb{A}(U_2)}(\mathbb{S},\Psi^U_2) \tag{2.94}$$

$$= \sum_{\mathbb{S}\in\mathcal{A}^{co}(U_1,\mathbb{V})} \sum_{\substack{\mathbb{C}\in\mathcal{A}^{cl}(U_1)\\(\mathbb{C}\supset\mathbb{S})}} \Delta^{U_1}_{\mathbb{S}}(\mathbb{C}) - \sum_{\mathbb{S}\in\mathcal{A}^{co}(U_2,\mathbb{V})} \sum_{\substack{\mathbb{C}\in\mathcal{A}^{cl}(U_2)\\(\mathbb{C}\supset\mathbb{S})}} \Delta^{U_2}_{\mathbb{S}}(\mathbb{C}). \tag{2.95}$$

Since $\Delta^{U_1}_{\mathbb{S}}(\mathbb{C}) = \Delta^{U_2}_{\mathbb{S}}(\mathbb{C})$ for $\mathbb{C}\subset U_1\cap U_2$, $\mathbb{S}\subset U_1\cap U_2$ in accordance to Proposition 2.8 we get

$$\rho^{\mathcal{I}}_{U_1}(\mathbb{V}) - \rho^{\mathcal{I}}_{U_2}(\mathbb{V}) = \sum_{\mathbb{S}\in\mathcal{A}^{co}(U_1\cap U_2,\mathbb{V})} \sum_{\substack{\mathbb{C}\in\mathcal{A}^{cl}(U_1)\\ \mathbb{C}\notin\mathcal{A}^{cl}(U_2)}} \Delta^{U_1}_{\mathbb{S}}(\mathbb{C}) - \sum_{\mathbb{S}\in\mathcal{A}^{co}(U_1\cap U_2,\mathbb{V})} \sum_{\substack{\mathbb{C}\notin\mathcal{A}^{cl}(U_1)\\ \mathbb{C}\in\mathcal{A}^{cl}(U_2)}} \Delta^{U_2}_{\mathbb{S}}(\mathbb{C})$$

$$+ \sum_{\substack{\mathbb{S}\in\mathcal{A}^{co}(U_1,\mathbb{V})\\ \mathbb{S}\notin\mathcal{A}^{co}(U_2,\mathbb{V})}} \sum_{\mathbb{C}\in\mathcal{A}^{cl}(U_1)} \Delta^{U_1}_{\mathbb{S}}(\mathbb{C}) - \sum_{\substack{\mathbb{S}\notin\mathcal{A}^{co}(U_1,\mathbb{V})\\ \mathbb{S}\in\mathcal{A}^{co}(U_2,\mathbb{V})}} \sum_{\mathbb{C}\in\mathcal{A}^{cl}(U_2)} \Delta^{U_2}_{\mathbb{S}}(\mathbb{C}). \tag{2.96}$$

When we estimate each of these four sums using Lemma 2.9 as in the proof of (a) we get

$$|\rho^{\mathcal{I}}_{U_1}(\mathbb{V}) - \rho^{\mathcal{I}}_{U_2}(\mathbb{V})| \le 4\exp[-(\omega - \log 2)\|\mathbb{V}\| - \omega d(\mathbb{V}, U_1 \div U_2)]. \tag{2.97}$$

□

Further more, we will use the symbol lim fin cyl to denote the limit over directed sequence of cylinders with a finite base.

Lemma 2.11 *Let V cylinder not necessarily with a finite base and let the assumptions of Lemma 2.10 hold true. Then there exists a probability measure $P^{\mathcal{I}}_V$ on $\mathcal{I}(V)$ that recovers its correlation function $\rho^{\mathcal{I}}_V(\mathbb{V}) = P^{\mathcal{I}}_V(\{\, I\in\mathcal{I}(V) \mid \mathbb{W}(I)\supset\mathbb{V}\,\})$. Moreover*

$$\rho^{\mathcal{I}}_V(\mathbb{V}) = \operatorname*{lim\,fin\,cyl}_{U\nearrow V} \rho^{\mathcal{I}}_U(\mathbb{V}), \tag{2.98}$$

$P^{\mathcal{I}}_U$ converges weakly to $P^{\mathcal{I}}_V$ and $\rho^{\mathcal{I}}_V$ has all properties have been proven for cylinder with finite base in Lemma 2.10.

Proof: The existence of limit $\rho^{\mathcal{I}}_V = \text{lim fin cyl}_{U\nearrow V}\,\rho^{\mathcal{I}}_U$ is a simple consequence of Lemma 2.10 (b). Naturally the limit satisfies the properties (a),(b) from that lemma.

We must now prove a weak convergence of $P^{\mathcal{I}}_U$ to finish the proof. Thanks to Lemma 2.1, the probabilities $P^{\mathcal{I}}_U$ can be understood as probabilities on $\mathcal{W}^{co}$, in fact on $\mathcal{W}^a = \mathcal{W}^{co}$ for cylinders U with finite bases.

The probability $P_V^{\mathcal{I}}$ is defined uniquely by its values on sets of the form

$$\mathcal{M}_{M,\mathbb{V}} = \{\, \mathbb{V}' \in \mathcal{W}^{co}(V) =\mid \mathbb{V}' \cap M = \mathbb{V} \,\} \tag{2.99}$$

for finite sets $M \in \mathcal{W}(V)$ and $\mathbb{V} \in \mathcal{W}^{co}(V)$. Since

$$\mathcal{M}_{M,\mathbb{V}} = \mathcal{M}_{\mathbb{V}} \Big\backslash \bigcup_{w \in M \setminus \mathbb{V}} \mathcal{M}_{\mathbb{V} \cup \{w\}}, \tag{2.100}$$

where $\mathcal{M}_{\mathbb{V}} = \{\, \mathbb{V}' \in \mathcal{W}^{co}(V) \mid \mathbb{V}' \supset \mathbb{V} \,\}$ for $\mathbb{V} \subset M$, and

$$\bigcap_{w \in \mathbb{V}'} \mathcal{M}_{\mathbb{V} \cup \{w\}} = \begin{cases} \mathcal{M}_{\mathbb{V} \cup \mathbb{V}'} & \text{for } \mathbb{V} \cup \mathbb{V}' \text{ compatible} \\ \emptyset & \text{otherwise} \end{cases} \tag{2.101}$$

one has for any probability P on $\mathcal{W}^{co}(V)$

$$\begin{aligned} P(\mathcal{M}_{M,\mathbb{V}}) &= P(\mathcal{M}_{\mathbb{V}}) - P\Big(\bigcup_{w \in M \setminus \mathbb{V}} \mathcal{M}_{\mathbb{V} \cup \{w\}} \Big) && (2.102) \\ &= P(\mathcal{M}_{\mathbb{V}}) - \sum_{\substack{\mathbb{V}' \subset M \setminus \mathbb{V} \\ \mathbb{V}' \neq \emptyset}} (-1)^{|\mathbb{V}'|+1} P(\mathcal{M}_{\mathbb{V} \cup \mathbb{V}'}) && (2.103) \\ &= \sum_{\mathbb{V}' \subset M \setminus \mathbb{V}} (-1)^{|\mathbb{V}'|} P(\mathcal{M}_{\mathbb{V} \cup \mathbb{V}'}). && (2.104) \end{aligned}$$

Thus it suffices to verify the convergence $P_U^{\mathcal{I}} \to P_V^{\mathcal{I}}$ only on $\mathcal{M}_{\mathbb{V}}$. However, by definition of $\mathcal{M}_{\mathbb{V}}$ we have

$$P_U^{\mathcal{I}}(\mathcal{M}_{\mathbb{V}}) = \rho_V^{\mathcal{I}}(\mathbb{V}), \tag{2.105}$$

and thus

$$P_U^{\mathcal{I}}(\mathcal{M}_{\mathbb{V}}) = \rho_U^{\mathcal{I}}(\mathbb{V}) \to \rho_V^{\mathcal{I}}(\mathbb{V}) = P_V^{\mathcal{I}}(\mathcal{M}_{\mathbb{V}}). \tag{2.106}$$

□

The following proposition describes a very important property of $P_V^{\mathcal{I}}$.

Proposition 2.12 *Let*

$$\frac{\log q}{2d} \geq c_1 + \frac{1}{2} \log 4c_{d-1}. \tag{2.107}$$

Then

$$P_V^{\mathcal{I}}(\mathcal{W}^a(V)) = P_V^{\mathcal{I}}(\mathcal{I}^a(V)) = 1. \tag{2.108}$$

Proof: It is simple to notice that $P_V^{\mathcal{I}}(\mathcal{W}^{co}(V)) = 1$, since the set $\mathcal{W}(V) \setminus \mathcal{W}^{co}(V)$ is covered by the countable union of sets of collections of walls that are "incompatible at some $i \in \mathbb{Z}_\star^d$."

To verify that $P_V^{\mathcal{I}}(\mathcal{W}^a(V)) = 1$ we use a usual proof. First, we consider the half-line p from a fixed $i \in I_0$, $i \in \mathbb{Z}_\star^d$ parallel to a fixed coordinate axis in $\mathbb{Z}^d$ such that $p \subset I_0$ and the wall w such that $i \in \mathrm{Int}_{I_0} \pi(w)$ and $\|w\| = n$. There are less than n possibilities for p to intersect $\pi(w)$, because $\|\pi(w)\| \leq \|w\|$, and thus

$$\begin{aligned} &P_V^{\mathcal{I}}(\{ \mathbb{V} \mid i \in \mathrm{Int}_{I_0} \pi(w), \|w\| = n, w \in \mathbb{V} \}) \\ &\qquad \leq n P_V^{\mathcal{I}}(\{ \mathbb{V} \mid i \in \pi(w), \|w\| = n, w \in \mathbb{V} \}). \end{aligned} \tag{2.109}$$

Hence, the probability that a site i is inside at least n walls may be bound by

$$\sum_{m=n}^{\infty} P_V^{\mathcal{I}}(\{ \mathbb{V} \mid i \in \mathrm{Int}_{I_0} \pi(w), \|w\| = m, w \in \mathbb{V} \})$$

$$\leq \sum_{m=n}^{\infty} m \sum_{\substack{w;\pi(w)\ni i \\ \|w\|=m}} \rho_V^{\mathcal{I}}(w) \tag{2.110}$$

$$\leq \sum_{m=n}^{\infty} m[c(d-1)]^m \exp[-(2\omega - \log 2)m] \tag{2.111}$$

$$\leq \sum_{m=n}^{\infty} 2^m [c(d-1)]^m \exp[-(2\omega - \log 2)m] \tag{2.112}$$

$$= \frac{\exp[-(2\omega - \log 4c_{d-1})n]}{1 - \exp[-(2\omega - \log 4c_{d-1})]}, \tag{2.113}$$

since the length of the n^{th} wall "encircling" i is at least n. Then the probability that the site i is "encircled" by an infinite number of wall is bound by

$$\lim_{n\to\infty} \frac{\exp[-(2\omega - \log 4c_{d-1})n]}{1 - \exp[-(2\omega - \log 4c_{d-1})]} = 0 \tag{2.114}$$

and so $P_V^{\mathcal{I}}(\mathcal{W}^a(V)) = 1$. □

We proved that there exists almost surely the interface between ordered and disordered phase in an arbitrary cylinder in the random-cluster model with boundary condition ξ. Since in Chapter 1 we proved that in every volume there are almost surely only finite contours enclosing the islands of opposite phase, we can dedicate from these two facts that there is almost surely only one infinite ordered component. This fact will be very important in the next chapter.

In the following proposition we will prove the existence of the limit random-cluster measure obtained as a limit $\lim\mathrm{fin\,cyl}_{U\nearrow V}\, \mu_U^{\xi}$.

Proposition 2.13 *Let the assumption of Proposition 2.12 holds true then the limit*

$$\mu(\cdot) = \lim_{U\nearrow V}\mathrm{fin\,cyl}\, \mu_U^{\xi}(\cdot) \tag{2.115}$$

exists and is translation non-invariant.

Proof: Let φ be a cylindrical function living on the B_Λ. From Lemma 2.11 we know that for arbitrary cylinder V there is a probability measure $P_V^{\mathcal{I}}$ on $\mathcal{I}(V)$ and due to Proposition 2.12 there is almost surely the interface and thus we can write

$$\mu_V^\xi(\varphi) = \int_{I\in\mathcal{I}(V)} \mu_V(\varphi \mid I)\, dP_V^{\mathcal{I}}(I) \tag{2.116}$$

for every cylinder. The conditional probability $\mu_V(\varphi \mid I)$ has a sense for arbitrary cylinder because it is a product of measures in upper and lower part of cylinder with ordered and disordered boundary condition and these measures exist according to Proposition 1.8.

Thus, for showing this theorem is true we must prove

$$\Big|\int_{I_U\in\mathcal{I}(U)} \mu_U(\varphi \mid I_U)\, dP_U^{\mathcal{I}}(I_U) - \int_{I_V\in\mathcal{I}(V)} \mu_V(\varphi \mid I_V)\, dP_V^{\mathcal{I}}(I_V)\Big| \le \varepsilon \tag{2.117}$$

for U, V large enough, both with finite base.

With respect to the note about $\mu_V(\varphi \mid I)$ we can write

$$\int_{I_U\in\mathcal{I}(U)} |\mu_U(\varphi \mid I_U) - \mu_V(\varphi \mid I_U)| dP_U^{\mathcal{I}}(I_U) \le \frac{\varepsilon}{2}. \tag{2.118}$$

To prove that

$$\Big|\int_{I_U\in\mathcal{I}(U)} \mu_V(\varphi \mid I_U)\, dP_U^{\mathcal{I}}(I_U) - \int_{I_V\in\mathcal{I}(V)} \mu_V(\varphi \mid I_V)\, dP_V^{\mathcal{I}}(I_V)\Big| \le \frac{\varepsilon}{2} \tag{2.119}$$

we define the function

$$f_k(I) = \chi_{k,\Lambda}(I)\mu_V(\varphi \mid I) \tag{2.120}$$

for every $k \in \mathbb{N} \cup \infty$, and where $\chi_{k,\Lambda}(I) = 1$ when for every wall from $\mathbb{W}(I)$ that intersects $\tilde{\pi}(B_\Lambda)$ holds $\|w\| \le k$, and $\chi_{k,\Lambda}(I) = 0$ otherwise. Using Lemma 2.10 (a) we can show that the limit

$$\lim_{k\to\infty} \int_{I_{\bar V}\in\mathcal{I}(\bar V)} f_k(I) dP_{\bar V}^{\mathcal{I}}(I_{\bar V}) \tag{2.121}$$

exists and is equal to

$$\int_{I_{\bar V}\in\mathcal{I}(\bar V)} f_\infty(I) dP_{\bar V}^{\mathcal{I}}(I_{\bar V}) = \int_{I_{\bar V}\in\mathcal{I}(\bar V)} \mu_V(\varphi \mid I_{\bar V})\, dP_{\bar V}^{\mathcal{I}}(I_{\bar V}) \tag{2.122}$$

for every finite-base cylinder $\bar V \subseteq V$ large enough. For I' converging to I we have $f_k(I')$ converges to $f_k(I)$, where I, $I' \in \mathcal{I}^a$ because $d(B_\Lambda, U_\diamond(I') \div U_\diamond(I))$ converges to infinity and we can use Lemma 2.10 (b). Hence, f_k is continuous and according to the weak convergence of $P_U^{\mathcal{I}}$ demonstrated in Lemma 2.11 we have

$$\Big|\int_{I_U\in\mathcal{I}(U)} f_k(I_U)\, dP_U^{\mathcal{I}}(I_U) - \int_{I_V\in\mathcal{I}(V)} f_k(I_V)\, dP_V^{\mathcal{I}}(I_V)\Big| \le \frac{\varepsilon}{2}. \tag{2.123}$$

The existence of limit random-cluster measure comes from (2.118)–(2.123). The assertion that this limit is not translation invariant is a simple consequence of existence of interface. □

Chapter 3

Proof of DLR property of translation non-invariant measure

In Section 1.2 we showed two possibilities of defining the infinite-volume random-cluster measure. In the previous chapter we found, using the technique from [5], that there exists a limit random-cluster measure in $\mathbb{Z}^d$ with boundary condition ξ. The aim of this chapter is to prove that the measure we have obtained fulfills the condition of the second definition.

Lemma 3.1 *Let μ^ξ be the limit random-cluster measure that we have found in the previous chapter, $\Lambda \subset \mathbb{Z}^d$ finite, A be a cylinder event defined in terms of edges from B_Λ and let us write μ_Λ instead of μ_{B_Λ}. Then the random variable $g(\omega) = \mu_\Lambda^\omega(A)$ is μ^ξ-a.s. continuous, using the product topology on its domain Ω.*

Proof: As we have mentioned there is μ^ξ-a.s. the only one infinite ordered component. Let us define the discontinuity set of the random variable $g(\omega)$ by

$$D = \bigcap_{\Delta} \Big\{ \omega \mid \sup_{\zeta : \zeta_{B_\Delta} = \omega_{B_\Delta}} |g(\zeta) - g(\omega)| > 0 \Big\}, \tag{3.1}$$

where the intersection is over all $\Delta \subset \mathbb{Z}^d$ containing Λ. For any ζ the difference $|g(\zeta) - g(\omega)|$ can be nonzero if and only if there exist two points i, $j \in \Lambda \cap \mathcal{B}_\Lambda = \partial\Lambda$ such that both i and j are joined to $\partial\Delta$ by a path using ordered edges of ω lying in $B_\Delta \setminus B_\Lambda$, but i is not joined to j by such path. Note that if this event occurs for no i, j, then $\bar{D}_{B_\Lambda}(\omega') = \bar{D}_{B_\Lambda}(\omega)$ for all ω' which agree with ω on B_Δ, so that $g(\zeta) = g(\omega)$. Denoting the last event by $D_{\Lambda,\Delta}$ we have that

$$D \subseteq \bigcap_{\Delta} D_{\Lambda,\Delta}. \tag{3.2}$$

Therefore,

$$\mu^\xi(D) \le \mu^\xi\Big(\bigcap_{\Delta} D_{\Lambda,\Delta} \Big). \tag{3.3}$$

However,

$$\bigcap_{\Delta} D_{\Lambda,\Delta} \subseteq \{(B_\Lambda)^C \text{ contains two or more infinite ordered components}\}, \tag{3.4}$$

is an event with zero probability as we have proved. □

Theorem 3.2 *The limit random-cluster measure μ^ξ have been found in the previous chapter as a limit*

$$\mu^\xi = \lim_{\Lambda \to \mathbb{Z}^d} \mu^\xi_\Lambda \tag{3.5}$$

is a random-cluster measure i.e.

$$\mu^\xi(A \mid \mathscr{T}_\Lambda) = \mu^{\cdot}_\Lambda(A), \quad \mu^\xi\text{-a.s., for all } A \in \mathscr{F} \text{ and finite } \Lambda. \tag{3.6}$$

Proof: Let A be a cylinder event defined in terms of the states of bonds in B_Λ and let $\Delta \supset \Lambda$. For a finite Δ we first prove that

$$\mu^{\cdot}_\Lambda(A) = \mu^\xi_\Delta(A \mid \mathscr{T}_\Lambda), \qquad \mu^\xi_\Delta\text{-a.s.} \tag{3.7}$$

We can rewrite the random-cluster measure μ^ξ_Δ using the definition of finite volume random-cluster measure (1.12) and the notation $w_F(X) = p^{|X|}(1-p)^{|F \setminus X|}$, where F is an arbitrary set of bonds from B,

$$\begin{aligned}
\mu^\xi_\Delta(A \mid \mathscr{T}_\Lambda) &= \frac{\sum_{X_\Delta} \chi_A(X) w_{B_\Delta}(X) q^{\bar{D}^\xi_\Delta(X)}}{\sum_{X_\Delta} w_{B_\Delta}(X) q^{\bar{D}^\xi_\Delta(X)}} & (3.8)\\
&= \frac{\sum_{X_{\Delta\setminus\Lambda}} w_{(B_\Delta \setminus B_\Lambda)}(X) \sum_{X'_\Lambda} \chi_A(X) w_{B_\Lambda}(X') q^{\bar{D}^\xi_\Delta(X \circ X')}}{\sum_{X_{\Delta\setminus\Lambda}} w_{(B_\Delta \setminus B_\Lambda)}(X) \sum_{X'_\Lambda} w_{B_\Lambda}(X') q^{\bar{D}^\xi_\Delta(X \circ X')}} & (3.9)\\
&= \frac{q^{\bar{D}_{\Lambda,\Delta}(X|\xi)} \sum_{X'_\Lambda} \chi_A(X) w_{B_\Lambda}(X') q^{\bar{D}^\xi_\Lambda(X \circ X')}}{q^{\bar{D}_{\Lambda,\Delta}(X|\xi)} \sum_{X'_\Lambda} w_{B_\Lambda}(X') q^{\bar{D}^\xi_\Lambda(X \circ X')}} = \mu^{\cdot}_\Lambda(A), & (3.10)
\end{aligned}$$

where $X_\Delta = X_{B_\Delta}$ is a subset of B_Δ, $X_{\Delta\setminus\Lambda} = X_{B_{\Delta\setminus\Lambda}}$ subset of $B_{\Delta\setminus\Lambda}$, χ_A is the characteristic function of event A and we use $\bar{D}^\xi_{\Lambda,\Delta}(X)$ to denote the number of components of graph $(\mathbb{Z}^d, X_{B_\Delta} \circ \xi_{B^C_\Delta})$ that intersect at least one bond from B_Δ but no one from B_Λ.

Let us consider a cylinder event B in $\mathscr{T}_\Lambda$. According to Lemma 3.1 the function $\chi_B(X)\mu^X_\Lambda(A)$ is μ^ξ-a.s. continuous. Hence,

$$\begin{aligned}
\mu^\xi(\chi_B(\cdot)\mu^{\cdot}_\Lambda(A)) &= \lim_{\Delta \nearrow \mathbb{Z}^d} \mu^\xi_\Delta(\chi_B(\cdot)\mu^{\cdot}_\Lambda(A)) & (3.11)\\
&= \lim_{\Delta \nearrow \mathbb{Z}^d} \mu^\xi_\Delta(\chi_B(\cdot)\mu^\xi_\Delta(A \mid \mathscr{T}_\Lambda)) & (3.12)\\
&= \lim_{\Delta \nearrow \mathbb{Z}^d} \mu^\xi_\Delta(A \cap B) = \mu^\xi(A \cap B). & (3.13)
\end{aligned}$$

where the last but one equality can be proven from the definition of finite volume random-cluster measure in a similar way as used in proof of (3.7).

Since $\mathscr{T}_\Lambda$ is generated by the collection of all such B, we deduce that

$$\mu^\xi(A \mid \mathscr{T}_\Lambda) = \mu^{\cdot}_\Lambda(A), \quad \mu^\xi\text{-a.s., for all } A \in \mathscr{F} \text{ and finite } \Lambda. \tag{3.14}$$

Appendix A

Polymer models and cluster expansion

In this appendix we will summarize standard facts about contour (polymer) models and cluster expansion. The first step will be formulating them in a very abstract form. Throughout the whole Appendix A we follow the article [7].

We will consider a countable set $\mathbb{K}$ of polymers. Let ι be a reflexive and symmetric relation. We call a pair γ_1, γ_2 incompatible (compatible) if and only if $(\gamma_1, \gamma_2) \in \iota$ ($(\gamma_1, \gamma_2) \notin \iota$). We use notation $\gamma_1 \iota \gamma_2$ for incompatible polymers. By $\mathcal{K}^{co}$, $\mathcal{K}^{co,f}$ we denote the set of all (finite) collections $\partial \subset \mathbb{K}$ of mutually compatible polymers. Considering a contour functional $\Phi : \mathbb{K} \mapsto \mathbb{C}$, we denote $\Phi(\partial) = \prod_{\gamma \in \partial} \Phi(\gamma)$ for each $\partial \in \mathcal{K}^{co,f}$

For each finite $\mathbb{L} \subset \mathbb{K}$ we introduce the partition function

$$\mathcal{Z}(\mathbb{L}, \Phi) = \sum_{\partial \in \mathcal{K}^{co}(\mathbb{L})} \Phi(\partial), \tag{A.1}$$

where $\mathcal{K}^{co}(\mathbb{L}) = \{\partial \in \mathcal{K}^{co} \mid \partial \subset \mathbb{L}\}$. We also define the corellation function $\rho_{\mathbb{L}}(\partial)$ as

$$\rho_{\mathbb{L}}(\partial, \Phi) = \Big[\sum_{\partial' : \partial \subset \partial' \in K^{co}(\mathbb{L})} \Phi(\partial') \Big] / \mathcal{Z}(\mathbb{L}, \Phi). \tag{A.2}$$

We use $\mathcal{K}^f(\mathbb{L})$ to denote the set of all finite collections from $\mathbb{L}$, $\mathcal{K}^f = \mathcal{K}^f(\mathbb{K})$ and for $\mathbb{C} \in \mathcal{K}^f$ we write $\gamma \iota \mathbb{C}$ if there is $\gamma' \in \mathbb{C}$ such that $\gamma \iota \gamma'$. We call $\mathbb{C}$ cluster if it is not decomposable into two nonempty sets, $\mathbb{C} = \mathbb{C}_1 \cup \mathbb{C}_2$, such that every pair $\gamma_1 \in \mathbb{C}_1$, $\gamma_2 \in \mathbb{C}_2$ is compatible. The set of all clusters will be denoted by $\mathcal{K}^{cl}$.

Now the main theorem of the cluster expansion appears. It is the copy of Theorem B.1 from [5] or the conclusion of the main theorem and proposition from [7].

Theorem A.1 *Let functions $a : \mathbb{K} \mapsto [0, \infty)$, $l : \mathbb{K} \mapsto [0, \infty)$ and $\Phi : \mathbb{K} \mapsto \mathbb{C}$ and number $\omega \geq 0$ be such that*

$$\sum_{\gamma' : \gamma' \iota \gamma} e^{a(\gamma') + \omega l(\gamma')} |\Phi(\gamma')| \leq a(\gamma) \tag{A.3}$$

for each $\gamma \in \mathbb{K}$. Then $\mathcal{Z}(\mathbb{L}, \Phi) \neq 0$ for each finite $\mathbb{L} \subset \mathbb{K}$ and:

(i) There exists a unique function $\Phi^T : \mathcal{K} \mapsto \mathbb{C}$ such that

$$\log Z(\mathbb{L}, \Phi) = \sum_{\mathbb{C} \in \mathcal{K}^f(\mathbb{L})} \Phi^T(C) \tag{A.4}$$

for each finite $\mathbb{L} \subset \mathbb{K}$. Moreover, the function Φ^T is given by the formula

$$\Phi^T(C) = \sum_{\mathbb{B}:\mathbb{B} \subset \mathbb{C}} (-1)^{|\mathbb{C}|-|\mathbb{B}|} \log \mathcal{Z}(\mathbb{B}, \Phi), \tag{A.5}$$

the estimate

$$\sum_{\mathbb{C} \iota \gamma} \left| \Phi^T(\mathbb{C}) \right| e^{\omega l(\mathbb{C})} \leq a(\gamma) \tag{A.6}$$

holds for each $\gamma \in \mathbb{K}$ with $l(\mathbb{C}) \leq \sum_{\gamma \in \mathbb{C}} l(\gamma)$. We have $\Phi^T(\mathbb{C}) = 0$ whenever $\mathbb{C} \notin \mathcal{K}^{cl}$.

(ii) For every $\partial \in \mathcal{K}^{f,co}$ there exists a unique function $\Delta_\partial : \mathcal{K}^f \mapsto \mathbb{C}$ such that

$$\rho_{\mathbb{L}}(\partial, \Phi) = \sum_{\mathbb{C} \subset \mathbb{L}} \Delta_\partial(\mathbb{C}) \tag{A.7}$$

for each finite $\mathbb{L} \subset \mathbb{K}$. Moreover,

$$\Delta_\partial(\mathbb{C}) = \sum_{\mathbb{B}:\mathbb{B} \subset \mathbb{C}} (-1)^{|\mathbb{C}|-|\mathbb{B}|} \rho_{\mathbb{B}}(\partial, \Phi) \tag{A.8}$$

for each finite $\mathbb{C} \subset \mathbb{K}$ and

$$\sum |\Delta_\partial(\mathbb{C})| e^{\omega l(\mathbb{C})} \leq e^{a(\partial) + \omega l(\partial)} |\Phi(\partial)| \tag{A.9}$$

with $a(\partial) = \sum_{\gamma \in \partial} a(\gamma)$. We have $\Delta_\emptyset(\emptyset) = 1$ and $\Delta_\emptyset(\mathbb{C}) = 0$ for $\mathbb{C} \neq \emptyset$. The function Δ_∂ satisfies a factorization property:

$$\Delta_\partial(\mathbb{C}) = \Delta_{\partial_1}(\mathbb{C}_1) \Delta_{\partial_2}(\mathbb{C}_2) \tag{A.10}$$

whenever $\partial = \partial_1 \cup \partial_2$, $\mathbb{C} = \mathbb{C}_1 \cup \mathbb{C}_2$, and all contours from $\mathbb{C}_1 \cup \partial_1$ are compatible with those from $\mathbb{C}_2 \cup \partial_2$.

For proof see [5] or [7] and their references.

This theorem in its abstract form is useful while working with an aggregate contour model. To simplify our work with contour model that arises after rewriting the random-cluster model, we introduce another theorem, that is, in general, the simple modification of the previous one.

We will use the notation from Section 1.3, $\diamond$ for o or d, as the functions $a(\gamma)$ and $l(\gamma)$ we will use the length $\|\gamma\|$. Two contours will be incompatible if they intersect themselves. It is easy to verify that there exists a constant c such that

$$|\{\gamma \in \mathbb{K}_\diamond \mid \gamma \ni i, \|\gamma\| = n\}| \leq c^n \tag{A.11}$$

for every i from set $\mathbb{Z}^d_\star = \{x \in \mathbb{R}^d \mid \exists \langle i, j \rangle \in B, x = \frac{1}{4}(i + 3j)\}$. Let $m_\diamond$ denote the minimal length of contour from $\mathbb{K}_\diamond$.

Theorem A.2 *Let $|\Phi_\diamond(\gamma)| \le e^{-\tau\|\gamma\|}$ for each $\gamma \in \mathbb{K}_\diamond$ with $\tau \ge 1 + \log(2c)$. Then*

(i), (ii) The statements (i), (ii) of Theorem A.1 are fulfilled with the estimates (A.6) and (A.9) replaced by

$$\sum_{\substack{\mathbb{C}\in\mathcal{K}_q^{cl}\\ \mathbb{C}\ni i}} |\Phi_\diamond^T(\mathbb{C})| e^{\omega\|C\|} \le 1 \tag{A.12}$$

whenever

$$\omega \le \tau - \left[1 + \log(2c) + \frac{\log m_\diamond}{m_\diamond}\right] \tag{A.13}$$

(here $\|\mathbb{C}\| = \sum_{\gamma\in\mathbb{C}} \|\gamma\|$) and by

$$\sum |\Delta_\partial^\diamond(\mathbb{C})| e^{\omega\mathbb{C}} \le e^{(\omega+1)\|\partial\|}\, \Phi_\diamond(\partial) \tag{A.14}$$

whenever $\omega \le \tau - [1 + \log(2c)]$.

(iii) Whenever $V \subset \mathbb{Z}^d$ and $\partial \in \mathcal{K}_\diamond^{f,co}(V)$, the limit over finite $U \subset V$, ordered by inclusion, of $\rho_{\diamond,U}(\partial)$ exists

$$\lim_{U\nearrow V} \rho_{\diamond,U}(\partial,\Phi) = \rho_{\diamond,V}(\partial,\Phi). \tag{A.15}$$

If $\Phi(\gamma) \ge 0$ for every $\gamma \in \mathbb{K}_\diamond$, there exists a unique σ-additive probability measure $P_{\diamond,V}$ on $\mathcal{K}_\diamond^{co}(V)$ such that

$$P_{\diamond,V}(\mathcal{K}_\diamond^{co}(\partial, V)) = \rho_{\diamond,V}(\partial,\Phi) \tag{A.16}$$

for each $\partial \in \mathcal{K}_\diamond^{co}(V)$. Moreover, $P_{\diamond,V}(K_\diamond^a(V)) = 1$ and $P_{\diamond,V}$ is the weak limit

$$P_{\diamond,V} = \lim_{U\nearrow V} P_{\diamond,U}. \tag{A.17}$$

(iv) Assuming further that Φ is translation invariant, one has for each simply connected finite $V \subset \mathbb{Z}^d$

$$\log \mathcal{Z}(\mathbb{K}_\diamond(V),\Phi) = p(\Phi_\diamond)|\mathbb{B}_V| - \sum_{\substack{\mathbb{C}\in\mathcal{K}_\diamond^{cl}\\ \mathbb{C}\cap(\mathbb{B}_V)^C\neq\emptyset}} \Phi_\diamond^T(\mathbb{C}) \frac{|\mathbb{C}\cap\mathbb{B}_V|}{|\mathbb{C}\cap B|} \tag{A.18}$$

with

$$p(\Phi_\diamond) = 2\sum_{\substack{\mathbb{C}\in\mathcal{K}_\diamond^{cl}\\ \mathbb{C}\ni i}} \frac{\Phi_\diamond^T(\mathbb{C})}{|\mathbb{C}\cap B|} \tag{A.19}$$

and

$$|\log \mathcal{Z}(\mathbb{K}_\diamond(V),\Phi_\diamond) - p(\Phi_\diamond)|\mathbb{B}_V|| \le [\exp(-\omega m_\diamond)]|\partial\mathbb{B}_\Lambda| \tag{A.20}$$

whenever

$$\omega \le \tau - \left[1 + \log(2c) + \frac{\log m_\diamond}{m_\diamond}\right]. \tag{A.21}$$

Proof: It is trivial to verify the assumption of Theorem A.1 from $|\Phi_\diamond(\gamma)| \leq e^{-\tau\|\gamma\|}$. Then (i) and (ii) follows directly from Theorem A.1. For proof of (iii) see [5]. Finally, the statement (iv) can be proven by direct application of (A.4) and (A.12). We only notice that there are two points from $\mathbb{Z}^d_\star$ on each $b \in B$, $\|\gamma\| = |\gamma \cap \mathbb{Z}^d_\star|$ and we use $\partial\mathbb{B}_\Lambda$ to denote the set of bonds from $\mathbb{B}^C$ sharing a vertex with a bond from $\mathbb{B}$. $\square$

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