

Markovian dynamics of exchangeable arrays

Jiří Černý

*Department of Mathematics and Computer Science
University of Basel
Spiegelgasse 1
4051 Basel, Switzerland
jiri.cerny@unibas.ch*

Anton Klimovsky

*Fakultät für Mathematik
Universität Duisburg-Essen
Thea-Leymann-Str. 9
D-45117 Essen, Germany
anton.klymovskiy@uni-due.de*

We study Markov processes with values in the space of general two-dimensional arrays whose distribution is exchangeable. The results of this paper are inspired by the theory of exchangeable dynamical random graphs developed by H. Crane in [4, 5].

1. Introduction

The goal of this paper is to survey some of the recent results on the Markovian dynamics of exchangeable random graphs due to Harry Crane [4, 5] and generalise them to the context of dynamics of exchangeable random arrays whose entries do not necessarily take values in a finite discrete set.

The paper extends the results presented by the authors in the learning session “Genealogies of particles on dynamic random networks” during the Programme “Genealogies of Interacting Particle Systems” of Institute for Mathematical Sciences in August 2017. The learning session concentrated on various aspects of the dynamics of random graphs and, in particular, of particle systems on such graphs. While the original theory due to H. Crane

cannot be applied directly in this context, the results of this paper could be relevant, e.g., for exchangeable Markovian dynamics of particle systems on weighted exchangeable dynamical random graphs.

Our results closely follow [4, 5]. However, as we cannot use the fact that the entries of the array take values in a finite space, some of the proofs require non-trivial modifications, which, in our opinion, sometimes make them cleaner. Many of the results of [4, 5] depend very strongly on the finiteness of the entry space and can't be proved easily in the general context. Those are omitted here.

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2. Exchangeable random arrays

We will consider arrays with values in an arbitrary Polish space S . This space will be endowed with its Borel σ -field $\mathcal{B}(S)$ and a compatible metric d_S , which we assume to be bounded by 1. We write $\mathcal{P}(S)$ for the set of all probability measures on $(S, \mathcal{B}(S))$ endowed with the topology of weak convergence, which is a Polish space as well.

A random S -valued array is a collection $Y = (Y_{ij})_{i,j \in \mathbb{N}}$ of S -valued random variables on some probability space (Ω, \mathcal{A}, P) . Otherwise said, Y is $\mathbb{S} := S^{\mathbb{N}^2}$ valued random variable. We endow \mathbb{S} with the product topology and the compatible metric $d_{\mathbb{S}}(y, y') = \sum_{i,j \in \mathbb{N}} 2^{-i-j} d_S(y_{ij}, y'_{ij})$.

For arbitrary set $A \subset \mathbb{N}$, we define $Y|_A = (Y_{ij})_{i,j \in A}$ to be restriction of Y to the index set A . In particular, with $[n] := \{1, \dots, n\}$, $Y|_{[n]}$ is its restriction to the first n coordinates, $Y|_{[n]} \in \mathbb{S}_n := S^{n^2}$.

Similarly, for every probability distribution ν on \mathbb{S} (or on \mathbb{S}_m , $m \geq n$), we denote by $\nu|_{[n]}$ its image under the canonical restriction from \mathbb{S} (or \mathbb{S}_m) to \mathbb{S}_n . It is a known fact that a sequence of measures $\mu^k \in \mathcal{P}(\mathbb{S})$ converges weakly to $\mu \in \mathcal{P}(\mathbb{S})$, iff all its restrictions $\mu^k|_{[n]}$, converge weakly in $\mathcal{P}(\mathbb{S}_n)$, or equivalently $\mu^k(f) \rightarrow \mu(f)$, for every bounded continuous cylinder function f .

Let Σ be the set of all permutations of integers, that is the set of all bijections of \mathbb{N} which fix all but finitely many values; Σ_n denotes the set of all permutations of $[n]$. For an array Y and $\pi = (\pi_1, \pi_2) \in \Sigma^2$, we define a new array Y^π by $Y_{ij}^\pi = Y_{\pi_1(i)\pi_2(j)}$. For $\pi \in \Sigma$, we also define Y^π by

$Y_{ij}^\pi = Y_{\pi(i)\pi(j)}$. An array Y is called *exchangeable* if

$$Y \stackrel{\text{law}}{=} Y^\pi, \quad \text{for every } \pi \in \Sigma^2. \tag{2.1}$$

An array Y is called *weakly exchangeable* if it is symmetric (i.e., $Y_{ij} = Y_{ji}$) and

$$Y \stackrel{\text{law}}{=} Y^\pi, \quad \text{for every } \pi \in \Sigma. \tag{2.2}$$

The key result of the theory of random arrays is their characterisation due to Aldous [1] and Hoover [6] which can be viewed as “two-dimensional version” of de Finetti’s theorem.

Theorem 2.1: (a) *If $(Y_{ij})_{i,j \in \mathbb{N}}$ is a S -valued exchangeable array, then there exists a measurable function $f: [0, 1]^4 \rightarrow S$ such that $Y \stackrel{\text{law}}{=} Y^*$, where*

$$Y_{ij}^* = f(U, U_i, U'_j, U_{ij}), \tag{2.3}$$

and U , $(U_i)_{i \in \mathbb{N}}$, $(U'_i)_{i \in \mathbb{N}}$, and $(U_{ij})_{i,j \in \mathbb{N}}$ are independent collections of i.i.d. $\text{Uniform}([0, 1])$ random variables.

(b) *If $(Y_{ij})_{i,j \in \mathbb{N}}$ is a S -valued weakly exchangeable array, then the analogous statement holds with a function $f: [0, 1]^4 \rightarrow S$ satisfying $f(\cdot, x, y, \cdot) = f(\cdot, y, x, \cdot)$, and with*

$$Y_{ij}^* = Y_{ji}^* = f(U, U_i, U_j, U_{ij}), \quad i \geq j. \tag{2.4}$$

The representing function f of the Aldous-Hoover theorem is not uniquely determined. E.g., in the case (a), if two functions f and f' satisfy $f(a, b, c, d) = f(T_1(a), T_2(b), T_3(c), T_4(d))$ for some measure preserving transformations T_1, \dots, T_4 of $[0, 1]$, then the corresponding exchangeable arrays have the same distribution.

A (weakly) exchangeable array is called *dissociated* if

$$(Y_{ij} : i, j \leq n) \text{ is independent of } (Y_{ij} : i, j > n), \text{ for each } n. \tag{2.5}$$

It is obvious that if the function f in the representation of Theorem 2.1 does not depend on the first coordinate, then Y is dissociated. Converse statement hold as well, see Corollary 14.13 in [2].

Dissociated arrays play a similar role as i.i.d. sequences do in the theory of exchangeable sequences: Every (weakly) exchangeable array is a mixture of (weakly) exchangeable dissociated arrays. To state this more formally, we need more definitions.

A set $A \in \mathcal{B}(S)$ is called *exchangeable* if $A = A^\pi$ for every $\pi \in \Sigma^2$, where $A^\pi = \{x^\pi : x \in A\}$ and $x_{ij}^\pi = x_{\pi_1(i)\pi_2(j)}$. The collection $\mathcal{E}_S \subset \mathcal{B}(S)$ of all

exchangeable sets is called the *exchangeable σ -field*. For an exchangeable array Y , we define $\mathcal{E}_Y = \{A \in \mathcal{A} : Y(A) \in \mathcal{E}_S\}$. We use $\mathcal{D}_S \subset \mathcal{P}(\mathbb{S})$ to denote the set of all distributions of dissociated exchangeable arrays, which is a closed subset of $\mathcal{P}(\mathbb{S})$. We write $\tilde{\mathcal{D}}_S$ for the set of all distributions of dissociated weakly exchangeable arrays.

The following proposition follows from Proposition 14.8 and Theorem 12.10 of [2].

Proposition 2.2: (a) *A (weakly) exchangeable array Y is dissociated iff $P(A) \in \{0, 1\}$ for every $A \in \mathcal{E}_Y$, that is its exchangeable σ -field is P -trivial.*

(b) *Let Y be an exchangeable array and α its regular conditional distribution given \mathcal{E}_Y . Then, $\alpha(\omega) \in \mathcal{D}_S$ for P -a.e. ω . Moreover, the distribution μ_Y of Y can be written as*

$$\mu_Y(\cdot) = \int_{\mathcal{D}_S} \nu(\cdot) \Lambda_Y(d\nu) \tag{2.6}$$

for a uniquely determined probability measure Λ_Y on \mathcal{D}_S .

Important feature of exchangeable arrays is that regular conditional distribution α of Y given \mathcal{E}_Y can be recovered from a realisation of Y by the following procedure. For $m \geq n$, and $y \in \mathbb{S}$, let $t_m^{y,n} \in \mathcal{P}(\mathbb{S}_n)$ be defined by

$$t_m^{y,n} = \frac{1}{((m)_n)^2} \sum_{\psi_1, \psi_2} \delta_{(y_{\psi_1(i), \psi_2(j)})_{i, j \in [n]}} \tag{2.7}$$

where the sum runs over all injections $\psi_1, \psi_2: [n] \rightarrow [m]$ and $(m)_n = m(m-1) \dots (m-n+1)$. $t_m^{y,n}$ can be viewed as the empirical distribution of $n \times n$ sub-arrays in the array $y|_{[m]}$. We further define

$$t^{y,n} = \lim_{m \rightarrow \infty} t_m^{y,n} \tag{2.8}$$

whenever this limit exists in the weak sense, and set $|y| = (t^{y,n})_{n \geq 1}$ whenever all $t^{y,n}$, $n \geq 1$, exist.

It follows from the construction that the probability measures $t_m^{y,n}$, $n = 1, \dots, m$, are consistent in the sense that $t_m^{y,n}|_{[n-1]} = t_m^{y,n-1}$ for every $2 \leq n \leq m$. This consistency transfers to the limit, that is

$$t^{y,n}|_{[n-1]} = t^{y,n-1}, \quad \text{for every } n \geq 2. \tag{2.9}$$

Therefore, in view of Kolmogorov's extension theorem, $|y|$, when it exists, can be viewed as an element of $\mathcal{P}(\mathbb{S})$.

It will be suitable to define $|y| := \partial$ for $y \in \mathbb{S}$ for which some of the limits $t^{y,n}$ does not exist. Here, ∂ is an arbitrary symbol, and we then view $|y|$ as a map from \mathbb{S} to $\mathcal{D}_S^* := \mathcal{D}_S \cup \{\partial\}$.

In the weakly exchangeable case, we set $\tilde{t}_m^{y,n}$ by

$$\tilde{t}_m^{y,n} = \frac{1}{(m)_n} \sum_{\psi} \delta_{(y_{\psi(i),\psi(j)})_{i,j \in [n]}}, \tag{2.10}$$

where the sum runs over all injections π from $[n]$ to $[m]$. We then define $\tilde{t}^{y,n}$ and $|y| = (\tilde{t}^{y,n})_{n \geq 1}$ analogously as in the exchangeable case.

The next proposition establishes the connection between $|Y|$ and its regular conditional distribution α .

Proposition 2.3: *Let Y be an (weakly) exchangeable array and α its regular conditional distribution given \mathcal{E}_Y . Then, for P -a.e. ω , $|Y(\omega)| = \alpha(\omega)$. In particular, $|Y(\omega)| \in \mathcal{D}_S$ (resp. $|Y(\omega)| \in \tilde{\mathcal{D}}_S$), P -a.s.*

As can be seen from the previous results, the differences between exchangeable and weakly exchangeable arrays are mostly a matter of notation. That is why, from now on, we mostly focus on the exchangeable case; the corresponding statements for the weakly exchangeable case can be derived easily.

2.1. Relation to exchangeable graphs and graph limits

The above construction is a straightforward generalisation of the graph limit construction from the theory of dense random graphs, which we recall briefly.

A (vertex) *exchangeable random graph* is a random graph G with countably many vertices labelled by \mathbb{N} whose distribution is invariant under permutations of the labels. By considering the adjacency matrix $(G_{ij})_{i,j \in \mathbb{N}}$ of this graph, it can be viewed as a $\{0, 1\}$ -valued weakly exchangeable array whose diagonal entries are 0.

Graph limits were introduced by Lovász and Szegedy [7] while studying sequences of dense graphs. They encode the limiting density of finite subgraphs in an infinite graph. Formally, let \mathcal{G}_n be the set of all graphs with n vertices labelled by $[n]$. For $m \geq n$ and $F \in \mathcal{G}_n$ and $G \in \mathcal{G}_m$, let $\text{ind}(F, G)$ be the number of injections $\psi: [n] \rightarrow [m]$ such that $G_{\psi(i)\psi(j)} = F_{ij}$ for all $i, j \in [n]$. Then, for every infinite graph G with vertices labelled by \mathbb{N} , one can define the “density of F in G ”

$$t(F, G) = \lim_{m \rightarrow \infty} \frac{\text{ind}(F, G|_{[m]})}{(m)_n}, \quad F \in \mathcal{G}_n. \tag{2.11}$$

It can be checked easily that $t(\cdot, G)$, restricted to \mathcal{G}_n , if it exists, is a probability measure on \mathcal{G}_n . This probability measure, in fact, coincides with the measure $t^{G,n}$ that was introduced in (2.8), when graphs are identified when their adjacency matrices.

By construction, every $t(\cdot, G)$ is invariant under action of Σ ,

$$t(F^\pi, G) = t(F, G), \quad \text{for every } F \in \mathcal{G}_n, \pi \in \Sigma_n. \quad (2.12)$$

Similarly, the following consistency relation, corresponding to (2.9) above, holds:

$$t(F, G) = \sum_{\bar{F} \in \mathcal{G}_m: \bar{F}|_{[n]} = F} t(\bar{F}, G). \quad (2.13)$$

That means that $(t(F, G))_{F \in \cup_n \mathcal{G}_n}$, if it exists for every $F \in \cup_n \mathcal{G}_n$, can be viewed (again in the sense of Kolmogorov’s extension theorem) as a distribution of a random graph, which must be exchangeable due to (2.12). This distribution corresponds to $|y|$ of the previous section.

3. Dynamics of exchangeable arrays

We now turn to the main goal of this paper, the investigation of processes $X = (X(t))_{t \in T}$ taking values in the space \mathbb{S} of two-dimensional S -valued arrays. Here, T denotes the set of times which can be both discrete, $T = \mathbb{N}_0$, or continuous $T = [0, \infty)$.

In the continuous-time case, we assume that the sample paths of X are càdlàg. Since we endowed \mathbb{S} with the product topology, this is the case iff every restriction $X|_{[n]}$ has càdlàg paths in \mathbb{S}_n , or equivalently, $t \mapsto X_{ij}(t)$ is càdlàg for every $i, j \in \mathbb{N}$. We write, $D(S)$ for the space of all càdlàg functions from T to S , endowed with the usual Skorokhod topology. The previous reasoning implies that $D(\mathbb{S}) = D(S)^{\mathbb{N}^2}$.

A \mathbb{S} -valued process X is called exchangeable, if

$$X^\pi := (X^\pi(t))_{t \in T} \stackrel{\text{law}}{=} X, \quad \text{for every } \pi \in \Sigma^2. \quad (3.1)$$

Equivalently, viewing X as an array of functions $(t \mapsto X_{ij}(t))_{i,j \in \mathbb{N}}$, it is often useful to regard X as an exchangeable $D(S)$ -valued array.

The process X is a *Markov process* when the Markov property holds, that is the past $(X(s))_{s \leq t}$ and the future $(X(s))_{s \geq t}$ are conditionally independent given the present $X(t)$ for all $t \in T$.

We call a S -valued Markov process X *exchangeable* if

- (a) its initial state $X(0)$ is a S -valued exchangeable array, that is

$$X(0)^\pi \stackrel{\text{law}}{=} X(0). \tag{3.2}$$

- (b) its transition kernels

$$p_{s,t}(x, A) := P[X(t) \in A \mid X(s) = x], \quad s < t \in T, A \in \mathcal{B}(\mathbb{S}), \tag{3.3}$$

are invariant under action of Σ^2 , that is for every $\pi \in \Sigma^2$, $s < t \in T$, $x \in \mathbb{S}$, and $A \in \mathcal{B}(\mathbb{S})$

$$p_{s,t}(x^\pi, A^\pi) = p_{s,t}(x, A). \tag{3.4}$$

For convenience, we mostly omit “ S -valued” from the terminology and say, e.g., “exchangeable Markov process” instead of “ S -valued exchangeable Markov process”.

The first theorem of this section implies that the “projection” of X to the space of dissociated arrays is a.s. well defined. Remark that Markov property is not assumed.

Theorem 3.1: *Let X be an exchangeable process with càdlàg sample paths. Then, P -a.s., $|X(t)| \in \mathcal{D}_S$, for all $t \in T$.*

Proof: In the discrete-time case, it suffices to observe that $X(t)$ is an exchangeable S -valued array for every $t \in \mathbb{N}_0$. By Proposition 2.3, $|X(t)| \in \mathcal{D}_S$, P -a.s., and the claim follows, since \mathbb{N}_0 is countable.

In the continuous-time case, we first assume that the distribution of X is dissociated, and use the fact that X can be viewed as $D(S)$ -valued array. Using Proposition 2.3, with $D(S)$ in place of S , we see that for every $n \in \mathbb{N}$, the sequence $t_m^{X,n}$ of probability measures on $D(\mathbb{S}_n)$ converges weakly as $m \rightarrow \infty$ P -a.s. to some $t^{X,n} \in \mathcal{P}(D(\mathbb{S}_n))$. Moreover, since we assume that X is dissociated, $t^{X,n}$ is a.s. deterministic and coincides with the distribution of $X|_{[n]}$.

Let J_n^X be the (deterministic) set of times defined by

$$J_n^X = \{t \in T : t^{X,n}(\{x \in D(\mathbb{S}_n) : t \text{ is a jump point of } x\}) > 0\}. \tag{3.5}$$

By the general theory of probability measures on Skorokhod spaces, see Chapter 15 in [3], J_n^X is at most countable. Therefore, using the same argument as in the discrete case, P -a.s., $|X(t)| \in \mathcal{D}_S$ for all $t \in \cup_n J_n^X$.

For $t \in T \setminus \cup_n J_n^X$, the coordinate projections $\phi_t : D(\mathbb{S}_n) \ni x \mapsto x(t) \in \mathbb{S}_n$ are $t^{X,n}$ -a.s. continuous. By Theorem 5.1 of [3], the weak convergence of $t_m^{X,n}$ then implies the existence of the weak limit $t^{X(t),n} := \phi_t \circ t^{X,n} = \lim_{m \rightarrow \infty} \phi_t \circ t_m^{X,n} = \lim_{m \rightarrow \infty} t_m^{X(t),n}$. The limit measures $t^{X(t),n} \in \mathcal{P}(\mathbb{S}_n)$

are consistent and dissociated, as $t_m^{X,n}$ are, and thus determine a probability measure $|X(t)| \in \mathcal{D}_S$, P -a.s., simultaneously for all $t \in T \setminus \cup_n J_n^X$.

The last two paragraphs together imply that for a dissociated X , $P[|X(t)| \in \mathcal{D}_S \text{ for all } t \in T] = 1$.

A general exchangeable càdlàg process X can be written as a mixture of dissociated processes by conditioning on \mathcal{E}_X , by Proposition 2.2. Therefore,

$$\begin{aligned} &P[|X(t)| \in \mathcal{D}_S \text{ for all } t \in T] \\ &= \int_{\Omega} P[|X(t)| \in \mathcal{D}_S \text{ for all } t \in T \mid \mathcal{E}_X](\omega)P(d\omega). \end{aligned} \tag{3.6}$$

Under $P[\cdot \mid \mathcal{E}_X]$, the law of X is dissociated, and thus the integrand equals 1, a.s., by the previous paragraph. This completes the proof. \square

From Proposition 2.3, we know that $|X(t)|$ is a regular conditional distribution of $X(t)$ given its exchangeable σ -field $\mathcal{E}_{X(t)}$. For processes, there is another exchangeable σ -field, \mathcal{E}_X , associated to the whole process,

$$\mathcal{E}_X = \{A \in \mathcal{A} : X(A) \in \mathcal{E}_{D(S)}\}, \tag{3.7}$$

where $\mathcal{E}_{D(S)}$ is defined as \mathcal{E}_S with $D(S)$ playing the role of S . Obviously, $\mathcal{E}_X \supset \mathcal{E}_{X(t)}$ for every $t \in T$, and $\mathcal{E}_{X(t)} = \mathcal{E}_X \cap \sigma(X(t))$. We now show that $|X(t)|$ is also a regular conditional distribution of $X(t)$ given \mathcal{E}_X .

Lemma 3.2: *Let α^X be the regular conditional distribution of X given \mathcal{E}_X . Then, P -a.s.,*

$$\alpha^X(\omega, X(t) \in \cdot) = |X(t)|(\omega, \cdot). \tag{3.8}$$

or, equivalently, denoting by ϕ_t the projection $D(\mathbb{S}) \ni x \mapsto x(t) \in \mathbb{S}$,

$$\phi_t \circ \alpha^X = |X(t)|. \tag{3.9}$$

Proof: Heuristically, the proof uses the fact that $|X(t)|$ is a dissociated distribution, and dissociated distributions are extremal in the set of all exchangeable distributions.

By properties of regular conditional distributions, for every $C \in \mathcal{E}_X$, and every bounded measurable $f: \mathbb{S} \rightarrow \mathbb{R}$,

$$E[\mathbf{1}_C f(X(t))] = \int_{\Omega} \mathbf{1}_C(\omega) \alpha^X(\omega, f \circ \phi_t) P(d\omega). \tag{3.10}$$

By conditioning w.r.t. $\mathcal{E}_{X(t)}$, we obtain

$$\int_{\Omega} P(d\omega') \int_{\Omega} P(d\omega \mid \mathcal{E}_{X(t)})(\omega') \mathbf{1}_C(\omega) \alpha^X(\omega, f \circ \phi_t). \tag{3.11}$$

Choosing $C \in \mathcal{E}_{X(t)} \subset \mathcal{E}_X$ and using that $\mathbf{1}_C(\omega) = \mathbf{1}_C(\omega')$, $P(\cdot | \mathcal{E}_{X(t)})(\omega')$ -a.s., in this case, we get

$$\int_{\Omega} P(d\omega') \mathbf{1}_C(\omega') \int_{\Omega} P(d\omega | \mathcal{E}_{X(t)})(\omega') \alpha^X(\omega, f \circ \phi_t), \quad (3.12)$$

Observe that, as function of ω' , the inner integral is $\mathcal{E}_{X(t)}$ measurable. Therefore,

$$\int_{\Omega} P(d\omega | \mathcal{E}_{X(t)})(\phi_t \circ \alpha^X)(\omega) \quad (3.13)$$

is a version of regular conditional distribution of $X(t)$ given $\mathcal{E}_{X(t)}$, that is it equals $|X(t)|$, P -a.s. However, $|X(t)|$ is dissociated, and thus extremal in the set of all exchangeable probability distributions. Therefore, necessarily, $(\phi_t \circ \alpha^X)(\omega) = |X(t)|(\omega)$ must hold true for $P(\cdot | \mathcal{E}_{X(t)})$ -a.e. ω . This then implies that $\phi_t \circ \alpha^X = |X(t)|$, P -a.s., as claimed. \square

Theorem 3.3: *The projection $|X| = (|X(t)|)_{t \in T}$ has P -a.s. càdlàg sample paths.*

Proof: Assume first that the distribution of X is dissociated, that is \mathcal{E}_X is trivial. Then $|X(t)|$ exists P -a.s. simultaneously for all $t \in T$, and when it exists, then it coincides with the distribution of $X(t)$. Since the trajectories of X are càdlàg, $\lim_{s \downarrow t} X(s) = X(t)$ pointwise and thus in distribution, implying $|X(t)|$ is right continuous.

On the other hand, let $X_{ij}^-(t) = \lim_{s \uparrow t} X_{ij}(s)$. Then, $X^-(t)$ is exchangeable array, and, by the same arguments as in the proof of Theorem 3.1, one can show that $|X^-(t)|$ exists a.s. simultaneously for all $t \in T$. As the distribution of X is dissociated, $|X^-(t)|$ agrees a.s. with the distribution of $X^-(t)$. Repeating the argument from the first part of the proof, we obtain $\lim_{s \uparrow t} |X(s)| = |X^-(t)|$.

For a general exchangeable process X , we write its distribution as a mixture of dissociated distributions by conditioning on \mathcal{E}_X ,

$$P(X \in \cdot) = \int_{\Omega} P(d\omega) P(X \in \cdot | \mathcal{E}_X)(\omega). \quad (3.14)$$

Under $P(\cdot | \mathcal{E}_X)$, the distribution of X is dissociated, and $|X(t)|$ agrees with regular conditional distribution of $X(t)$ given \mathcal{E}_X , by Lemma 3.2. So, by the previous reasoning, $P(t \mapsto |X(t)|)$ is càdlàg $| \mathcal{E}_X = 1$ a.s., and the claim follows. \square

Theorem 3.4: *Let X be an exchangeable Markov process. Then, $|X|$ is a \mathcal{D}_S^* -valued Markov process with a.s. càdlàg sample paths.*

Proof: The càdlàg property follows from Theorem 3.3. We thus need to show that the Markov property is preserved by the map $\mathbb{S} \ni y \mapsto |y| \in \mathcal{D}_S^*$. To this end, it is sufficient to show that for every $s < t$, and $A \subset \mathcal{D}_S$ measurable

$$p_{s,t}(x, A^{\leftarrow}) = p_{s,t}(x', A^{\leftarrow}), \quad \text{for every } x, x' \in \mathbb{S} \text{ with } |x| = |x'|, \quad (3.15)$$

where $A^{\leftarrow} = \{x \in \mathbb{S} : |x| \in A\}$.

To prove this, observe first $A^{\leftarrow} = (A^{\leftarrow})^\pi$ and thus, by the exchangeability (3.3) of the transition kernel

$$p_{s,t}(x, A^{\leftarrow}) = p_{s,t}(x^\pi, (A^{\leftarrow})^\pi) = p_{s,t}(x^\pi, A^{\leftarrow}), \quad \text{for all } \pi \in \Sigma^2. \quad (3.16)$$

Hence, $x \mapsto p_{s,t}(x, A^{\leftarrow})$ is \mathcal{E}_S measurable. In addition, by the same arguments as in Corollary 3.10 of [2], \mathcal{E}_S agrees with the σ -field generated by the map $x \mapsto |x|$, which implies the claim. \square

Is it clear?

4. Jumps of discrete-time Markov processes

In this and the next section, we study in detail the structure of the jumps of time-homogeneous exchangeable Markov processes. We first consider processes in discrete time, where the situation is rather simple.

Lemma 4.1: *Let X be an exchangeable Markov process in discrete time. Then, the array $J_{ij}(t) = \mathbf{1}\{X_{ij}(t-1) \neq X_{ij}(t)\}$ encoding its jumps at time $t \geq 1$ is also exchangeable. As consequence, only the following two possibilities occur a.s.*

- X is constant at t , that is $X_{ij}(t-1) = X_{ij}(t)$ for all $(i, j) \in \mathbb{N}^2$.
- There is a positive proportion of entries which jump, that is

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{1 \leq i, j \leq n} \mathbf{1}\{X_{ij}(t-1) \neq X_{ij}(t)\} > 0. \quad (4.1)$$

Proof: The exchangeability of $J_{ij}(t)$ follows directly from the exchangeability of X . $J(t)$ is then $\{0, 1\}$ -valued exchangeable array. The proportion of ones in this array equals $t^{J,1}(\{1\})$ in the notation of Section 2. Therefore, by Proposition 2.3, it exists a.s. If it is zero, then the array J_{ij} must be identically 0, a.s. Otherwise, there is a positive proportion of entries that jump. \square

4.1. Restrictions of Markov exchangeable processes are not Markov

If one is interested not only in the occurrence of jumps, but also in their “sizes”, this argument can be pushed even further, similarly as in [4]. For $t \geq 1$, consider S^2 -valued array $Z_{ij} := (X_{ij}(t-1), X_{ij}(t))$, which is again exchangeable. By Proposition 2.3 (with S^2 in place of S), for every $n \in \mathbb{N}$, the limit $t^{Z,n} \in \mathcal{P}(\mathbb{S}_n^2)$ exists a.s.

The measure $t^{Z,n}$ can be used to construct a new Markov transition kernel q_n on \mathbb{S}^n , by disintegrating $t^{Z,n}$ with respect to its first marginal $t^{X(t-1),n}$,

$$t^{Z,n}(dy_1, dy_2) = t^{X(t-1),n}(dy_1)q_{t-1,t}^n(y_1, dy_2), \tag{4.2}$$

or, in the case when S is finite, simply by defining

$$q_{t-1,t}^n(y_1, y_2) = \frac{t^{Z,n}(\{(y_1, y_2)\})}{t^{X(t-1),n}(\{y_1\})}, \quad y_1, y_2 \in \mathbb{S}_n. \tag{4.3}$$

Since, by Proposition 2.3, $t^{X,n}$ agrees with the distribution of $X|_{[n]}$ given \mathcal{E}_X , it is tempting to interpret the kernels q_n as transition kernels of $X|_{[n]}$ (at least conditionally on \mathcal{E}_X), as is done in [4]: Proposition 4.8 of [4] contains, among others, the following claim (stated in the notation of the present paper):

Let $X = (X_t)_{t \in T}$ be a time-homogeneous exchangeable Markov process, with T being finite. Conditioned on \mathcal{E}_X , X is dissociated, and, moreover, for every $n \in \mathbb{N}$, the restriction $X|_{[n]}$ of X to \mathbb{S}_n is (conditionally) a time-inhomogeneous Markov chain with transition probabilities $q_{t-1,t}^n$.

We now provide a counterexample for a part of this claim, namely that $X|_{[n]}$ is (conditionally) Markov. We will also see that the transition kernel of $X|_{[n]}$ is not q^n .

Example 4.2: We work in the settings exchangeable random graphs, similarly as in [4]. That is $X_{ij}(t)$ denotes the adjacency matrix of a random exchangeable graph, which can thus be viewed as $\{0, 1\}$ -valued weakly exchangeable array with zeros on the diagonal. We fix $T = \{0, 1, \dots, N\}$ for a large N .

To construct the process, let $\xi_i, i \in \mathbb{N}$, be i.i.d. Bernoulli($\frac{1}{2}$) random variables. In the initial configuration $X(0)$, we draw an edge between ver-

tices $i \neq j$ (i.e., we set $X_{ij}(0) = 1$) with probability $p_{ij}(\xi)$, where

$$p_{ij}(\xi) = \begin{cases} \frac{1}{4}, & \text{if } \xi_i = \xi_j = 0, \\ \frac{1}{2}, & \text{if } \xi_i \neq \xi_j, \\ \frac{3}{4}, & \text{if } \xi_i = \xi_j = 1. \end{cases} \quad (4.4)$$

All edges are drawn independently.

To define the dynamics, for every $x \in \mathbb{S}$, we define

$$\xi_i(x) = \mathbf{1} \left\{ \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n x_{ij} > \frac{1}{2} \right\}. \quad (4.5)$$

Given the configuration of X at time t , we construct $X(t+1)$ as follows

- If $\xi_i(X(t)) = \xi_j(X(t)) = 0$, then X_{ij} does not change, that is $X_{ij}(t+1) = X_{ij}(t)$.
- Otherwise, X_{ij} is refreshed according to $p_{ij}(X(t))$, that is $X_{ij}(t)$ is a Bernoulli($p_{ij}(\xi(X(t)))$) random variable, chosen independently of all other $X_{ij}(t)$'s.

It is easy to see that the process X is exchangeable. And, by construction, it is obviously Markov. In addition, the law of large numbers implies that $\xi_i(X(0)) = \xi_i$ a.s., and thus $X(1)$, and inductively also $X(t)$, $t \geq 1$, have the same distribution as $X(0)$.

The exchangeable σ -field \mathcal{E}_X is trivial in this example, since X is dissociated by construction. Hence, conditioning on \mathcal{E}_X does not have any effect.

On the other hand, the functions $\xi_i(X(t))$ cannot be determined from any finite restriction $X(t)|_{[n]}$. That is, ξ 's are “hidden variables” for the restriction $X|_{[n]}$, and while conditionally on ξ , $X|_{[n]}$ is Markov, it is not Markov unconditionally.

To prove this, fix $n = 2$, that is consider only the state of the edge connecting the vertices 1 and 2. Then, by an easy computation taking into account all possible values of ξ_1 and ξ_2 , we obtain that $P(X_{12}(t+1) = 1 \mid X_{12}(t) = 1) = \frac{21}{32}$. On the other hand, $P(X_{12}(N) = 1 \mid X_{12}(t) = 1, \forall t < N)$ can be made arbitrarily close to one by choosing N large, because if we know that $X_{12}(t) = 1$ for all $t < N$, then very likely $\xi_1 = \xi_2 = 0$ and thus

X_{12} never flips:

$$\begin{aligned} P(X_{12}(N) = 1 \mid X_{12}(t) = 1, \forall t < N) &= \frac{P(X_{12}(N) = 1, \forall t \leq N)}{P(X_{12}(N) = 1, \forall t < N)} \\ &= \frac{\frac{1}{4} \cdot \frac{1}{4} \cdot 1 + \frac{1}{2} \cdot \left(\frac{1}{2}\right)^N + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^N}{\frac{1}{4} \cdot \frac{1}{4} \cdot 1 + \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{N-1} + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^{N-1}} \xrightarrow{N \rightarrow \infty} 1. \end{aligned} \quad (4.6)$$

This implies that X_{12} is not Markov.

On the technical level, the problem of [4] lies in the fact that the relation (14) therein, which gives certain consistency for the kernels q_n , does not hold true, in general.

5. Jumps of continuous-time Markov processes

We now study exchangeable Markov processes in continuous time. Similarly as in discrete time (see Lemma 4.1), we describe the possible jumps of this process. The structure here is richer, because the process is indexed by an uncountable set of times. So, certain events which have probability 0 in the discrete settings can occur.

Theorem 5.1: *Let X be exchangeable Markov process with càdlàg paths in continuous time, and let $J \subset (0, \infty)$ be the (random) set of times when $t \mapsto X_t$ is discontinuous. Then, a.s., J can be written as a disjoint union $J = J^1 \dot{\cup} J^2 \dot{\cup} J^3$, where*

- J^1 is the set of times, where a positive proportion of entries of X jumps,

$$J^1 := \left\{ t > 0 : \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{1 \leq i, j \leq n} \mathbf{1}\{X_{ij}^-(t) \neq X_{ij}(t)\} > 0 \right\}, \quad (5.1)$$

- J^2 is the set of times, where a positive proportion of entries in one row or column of X jumps, $J^2 = J^{2,c} \dot{\cup} J^{2,r}$ with

$$\begin{aligned} J^{2,r} &= \left\{ t > 0 : \exists! i \in \mathbb{N} \text{ s.t. } X_{i'j}^-(t) = X_{i'j}(t), \forall i' \neq i, j \in \mathbb{N}, \right. \\ &\quad \left. \text{and } \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \mathbf{1}\{X_{ij}^-(t) \neq X_{ij}(t)\} > 0 \right\}, \end{aligned} \quad (5.2)$$

$$\begin{aligned} J^{2,c} &= \left\{ t > 0 : \exists! j \in \mathbb{N} \text{ s.t. } X_{ij'}^-(t) = X_{ij'}(t), \forall j' \neq j, i \in \mathbb{N}, \right. \\ &\quad \left. \text{and } \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{X_{ij}^-(t) \neq X_{ij}(t)\} > 0 \right\}, \end{aligned} \quad (5.3)$$

- J^3 is the set of times, where a unique entry jumps,

$$J^3 = \{t > 0 : \exists! i, j \in \mathbb{N} \text{ s.t. } X_{ij} \text{ is discontinuous at } t, \text{ and } X_{i'j'} \text{ is continuous at } t, \forall (i', j') \neq (i, j)\}. \quad (5.4)$$

Proof: We follow the proof of Theorem 3.6 of [4]. By conditioning on \mathcal{E}_X , we may assume without loss of generality that X is dissociated. Similarly as in the proof of Theorem 3.1, let J^X be the set of times when X jumps with positive probability,

$$J^X := \{t > 0 : P(X \text{ is discontinuous at } t) > 0\}. \quad (5.5)$$

As we remarked previously, this set is at most countable. Hence, by considering the arrays $(\mathbf{1}\{X_{ij}^-(t) \neq X_{ij}(t)\})_{ij}$, $t \in J^X$, and using the same arguments as in the proof of Lemma 4.1, we obtain that $J^X \subset J^1$.

We now consider times $t \in (0, \infty) \setminus J^X =: C^X$. We first claim that for such t , the proportion of entries that jump must be 0, that is $J^X \supset J^1$. Indeed, since X is dissociated, by Proposition 2.3, for every $t \in (0, \infty)$,

$$P(X_{12}^-(t) \neq X_{12}(t)) = \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{1 \leq i, j \leq n} \mathbf{1}\{X_{ij}^-(t) \neq X_{ij}(t)\}, \quad (5.6)$$

so if the right-hand side is positive, so must be the left-hand side, implying $t \in J^X$.

We further claim that at $t \in C^X$ it is a.s. impossible that two entries that are not in the same row or in the same column jump at the same time. To see this, fix $i \neq k$ and $j \neq l$ and write J_{ij} for the set of times when X_{ij} jumps. Then,

$$P(J_{ij} \cap J_{kl} \cap C^X \neq \emptyset) = E[P(J_{ij} \cap J_{kl} \cap C^X \neq \emptyset \mid J_{kl})]. \quad (5.7)$$

The set J_{kl} is at most countable and J_{ij} and J_{kl} are independent because X is dissociated. Therefore, the conditional probability in the last formula satisfies

$$\begin{aligned} P(J_{ij} \cap J_{kl} \cap C^X \neq \emptyset \mid J_{kl}) &\leq P(J_{ij} \cap C^X \neq \emptyset \mid J_{kl}) \\ &= P(J_{ij} \cap C^X \neq \emptyset) = 0, \end{aligned} \quad (5.8)$$

where the last equality follows from the definition of C^X . This yields the claim.

It remains to be shown that if X_{ij} jumps at t , then either it is the only entry that jumps, or that there is a positive proportion of entries that jump in i -th row or j -th column. To see this, it is sufficient to observe that $X_{i \cdot} := (X_{ij})_{j \in \mathbb{N}}$ is an exchangeable $D(S)$ -valued sequence. In general, X_i is

not Markov, but we do not need it to be. By conditioning on its exchangeable field \mathcal{E}_i (which, in general, is not related to \mathcal{E}_X), $X_{i\cdot}$ becomes an i.i.d. sequence, by de Finetti's theorem. We may then repeat the arguments of the previous paragraphs (applied to sequences instead of arrays) to show that $J^{2,r} = \dot{\cup}_{i \in \mathbb{N}} J_i^{2,r}$, where $J_i^{2,r}$ is the set of times when the row i jumps with a positive probability,

$$J_i^{2,r} = \{t > 0 : P(X_{i\cdot}^-(t) \neq X_{i\cdot}(t) \mid \mathcal{E}_i) > 0\}. \quad (5.9)$$

and, out of $J_i^{2,r}$ there are no simultaneous jumps of two entries X_{ij} and $X_{ij'}$ with $j \neq j'$, that is

$$P((J_{ij} \cap J_{ij'}) \setminus J_i^{2,r} \neq \emptyset \mid \mathcal{E}_i) = 0. \quad (5.10)$$

This completes the proof. \square

Inspection of the previous proof allows us to deduce the following claim about the discontinuities of the projection $|X|$.

Corollary 5.2: *Let $J^{|X|}$ be the set of times when $t \mapsto |X(t)|$ is discontinuous. Then, $J^{|X|} \subset J^1$, where J^1 is as in Theorem 5.1.*

The inclusion in the previous theorem might be strict. As an example, consider the process started from $X_{ij}(0)$ being i.i.d. Bernoulli($\frac{1}{2}$), where all entries are refreshed simultaneously by an independent i.i.d. Bernoulli($\frac{1}{2}$) array at times of jump of a standard Poisson process N_t . In this case, for every $t \geq 0$, $|X(t)|$ is the distribution of the i.i.d. Bernoulli($\frac{1}{2}$) array, that is $J^{|X|} = \emptyset$. On the other hand, J^1 agrees with the set of jumps of N_t .

6. The Feller property

In the last part of this paper, we discuss the conditions under which exchangeable \mathbb{S} -valued Markov processes in continuous time have the Feller property.

Recall that a time-homogeneous S -valued Markov process with transition kernels $p_t(\cdot, \cdot)$ is called Feller if

- (a) For every $g \in C_b(S)$, $t \geq 0$ and $y \in S$, the map $x \mapsto \int g(y)p_t(x, dy)$ is continuous.
- (b) For every $x \in S$ and $g \in C_b(S)$,

$$\lim_{t \downarrow 0} \int g(y)p_t(x, dy) = g(x). \quad (6.1)$$

It is easy to construct exchangeable Markov processes that are not Feller. E.g., the process considered in Example 4.2 obviously does not satisfy (a) of Feller property. To see this, take $g(y) = y_{12}$, $y \in \mathbb{S}$, and observe that for every $t > 0$ there is $\varepsilon_t > 0$ such that if $x_{12} = 1$, then

$$\int g(y)p_t(x, dy) \begin{cases} = 1, & \text{if } \xi_1(x) = 0 \text{ and } \xi_2(x) = 0, \\ < 1 - \varepsilon(t), & \text{otherwise.} \end{cases} \quad (6.2)$$

Inspecting, the definition (4.5) of $\xi_i(x)$, it is easy to see that it is not continuous function of x , and thus $X(t)$ is not Feller.

This example indicates one possibility how Feller property can be violated by exchangeable Markov processes: If the transition kernel depends on “non-local exchangeable quantities”, then the process is not Feller. We now show that this is essentially only mechanism, how the Feller property can be violated.

The following definition imposes very strong “locality” of the distribution of X .

Definition 6.1: An exchangeable Markov process X is called *consistent* if every restriction $X|_{[n]}$ to \mathbb{S}_n is Markov with respect to its own natural filtration.

Let $\mathcal{B}_n \subset \mathcal{B}(\mathbb{S})$ be the σ -field generated by the canonical projection from \mathbb{S} to \mathbb{S}_n . If X is consistent, then its transition kernel must satisfy

$$x \mapsto p_t(x, A) \text{ is } \mathcal{B}_n\text{-measurable for every } A \in \mathcal{B}_n \text{ and } t \geq 0, \quad (6.3)$$

or, in the case when S is finite,

$$P[X|_{[n]}(t) = y \mid X(0) = x] = P[X|_{[n]}(t) = y \mid X(0) = x'] \quad (6.4)$$

for every $t \geq 0$, $y \in \mathbb{S}_n$ and $x, x' \in \mathbb{S}$ such that $x|_{[n]} = x'|_{[n]}$.

Theorem 6.2: For a time-homogeneous exchangeable Markov process X , the following are equivalent:

- (i) X is consistent and every $X|_{[n]}$ is a Feller process on \mathbb{S}_n .
- (ii) X is Feller.

Proof: (i) \implies (ii): Part (a) of the Feller property is equivalent to $x \mapsto p_t(x, \cdot)$ is continuous when $\mathcal{P}(\mathbb{S})$ is endowed with the topology of weak convergence. As remarked in the introduction, this is equivalent to $x \mapsto \int g(y)p_t(x, dy)$ is continuous for every $t \geq 0$ and every cylinder function

g , that is for every \mathcal{B}_n -measurable g , $n \geq 1$. Since the restriction $X|_{[n]}$ is Markov by assumption, denoting by $p_t^n(\cdot, \cdot)$ its transition kernel, for $g \in \mathcal{B}_n$,

$$\int_{\mathbb{S}} g(y) p_t(x, dy) = \int_{\mathbb{S}_n} g(y) p_t^n(x|_{[n]}, dy) \quad (6.5)$$

Since $X|_{[n]}$ is assumed to be Feller, the right-hand side is a continuous function of $x|_{[n]}$. Since $x^k \rightarrow x$ in \mathbb{S} implies $x^k|_{[n]} \rightarrow x|_{[n]}$ in \mathbb{S}_n , the continuity of the left-hand side follows.

(ii) \implies (i): We first show that X is consistent, that it satisfy (6.3). This is equivalent to

$$\int g(y) p_t(x, dy) = \int g(y) p_t(x', dy) \quad (6.6)$$

for all x' with $x'|_{[n]} = x|_{[n]}$ and for all bounded continuous \mathcal{B}_n -measurable functions g . X is assumed to be Feller, so the right-hand side is continuous in x' , so it is sufficient to verify (6.6) for a dense set of x' satisfying $x'|_{[n]} = x|_{[n]}$.

To this end, we use the exchangeability (3.4). Let

$$\Sigma_{(n)} = \{\pi \in \Sigma : \pi(i) = i, \forall i \leq n\} \quad (6.7)$$

be the set of permutations of \mathbb{N} that coincide with the identity on $[n]$. Let g be a bounded continuous \mathcal{B}_n -measurable function. Then, $g(y) = g(y^\pi)$ for every $y \in \mathbb{S}$ and $\pi \in \Sigma_{(n)}^2$. Therefore, (3.4) implies that

$$\int g(y) p_t(x, dy) = \int g(y) p_t(x^\pi, dy), \quad \text{for every } \pi \in \Sigma_{(n)}^2. \quad (6.8)$$

Hence, to prove (6.6) it suffices show that there is $\bar{y} \in \mathbb{S}$ with $\bar{y}|_{[n]} = x|_{[n]}$ such that the set $\{\bar{y}^\pi : \pi \in \Sigma_{(n)}^2\}$ is dense in $\{x' : x'|_{[n]} = x|_{[n]}\}$.

We construct such \bar{y} by picking it randomly. To this end, let $U \subset S$ be a countable dense subset of S , and let ρ be a probability measure on U such that $\rho(x) > 0$ for every $x \in U$. Let Y be a \mathbb{S} -valued random variable on some auxiliary probability space $(\tilde{\Omega}, \tilde{P})$ such that $Y|_{[n]} = x|_{[n]}$ and Y_{ij} , $i > n$ or $j > n$, are i.i.d. ρ -distributed.

Then, for $m \geq n$ and $z \in \mathbb{S}_m$ such that $z|_{[n]} = x|_{[n]}$ and $z_{ij} \in U$ for $(i, j) \in [m]^2 \setminus [n]^2$, we have $\tilde{P}(Y|_{[m]} = z) > 0$. So, by the 0-1 law, there is \tilde{P} -a.s. $\pi \in \Sigma_{(n)}^2$ such that $Y^\pi|_{[m]} = z$. This implies that $\{Y^\pi : \pi \in \Sigma_{(n)}^2\}$ is dense in $\{x' : x'|_{[n]} = x|_{[n]}\}$, P -a.s., which is more than sufficient for the existence of \bar{y} of the last paragraph. This completes the proof that X is consistent.

The consistency then implies that $X|_{[n]}$ is Markov with respect to its natural filtration for every $n \geq 1$. The Feller property of $X|_{[n]}$ is then a direct consequence of the Feller property of X . \square

Remark 6.3: If S is finite, then \mathbb{S}_n is finite as well. Every càdlàg Markov process on a finite state space is Feller. Therefore, in this case, the consistency of X is equivalent to Feller property. This was proved in the exchangeable random graph case in [5].

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