CRITICAL AND NEAR-CRITICAL LEVEL-SET PERCOLATION OF THE GAUSSIAN FREE FIELD ON REGULAR TREES

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Abstract. For the Gaussian free field on a $(d+1)$-regular tree with $d \geq 2$, we study the percolative properties of its level sets in the critical and the near-critical regime. In particular, we show the continuity of the percolation probability, derive an exact asymptotic tail estimate for the cardinality of the connected component of the critical level set, and describe the asymptotic behaviour of the percolation probability in the near-critical regime.

1. Introduction

In this paper we study the level-set percolation for the discrete Gaussian free field on regular trees, with focus on its properties in the critical and the near-critical regime. Our results include the continuity of the percolation function, an exact asymptotic tail estimate for the cardinality of the connected component of the critical level set, and describe the asymptotic behaviour of the percolation probability in the near-critical case.

The level-set percolation of the Gaussian free field, in particular on $\mathbb{Z}^d$, is one of the most important and studied percolation models with long range dependencies, with first studies dating back to 1980s, [MS83, LS86, BLM87]. In the past decade, a new wave of results on this model was initiated by [RS13], where it was shown that, on $\mathbb{Z}^d$, this model exhibits a non-trivial percolation phase transition at a critical level $h^* = h^*(d)$ in any dimension $d \geq 3$. In the subsequent papers, see for instance [RS13, DRS14, PR15, DPR18, Szn19, CN20, GRS22, PS22], the sub- and supercritical phases of the model were understood thoroughly, often making use of additional natural critical points in order to work in a strongly sub-/super-critical regime. In the remarkable paper [DGRS20], it was then shown that all those critical points agree with h^*, that is the percolation phase transition is sharp (for a recent, simpler, and more general proof of the sharpness see [Mui22]).

Compared to the sub- and supercritical regime, the critical and near-critical regimes are much less understood. On $\mathbb{Z}^d$, the situation is to some extend similar to the Bernoulli percolation: it is not known whether the percolation probability is continuous at h^*, and the existence of various critical exponents is only conjectured. Only very recently, [Mui22] provided a first (conjecturally not-optimal) upper bound on the critical exponent β involved in near-critical asymptotics of the percolation probability.

Incidentally, the critical behaviour is much better understood on the related model of Gaussian free field on the metric graph of $\mathbb{Z}^d$, where the continuity of the percolation function is known [DW20, DPR22], and various critical exponents were computed in [DPR21].

Here, we study the critical behaviour in a considerably simpler situation, for the level-set percolation of the Gaussian free field on regular trees. This model was initially investigated in [Szn16] where the critical value h^* was characterised as the largest eigenvalue of certain integral operator, and a coupling with random interlacements was used to derive bounds on h^*, implying in particular that $0 < h^* < \infty$. Later, in [AČ20], the sub- and supercritical phase of the model was studied in detail. Their results include the continuity of the percolation probability away from the critical level h^*, and rather precise estimates for the cardinality of the connected components of the level sets in the sub- and super-critical phase. We complement these results with critical and near-critical estimates.

Similarly to [Szn16] and, in particular, to [AČ20], we will take advantage of a connection of the Gaussian free field on regular trees to certain multi-type branching processes (cf. Section 3 below). The analysis of these branching processes is not completely straightforward, as their type space is uncountable and unbounded and they do not satisfy the conditions used in the classical literature on branching processes [Har63, Mod71, AN72]. Fortunately, these conditions can be substituted by certain hypercontractivity estimates (cf. Proposition 3.2 below), which have already been featured in the previous works. We are also not aware of any results about near-critical multi-type branching process which resemble our analysis of the near-critical behaviour of the percolation probability.

2. Model and results

We now define our model. Let $\mathbb{T}$ be the infinite $(d+1)$-regular tree, with $d \geq 2$, rooted at an arbitrary fixed vertex $o \in \mathbb{T}$. On $\mathbb{T}$ we consider the Gaussian free field $\varphi = (\varphi_x)_{x\in\mathbb{T}}$, which is a centred Gaussian process whose covariance function agrees with the Green function of the simple random walk on $\mathbb{T}$, see (3.1) below for the precise definition. We use P to denote the law of this process on $\mathbb{R}^{\mathbb{T}}$, and, for $a \in \mathbb{R}$, we write P_a for the conditional distribution of φ given that $\varphi_o = a$,

$$P_a[\,\cdot\,] := P[\,\cdot \mid \varphi_o = a]. \tag{2.1}$$

(For an explicit construction of P_a see (3.3) and the paragraph below it.) Let further $\bar{o} \in \mathbb{T}$ be an arbitrary fixed neighbour of the root o and define the forward tree $\mathbb{T}^+$ by

$$\mathbb{T}^+ := \{x \in \mathbb{T} : \bar{o} \text{ is not contained in the geodesic path between } o \text{ and } x\}. \tag{2.2}$$

We analyse the percolation properties of the (super-)level sets of φ above level $h \in \mathbb{R}$, that is of

$$E^h_\varphi := \{x \in \mathbb{T} : \varphi_x \geq h\}. \tag{2.3}$$

In particular, we are interested in the connected component of this set containing the root o,

$$\mathcal{C}_o^h \coloneqq \{y \in \mathbb{T} : y \text{ is connected to } o \text{ in } E_\varphi^h\}, \quad h \in \mathbb{R}. \tag{2.4}$$

The critical height h^* of the level-set percolation is defined by

$$h^* = h^*(d) \coloneqq \inf\{h \in \mathbb{R} : P[|\mathcal{C}_o^h| = \infty] = 0\}. \tag{2.5}$$

It is well known that h^* is non-trivial, more precisely $h^* \in (0, \infty)$, see [Szn16, Corollary 4.5]. Moreover, as proved in [Szn16], h^* can be characterised with help of the operator norms of a certain family of non-negative operators $(L_h)_{h\in\mathbb{R}}$ acting on the space $L^2(\nu)$, where ν is the centred Gaussian measure with variance $\sigma_\nu^2 = d/(d-1)$. We give more details of this characterisation in Section 3 below. Here, we only define λ_h to be the largest eigenvalue of L_h and χ_h the corresponding normed eigenfunction, and recall that h^* is the unique solution to

$$\lambda_{h^*} = 1. \tag{2.6}$$

Since we will mostly deal with the critical case, we often abbreviate

$$\chi \coloneqq \chi_{h^*} \quad \text{and} \quad L \coloneqq L_{h^*}. \tag{2.7}$$

For $a, h \in \mathbb{R}$ we further introduce *conditioned* percolation and forward percolation probabilities by

$$\eta(h,a) \coloneqq P_a[|\mathcal{C}_o^h| = \infty] \quad \text{and} \quad \eta^+(h,a) \coloneqq P_a[|\mathcal{C}_o^h \cap \mathbb{T}^+| = \infty]. \tag{2.8}$$

It is known that both of these functions are identically 0 when $h > h^*$, and for $h < h^*$ they are strictly positive iff $a \in [h, \infty)$, see [Szn16, Proposition 3.3 and its proof].

Our first two results consider the behaviour of $\mathcal{C}_0^h$ at the critical height $h = h^*$. The first interim result shows that there is no percolation at h^*.

Theorem 2.1. *For all $a \in \mathbb{R}$,*

$$\eta(h^*, a) = \eta^+(h^*, a) = 0, \tag{2.9}$$

and, as consequence,

$$P[|\mathcal{C}_o^{h^*} \cap \mathbb{T}^+| = \infty] = P[|\mathcal{C}_o^{h^*}| = \infty] = 0. \tag{2.10}$$

As corollary of this theorem, we directly obtain the continuity of the percolation functions. This extends Theorem 5.1 of [AČ20], where it is shown that the functions $h \mapsto \eta(h,a)$ and $h \mapsto \eta^*(h,a)$ are left-continuous everywhere and continuous on $\mathbb{R} \setminus \{h^*\}$.

Corollary 2.2. *The functions $h \mapsto \eta(h,a)$ and $h \mapsto \eta^*(h,a)$ are continuous for every $a \in \mathbb{R}$.*

The second result considers the cardinality of $\mathcal{C}_o^{h^*}$ in the critical case, and describes its exact asymptotic tail behaviour. In particular it gives a probabilistic meaning to the eigenfunction χ.

Theorem 2.3. *For every $a \geq h^*$, as $t \to \infty$,*

$$P_a[|\mathcal{C}_o^{h^*} \cap \mathbb{T}^+| > t] = C_1 \chi(a) t^{-1/2}\big(1 + o(1)\big), \tag{2.11}$$

$$P_a[|\mathcal{C}_o^{h^*}| > t] = C_1 d^{-1}(d+1)\chi(a) t^{-1/2}\big(1 + o(1)\big), \tag{2.12}$$

where, denoting by $\langle \cdot, \cdot \rangle_\nu$ the scalar product on $L^2(\nu)$, the constant C_1 is given by

$$C_1 = \frac{1}{\Gamma(1/2)} \sqrt{\frac{2d}{d-1} \cdot \frac{\langle 1, \chi \rangle_\nu}{\langle \chi, \chi^2 \rangle_\nu}}. \tag{2.13}$$

Remark 2.4. Theorem 2.1 could be seen as a corollary to Theorem 2.3, but we prefer to state it separately, since the former theorem is used in the proof of the latter one.

Remark 2.5. Combining Theorem 2.3 with the stochastic domination (see (3.4) below), it follows that

$$a \mapsto \chi_{h^*}(a) \text{ is non-decreasing}, \tag{2.14}$$

which, to our knowledge, was not known previously.

Our third result considers the percolation probabilities in the near-critical supercritical regime. We are able to describe their asymptotic behaviour as $h \uparrow h^*$. As in Theorem 2.3, the limiting objects can be expressed in terms of the eigenfunction χ.

Theorem 2.6. *The percolation probabilities η and η^+ can be written as*

$$\eta^+(h, a) = C_2 (h^* - h)\big(\chi(a) + r_h^+(a)\big), \tag{2.15}$$

$$\eta(h, a) = C_2 \frac{d+1}{d} (h^* - h)\big(\chi(a) + r_h(a)\big), \tag{2.16}$$

where for an arbitrary $\varepsilon \in (0, 1]$ the reminder functions r_h and r_h^+ satisfy

$$\lim_{h \uparrow h^*} \|r_h\|_{L^{2-\varepsilon}(\nu)} = \lim_{h \uparrow h^*} \|r_h^+\|_{L^{2-\varepsilon}(\nu)} = 0, \tag{2.17}$$

and where, with Id *denoting the identity on $\mathbb{R}$, the constant C_2 is given by*

$$C_2 = 2 \cdot \frac{d-1}{d+1} \frac{\langle \mathrm{Id}, \chi^2 \rangle_\nu}{\langle \chi, \chi^2 \rangle_\nu}. \tag{2.18}$$

Remark 2.7. As a consequence of Theorem 2.6, the critical exponent β defined by $\beta = \lim_{h \uparrow h^*} \frac{\log P(|\mathcal{C}_0^h| = \infty)}{\log(h^* - h)}$ satisfies $\beta = 1$. This coincides with the conjectured value of this exponent for the Gaussian free field on $\mathbb{Z}^d$, as well as with its rigorously proved value on a large family of metric graphs, [DPR21, Corollary 1.2].

We now briefly discuss the structure of this article. In Section 3 we introduce more notation and collect useful known facts about the Gaussian free field on $\mathbb{T}$. The proof of Theorem 2.1, which used the techniques known from the theory of multi-type branching processes, and exploits the convergence of a certain non-negative martingale (see (3.15)), is given in Section 4. In Section 5 we give the proof of Theorem 2.3, using the Tauberian theory and investigating the Laplace transform of the cardinality of $\mathcal{C}_o^{h^*}$. Finally, in Section 6, we use results on bifurcations on Banach spaces together with the defining equation for the forward percolation probability introduced in [AČ20] (see (6.1)) to prove Theorem 2.6.

3. Notation and useful results

In this section we introduce the notation used throughout the paper and recall several known facts concerning the level set percolation of the Gaussian free field on trees.

As already stated in the introduction, we use $\mathbb{T}$ to denote the $(d+1)$-regular tree, $d \geq 2$, that is an infinite tree whose every vertex has exactly $d+1$ neighbours. For two vertices $x, y \in \mathbb{T}$ we use $d(x, y)$ to denote their usual graph distance. The tree is rooted at a fixed arbitrary vertex $o \in \mathbb{T}$, and $\bar{o} \in \mathbb{T}$ denotes a fixed neighbour of o. $\mathbb{T}^+$ denotes the forward tree as defined in (2.2).

We consider the Gaussian free field $\varphi = (\varphi_x)_{x \in \mathbb{T}}$ which is the centred Gaussian process on $\mathbb{T}$ whose covariance function is the Green function of the simple random walk on $\mathbb{T}$, that is

$$E[\varphi_x \varphi_y] = g(x, y) := \frac{1}{d+1} \mathbb{E}_x \Big[\sum_{k=0}^{\infty} 1_{X_k = y} \Big], \qquad x, y \in \mathbb{T}, \tag{3.1}$$

where $\mathbb{E}_x$ stands for the expectation with respect to the simple random walk $(X_k)_{k \geq 0}$ on $\mathbb{T}$ starting at $x \in \mathbb{T}$.

We frequently use the fact that that the Gaussian free field on $\mathbb{T}$ can be viewed as multi-type branching process with a continuous type space (see [Szn16, Section 3] and [AČ20, Section 2.1]). To this end, we define

$$\sigma_\nu^2 := \frac{d}{d-1} \quad \text{and} \quad \sigma_Y^2 := \frac{d+1}{d}, \tag{3.2}$$

and let $(Y_x)_{x \in \mathbb{T}}$ be a collection of independent centred Gaussian random variables on some probability space $(\Omega, \mathcal{A}, P')$ such that $Y_o \sim \mathcal{N}(0, \sigma_\nu^2)$ and $Y_x \sim \mathcal{N}(0, \sigma_Y^2)$ for $x \neq o$. We then recursively define another field $\widetilde{\varphi}$ on $\mathbb{T}$ by

(a) $\widetilde{\varphi}_o := Y_o$,

(b) for $x \neq o$, $\widetilde{\varphi}_x := \frac{1}{d} \widetilde{\varphi}_{\bar{x}} + Y_x$ where $\bar{x}$ is the direct ancestor of x in $\mathbb{T}$, that is the first vertex on the geodesic path from x to o different from x. (3.3)

As explained e.g. in [AČ20, (2.9)], the law of $(\widetilde{\varphi}_x)_{x \in \mathbb{T}}$ under P' agrees with the law P of the Gaussian free field φ. Therefore, we will always assume that the considered Gaussian free field is constructed in this way and will not distinguish between φ and $\widetilde{\varphi}$.

Representation (3.3) of φ can be used to give a concrete construction for the conditional probability P_a introduced in (2.1): It is sufficient to replace (a) in (3.3) by $\widetilde{\varphi}_0 = a$. In addition, (3.3) easily allows to construct a monotone coupling of P_a and P_b. As the result we obtain:

$$\text{If } a < b, \text{ then } P_b \text{ stochastically dominates } P_a, \tag{3.4}$$

that is $E_a[f(\varphi)] \le E_b[f(\varphi)]$ for every bounded increasing function $f : \mathbb{R}^{\mathbb{T}} \to \mathbb{R}$.

From the construction (3.3) it follows that the root o can be viewed as an initial particle of a multi-type branching process; its type is distributed as Y_0. Every particle $\bar{x}$ in this branching process has then d offsprings ($d+1$ if $\bar{x} = 0$) whose types are independently given by $\frac{1}{d}\varphi_{\bar{x}} + Y$, with $Y \sim N(0, \sigma_Y^2)$.

The branching process point of view can be adapted to $\mathcal{C}_o^h$, by considering the same multi-type branching process but killing all particles with type lower than h (and thus also not allowing them to have descendants themselves). Similarly, $\mathcal{C}_o^h \cap \mathbb{T}^+$ can be constructed the same way, with the only difference that in this case also the root node o has d potential descendants, instead of $d+1$. We denote by Z_n^h the n-th generation of this branching process

$$Z_n^h := \{x \in \mathcal{C}_o^h \cap \mathbb{T}^+ : d(o,x) = n\}, \qquad h \in \mathbb{R}, n \in \mathbb{N}. \tag{3.5}$$

We now recall more in detail the spectral machinery introduced in [Szn16] in order to characterise the critical value h^*. Let ν be a centred Gaussian measure on $\mathbb{R}$ with variance σ_ν^2 (as defined in (3.2)), and let Y be a centred Gaussian random variable with variance σ_Y^2. The expectation with respect to this random variable is denoted E_Y. We consider the Hilbert space $L^2(\nu) := L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$, and for every $h \in \mathbb{R}$ we define the operator L_h on $L^2(\nu)$ by

$$\begin{aligned} L_h[f](a) &:= 1_{[h,\infty)}(a)\, d\, E_Y\Big[1_{[h,\infty)}\Big(Y + \frac{a}{d}\Big) f\Big(Y + \frac{a}{d}\Big)\Big] \\ &= 1_{[h,\infty)}(a)\, d \int_{[h,\infty)} f(x) \rho_Y\Big(x - \frac{a}{d}\Big)\, \mathrm{d}x, \end{aligned} \tag{3.6}$$

where ρ_Y denotes the density of Y. We let λ_h to stand for the operator norm of L_h in $L^2(\nu)$,

$$\lambda_h := \|L_h\|_{L^2(\nu) \to L^2(\nu)}. \tag{3.7}$$

The following proposition summarises some known properties of the operator L_h as well as the connection between L_h and the critical height h^*.

Proposition 3.1 ([Szn16] Propositions 3.1, 3.3, Corollary 4.5)**.** *For all $h \in \mathbb{R}$, L_h is a self-adjoint, non-negative, Hilbert-Schmidt operator on $L^2(\nu)$, λ_h is its simple eigenvalue and there exists a unique $\chi_h \ge 0$ with unit $L^2(\nu)$-norm, continuous, strictly positive on $[h,\infty)$, vanishing on $(-\infty, h)$, such that*

$$L_h[\chi_h] = \lambda_h \chi_h. \tag{3.8}$$

Additionally, the map $h \mapsto \lambda_h$ is a decreasing homeomorphism from $\mathbb{R}$ to $(0,d)$ and h^* is the unique value in $\mathbb{R}$ such that $\lambda_{h^*} = 1$. Finally, for every $d \ge 2$,

$$0 < h^* < \infty. \tag{3.9}$$

Later we will need the following estimates on the norms of $L_h[f]$ which follow from the hypercontractivity of the Ornstein-Uhlenbeck semigroup, see (3.14) in [Szn16] and (4.12) in [AČ20].

Proposition 3.2. *For every $f \in L^2(\nu)$, $h \in \mathbb{R}$, $1 < p < \infty$ and $q \leq (p-1)d^2+1$,*

$$\big\|L_h[f]\big\|_{L^q(\nu)} \leq \left\|dE_Y\Big[f\Big(Y + \frac{\cdot}{d}\Big)\Big]\right\|_{L^q(\nu)} \leq d\,\|f\|_{L^p(\nu)}. \tag{3.10}$$

In particular (taking $p = 2$),

$$\big\|L_h[f]^k\big\|_{L^2(\nu)} \leq d\,\|f\|^k_{L^2(\nu)} \quad \text{for all } 1 \leq k \leq d^2+1. \tag{3.11}$$

The eigenfunctions χ_h of L_h were studied more in detail in [AČ20]. We will need the following proposition describing their behaviour. (Note that [AČ20] considers d-regular trees, and thus d in our setting corresponds to $d-1$ in [AČ20].)

Proposition 3.3 ([AČ20] Proposition 3.1)**.** *(a) There exists $c > 0$ such that*

$$\chi_h(a) \leq c a^{1-\log_d(\lambda_h)} \text{ for all } h \in \mathbb{R} \text{ and } a \geq d. \tag{3.12}$$

(b) For every $h \in \mathbb{R}$ there exists $c_h > 0$ such that

$$\chi_h(a) \geq c_h a^{1-\log_d(\lambda_h)} \text{ for all } a \geq h. \tag{3.13}$$

Finally, we introduce the filtration

$$\mathcal{F}_n := \sigma\big(\varphi_x : x \in \mathbb{T}^+,\ d(o,x) \leq n\big), \qquad n \geq 0, \tag{3.14}$$

and recall from [Szn16, (3.35)], that the $(\mathcal{F}_n)$-adapted process $M^h = (M^h_n)_{n\geq 0}$ defined by

$$M^h_n := \lambda_h^{-n} \sum_{x \in Z^h_n} \chi_h(\varphi_x) \tag{3.15}$$

is a non-negative martingale under P as well as under every P_a, $a \in \mathbb{R}$.

Throughout the paper we use the usual notation for the asymptotic relation of two functions: For functions f and g, we write $f(s) \sim g(s)$ as $s \to s_0$ if $\lim_{s\to s_0} \frac{f(s)}{g(s)} = 1$, and write $f(s) = o(g(s))$ as $s \to s_0$ if $\lim_{s\to s_0} \frac{|f(s)|}{g(s)} = 0$. We use $c, c', c_1, \dots$ to denote finite positive constants whose values may change from place to place and which can only depend on d. The dependence of these constants on additional parameters appears in the notation.

4. Percolation probability at the critical height

In this section, we will show Theorem 2.1 which states that there is no percolation at critical height h^*. Its proof uses arguments that are rather common in the context of branching processes and is given for sake of completeness. It exploits the fact that the martingale $(M^h_n)_{n\geq 0}$ introduced in (3.15) converges almost surely, which induces certain boundedness of the sizes of the generations Z^h_n (see (3.5)) as well as of the value of the field on them. This is then enough to show the almost sure finiteness of $\mathcal{C}_o^{h^*} \cap \mathbb{T}^+$.

To keep the notation simple, we often omit $h = h^*$ from the notation and write, e.g., $Z_n := Z_n^{h^*}$, $M_n := M_n^{h^*}$ and $\chi = \chi_{h^*}$. Let $\mathcal{A}$ be the event that $\mathcal{C}_o^{h^*} \cap \mathbb{T}^+$ has infinite size,

$$\mathcal{A} := \{|\mathcal{C}_o^{h^*} \cap \mathbb{T}^+| = \infty\}, \tag{4.1}$$

and let $\Phi_n := \max_{x \in Z_n} \varphi_x$ to be the maximum of the field over Z_n (with the convention that a maximum over the empty set is $-\infty$). For $H > 0$, $N \in \mathbb{N}$ we define the events

$$\mathcal{C}_H := \{\Phi_n \le H \text{ for all } n \ge 0\} \quad \text{and} \quad \mathcal{D}_N := \{|Z_n| \le N \text{ for all } n \ge 0\}. \tag{4.2}$$

We first show that for H and N large those events are typical.

Lemma 4.1. *For every $a \ge h^*$ and $\varepsilon > 0$ there is $H = H(a, \varepsilon) < \infty$ and $N = N(a, \varepsilon) < \infty$ so that*

$$P_a[\mathcal{C}_H] \ge 1 - \varepsilon, \tag{4.3}$$

$$P_a[\mathcal{D}_N] \ge 1 - \varepsilon. \tag{4.4}$$

Proof. From the almost sure convergence of the non-negative martingale M, it follows that

$$\text{for every } \varepsilon > 0 \text{ there is } N < \infty \text{ such that } P_a\Big[\sup_{n \ge 0} M_n \le N\Big] \ge 1 - \varepsilon. \tag{4.5}$$

Indeed, assume that the statement does not hold. Then, there exists a $\varepsilon_0 > 0$ such that the events $A_k := \{\sup M_n \ge k\}$ satisfy $P_a[A_k] > \varepsilon_0$ for all $k \in \mathbb{N}$. Since $A_{k+1} \subseteq A_k$, this implies $P_a[\sup M_n = \infty] > \varepsilon_0$ which contradicts the almost sure convergence of M to a finite limit.

To prove (4.3), observe that $\lambda_{h^*} = 1$ implies that $M_n = \sum_{x \in Z_n} \chi(\varphi_x) \ge \chi(\Phi_n)$. Therefore, setting $H' = \inf\{\chi(h) : h > H\}$ and using that $\Phi_n > H$ implies $\chi(\Phi_n) \ge H'$, we obtain

$$\mathcal{C}_H = \Big\{\sup \Phi_n \le H\Big\} \supseteq \Big\{\sup \chi(\Phi_n) < H'\Big\} \supseteq \Big\{\sup M_n < H'\Big\}. \tag{4.6}$$

By Proposition 3.3 $\lim_{x \to \infty} \chi(x) = \infty$, and thus for N as in (4.5) there is H' so that $H' \ge N + 1$. Estimate (4.3) then follows from (4.5) and (4.6).

Estimate (4.4) is proved similarly. By (3.13) there is $c > 0$ such that $c \le \chi(h)$ for every $h \in [h^*, \infty)$. Therefore, $M_n \ge c|Z_n|$, and thus $\mathcal{D}_N \supseteq \{\sup M_n \le cN\}$. Claim (4.5) then directly implies (4.4). □

We now argue that the events $\mathcal{C}_H$ and $\mathcal{D}_N$ exclude the percolation event $\mathcal{A}$.

Lemma 4.2. *$P_a[\mathcal{A} \cap \mathcal{C}_H \cap \mathcal{D}_N] = 0$ for every $a \ge h^*$, $H \ge h^*$, and $N \ge 1$.*

Proof. For given $H \ge h^*$ and $N > 0$, let Θ_n be the event that up to generation n, no generation of $\mathcal{C}_o^{h^*} \cap \mathbb{T}^+$ exceeds N and the field is bounded by H,

$$\Theta_n := \{|Z_k| \le N \text{ and } \Phi_k \le H \text{ for all } k \le n\}, \qquad n \ge 0, \tag{4.7}$$

and let $\mathcal{A}_n$ be the event that the n-th generation is non-empty,

$$\mathcal{A}_n \coloneqq \{|Z_n| > 0\}, \qquad n \geq 0. \tag{4.8}$$

The sequences $\mathcal{A}_n$ and Θ_n are decreasing, with $\cap_{n\geq 0}\Theta_n = \mathcal{C}_H \cap \mathcal{D}_N$, $\cap_{n\geq 0}\mathcal{A}_n = \mathcal{A}$. Therefore,

$$P_a[\mathcal{A} \cap \mathcal{C}_H \cap \mathcal{D}_N] = \lim_{n\to\infty} P_a[\mathcal{A}_n \cap \Theta_n]. \tag{4.9}$$

We will show that this limit is zero. Conditionally on the event $\Theta_n \cap \mathcal{A}_n$, the number of particles in Z_n is limited by N and their types are bounded by H. Therefore, by the stochastic domination (3.4), the conditional probability that Z_{n+1} is empty can be bounded from below by the probability that N independent particles of type H have no descendants,

$$P_a[\mathcal{A}^c_{n+1} \mid \mathcal{A}_n \cap \Theta_n] \geq P_H[|Z_1| = 0]^N \geq c > 0. \tag{4.10}$$

As consequence, since $\mathcal{A}_{n+1} \cap \Theta_{n+1} \subset \mathcal{A}_n \cap \Theta_n$,

$$\frac{P_a[\mathcal{A}_{n+1} \cap \Theta_{n+1}]}{P_a[\mathcal{A}_n \cap \Theta_n]} = P_a[\mathcal{A}_{n+1} \cap \Theta_{n+1} \mid \mathcal{A}_n \cap \Theta_n] \leq P_a[\mathcal{A}_{n+1} \mid \mathcal{A}_n \cap \Theta_n] \leq 1 - c. \tag{4.11}$$

Applying this bound inductively proves that the limit on the right-hand side of (4.9) is zero, completing the proof. □

With help of Lemmas 4.1, 4.2, it is straightforward to complete the proof of Theorem 2.1.

Proof of Theorem 2.1. By Lemma 4.1, for an arbitrary $\varepsilon > 0$ and $a \in \mathbb{R}$ there is H and N such that $P_a[\mathcal{C}_H] \geq 1 - \varepsilon$ and $P_a[\mathcal{D}_N] \geq 1 - \varepsilon$. Therefore, using also Lemma 4.2

$$0 = P_a[\mathcal{A} \cap \mathcal{C}_H \cap \mathcal{D}_N] \geq P_a[\mathcal{A}] - 2\varepsilon. \tag{4.12}$$

Since ε is arbitrary, this implies $P_a[\mathcal{A}] = 0$ as required. The second claim of the theorem follows from the equality

$$P[\mathcal{A}] = \int P_a[\mathcal{A}]\, \nu(\mathrm{d}a) = 0, \tag{4.13}$$

which holds due to (3.3). □

5. Distribution of the size of the critical cluster

In this section we prove Theorem 2.3 describing the asymptotic behaviour of the size of the connected clusters $|\mathcal{C}_o^{h^*}|$ and $|\mathcal{C}_o^{h^*} \cap \mathbb{T}^+|$ in the critical case $h = h^*$.

To this end we denote by T the total size of $\mathcal{C}_o^{h^*}$ restricted to the forward tree,

$$T \coloneqq |\mathcal{C}_o^{h^*} \cap \mathbb{T}^+|, \tag{5.1}$$

and let $\mathcal{L}_a(s)$ be its Laplace transform under P_a,

$$\mathcal{L}_a(s) \coloneqq E_a[e^{-sT}], \quad a \in \mathbb{R}, s \geq 0. \tag{5.2}$$

The proof of Theorem 2.3 is based on the following classical Tauberian theorem, that connects the asymptotic behaviour of the cumulative distribution function of a random variable at infinity and its Laplace transform near zero.

Proposition 5.1 (Corollary 8.1.7, [BGT89])**.** *Let X be a non-negative random variable with cumulative distribution function F and Laplace transform $\mathcal{L}(s) = E[e^{-sX}]$. For $0 \le \alpha < 1$ and a function $\ell : [0,\infty) \to [0,\infty)$ slowly varying at ∞ the following are equivalent:*

(a) $1 - \mathcal{L}(s) \sim \Gamma(1-\alpha)s^\alpha \ell(1/s)$ *as* $s \to 0^+$,

(b) $1 - F(t) \sim t^{-\alpha}\ell(t)$ *as* $t \to \infty$.

In view of this proposition, to show Theorem 2.3 we first need to control the asymptotic behaviour of $1 - \mathcal{L}_a(s)$.

Proposition 5.2. *For every $a \ge h^*$,*

$$\lim_{s\downarrow 0} s^{-1/2}\big(1 - \mathcal{L}_a(s)\big) = C_1\Gamma(1/2)\chi(a), \tag{5.3}$$

where C_1 was defined in (2.13).

We start with some basic observations and definitions that will eventually lead to the proof of this proposition. By Theorem 2.1, $P_a[T = \infty] = 0$ for every $a \in \mathbb{R}$, and thus

$$\lim_{s\downarrow 0} \big(1 - \mathcal{L}_a(s)\big) = 0. \tag{5.4}$$

Moreover, the Laplace transform $\mathcal{L}_a(s)$ satisfies the recursive equation

$$\mathcal{L}_a(s) = \begin{cases} e^{-s}\left(E_Y[\mathcal{L}_{\frac{a}{d}+Y}(s)]\right)^d, & \text{if } a \ge h^*, \\ 1, & \text{if } a < h^*, \end{cases} \tag{5.5}$$

where, as in (3.6), $Y \sim \mathcal{N}(0, \sigma_Y^2)$. To see this in the case $a \ge h^*$ (the other case is trivial), it is sufficient to write $T = 1 + T_1 + \cdots + T_d$, where T_i is the size of the intersection of $\mathcal{C}_0^{h^*}$ with the sub-tree of the i-th neighbour x_i of the root o, and observe that $T_1, \dots, T_d$ are conditionally independent given $\varphi_o = a$ with respective Laplace transforms

$$E_a[e^{-sT_i}] = E_a\big[E_a[e^{-sT_i} \mid \varphi_{x_i}]\big] = E_a[\mathcal{L}_{\varphi_{x_i}}(s)] = E_Y[\mathcal{L}_{\frac{a}{d}+Y}(s)], \tag{5.6}$$

where the last equality uses the branching process representation (3.3) of φ.

We further set

$$\gamma_s(a) \coloneqq 1 - \mathcal{L}_a(s), \tag{5.7}$$

and note that $\gamma_s(a) = 0$ for $a < h^*$, $\gamma_s(a) \in [0,1]$ for every $s \ge 0$ and $a \in \mathbb{R}$, and therefore for every $s \ge 0$, $\gamma_s \in L^2(\nu)$. By (5.5), using the operator $L = L_{h^*}$ from (3.6), for $a \ge h^*$,

$$1 - \gamma_s(a) = e^{-s}E_Y\Big[1 - \gamma_s\Big(\frac{a}{d} + Y\Big)\Big]^d = e^{-s}\Big(1 - \frac{1}{d}L[\gamma_s](a)\Big)^d. \tag{5.8}$$

Rearranging this equality implies that for $a \geq h^*$ and $s \geq 0$,

$$1 - e^{-s} - \gamma_s(a) + e^{-s} L[\gamma_s](a) = e^{-s} f(L[\gamma_s](a)), \tag{5.9}$$

where the function $f : [0, d] \to \mathbb{R}$ is defined by

$$f(x) = f_d(x) := \Big(1 - \frac{x}{d}\Big)^d - 1 + x = \sum_{k=2}^{d} \binom{d}{k} (-1)^k \Big(\frac{x}{d}\Big)^k. \tag{5.10}$$

Equation (5.9) will be the starting point for several proofs that follow.

We continue with a simple observation about the function f.

Lemma 5.3. *For any $d \geq 2$ there are constants c_1, c_2 such that for all $x \in [0, d]$,*

$$c_1 x^2 \leq f(x) \leq c_2 x^2. \tag{5.11}$$

Proof. From (5.10) it is easy to see that f is smooth, strictly convex on $[0, d]$ with $f(0) = f'(0) = 0$ and $f''(0) = c > 0$. It follows that $cx^2/2 \leq f(x) \leq 2cx^2$ for x in a certain interval $[0, \varepsilon]$, and that f is strictly positive and bounded on $(\varepsilon, d]$. From these two facts the lemma easily follows. □

In the remainder of this section we exclusively work in $L^2(\nu)$, and denote by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ the corresponding norm and scalar product. Since L is self-adjoint (see Proposition 3.1), $L^2(\nu)$ has an orthonormal basis consisting of the eigenfunctions $\{e_k\}_{k\geq 1}$ of L corresponding to the eigenvalues $\{\lambda_k\}_{k\geq 1}$. Since $h = h^*$, by Proposition 3.1 we may assume that $1 = \lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq 0$ with $\lambda_k \to 0$ as $k \to \infty$, and also $e_1 = \chi$. Therefore

$$\gamma_s = \sum_{k\geq 1} a_k(s) e_k, \qquad \text{with} \qquad a_k(s) := \langle \gamma_s, e_k \rangle. \tag{5.12}$$

By considering the first summand separately, we write γ_s as

$$\gamma_s = \alpha_s + \beta_s, \qquad \text{with} \qquad \alpha_s := a_1(s)\chi \text{ and } \beta_s := \sum_{k\geq 2} a_k(s) e_k. \tag{5.13}$$

Observe that

$$L[\gamma_s] = \sum_{k\geq 1} \lambda_k a_k(s) e_k = \alpha_s + \sum_{k\geq 2} \lambda_k a_k(s) e_k = \alpha_s + L[\beta_s]. \tag{5.14}$$

and thus

$$\langle \gamma_s - L[\gamma_s], \chi \rangle = 0. \tag{5.15}$$

Since $\gamma_s(a) \in [0, 1]$, the definition (3.6) of L and Lemma 5.3 imply that

$$0 \leq L[\gamma_s] \leq d \qquad \text{and} \qquad 0 \leq f(L[\gamma_s]) \leq c_2 L[\gamma_s]^2 \leq c_2 d^2. \tag{5.16}$$

From the pointwise convergence (5.4) of γ_s to 0, using successively the dominated convergence theorem and the continuity of L, it follows that

$$\lim_{s\downarrow 0} \gamma_s = \lim_{s\downarrow 0} L[\gamma_s] = \lim_{s\downarrow 0} f(L[\gamma_s]) = 0 \quad \text{in } L^2(\nu). \tag{5.17}$$

In particular $a_k(s) \to 0$ as $s \downarrow 0$ for all $k \geq 1$. Finally, since $\|\gamma_s - L[\gamma_s]\|^2 = \sum_{k\geq 2}(1-\lambda_k)^2 a_k(s)^2 \geq \sum_{k\geq 2}(1-|\lambda_2|)^2 a_k(s)^2 = (1-|\lambda_2|)^2\|\beta_s\|^2$, it holds that

$$\|\beta_s\| \leq \frac{1}{1-|\lambda_2|}\|\gamma_s - L[\gamma_s]\|. \tag{5.18}$$

The following three lemmas are the main preparatory steps for the proof of Proposition 5.2. They together show that α_s dominates β_s in norm and then estimate α_s precisely.

Lemma 5.4. *There is a constant $c < \infty$ such that $\|\alpha_s\|^2 \leq cs$ for all s small enough.*

Proof. Noting that $\chi = 0$ on $(-\infty, h^*)$ and applying $\langle \cdot, \chi\rangle$ on both sides of (5.9) yields

$$(1-e^{-s})\big(\langle 1, \chi\rangle - a_1(s)\big) = e^{-s}\big\langle f(L[\gamma_s]), \chi\big\rangle \quad \text{for } s \geq 0. \tag{5.19}$$

By (3.13), $\chi > c > 0$ on $[h^*, \infty)$. Therefore, using also Lemma 5.3, the right-hand side of (5.19) satisfies for $s \leq 1$

$$e^{-s}\langle f(L[\gamma_s]), \chi\rangle \geq c\langle L[\gamma_s]^2, \chi\rangle \geq c'\|L[\gamma_s]\|^2 \geq c'\|\alpha_s\|^2 = c'\alpha_1(s)^2, \tag{5.20}$$

where the last inequality follows from the orthogonal decomposition (5.14). Together with (5.19), this gives

$$(1-e^{-s})(\langle 1, \chi\rangle - a_1(s)) \geq c'\alpha_1(s)^2 \quad \text{for } s \leq 1. \tag{5.21}$$

Since, as $s \downarrow 0$, $(1-e^{-s}) \sim s$ and $a_1(s) \to 0$ this finishes the proof. □

Lemma 5.5. *There is a constant $c < \infty$ such that $\|\beta_s\| \leq cs$ for s small enough.*

Proof. We rearrange equation (5.9) to obtain

$$\gamma_s - L[\gamma_s] = (1-e^{-s})(1-L[\gamma_s]) - e^{-s}f(L[\gamma_s]) \quad \text{on} \quad [h^*, \infty). \tag{5.22}$$

Since the left-hand side is identically zero on $(-\infty, h^*)$, taking norms yields

$$\|\gamma_s - L[\gamma_s]\| \leq (1-e^{-s})\|1 - L[\gamma_s]\| + \|f(L[\gamma_s])\|. \tag{5.23}$$

Using (5.16), $1 - e^{-s} \leq s$, and Lemma 5.3, this implies

$$\|\gamma_s - L[\gamma_s]\| \leq (1+d)s + c\|L[\gamma_s]^2\|. \tag{5.24}$$

The norm on the right-hand side can be bounded using Proposition 3.2 and the inequality $(a+b)^2 \leq 2a^2 + 2b^2$,

$$\|L[\gamma_s]^2\| \leq d\|\gamma_s\|^2 \leq d(\|\alpha_s\| + \|\beta_s\|)^2 \leq 2d(\|\alpha_s\|^2 + \|\beta_s\|^2). \tag{5.25}$$

Combining (5.24), (5.25) with (5.18), we obtain that for a constant $c < \infty$ and all $s > 0$ small enough

$$\|\beta_s\| \leq cs + c\|\alpha_s\|^2 + c\|\beta_s\|^2. \tag{5.26}$$

By (5.17), $\lim_{s\to 0}\|\gamma_s\| = 0$ and thus $\lim_{s\to 0}\|\beta_s\| = 0$ as well. The claim of the lemma then follows easily from Lemma 5.4. □

Lemma 5.6. *It holds that* $\lim_{s\downarrow 0} s^{-1}a_1(s)^2 = (C_1\Gamma(1/2))^2$*, where* C_1 *was defined in* (2.13)*.*

Proof. We start by proving the estimate

$$\left\|(1-e^{-s})1_{[h^*,\infty)} - \gamma_s + L[\gamma_s] - \frac{1}{d^2}\binom{d}{2}L[\alpha_s]^2\right\| \le cs^{3/2} \tag{5.27}$$

holding for some constant $c < \infty$ and all s small enough: Rearranging (5.9) and subtracting $\frac{1}{d^2}\binom{d}{2}L[\alpha_s]^2$ on both sides shows that, on $[h^*,\infty)$,

$$\begin{aligned}(1-e^{-s}) - (\gamma_s - L[\gamma_s]) - \frac{1}{d^2}\binom{d}{2}L[\alpha_s]^2 \\ = (1-e^{-s})\big(L[\gamma_s] - f(L[\gamma_s])\big) + f(L[\gamma_s]) - \frac{1}{d^2}\binom{d}{2}L[\alpha_s]^2.\end{aligned} \tag{5.28}$$

After taking norms, using again that $s \sim (1-e^{-s})$ as $s \downarrow 0$, this implies that

$$\begin{aligned}\left\|(1-e^{-s})1_{[h^*,\infty)} - (\gamma_s - L[\gamma_s]) - \frac{1}{d^2}\binom{d}{2}L[\alpha_s]^2\right\| \\ \le cs\|L[\gamma_s]\| + cs\|f(L[\gamma_s])\| + \left\|f(L[\gamma_s]) - \frac{1}{d^2}\binom{d}{2}L[\alpha_s]^2\right\|\end{aligned} \tag{5.29}$$

for some constant c and s small enough. By Lemmas 5.4 and 5.5, $\|L[\gamma_s]\| \le \|\gamma_s\| \le \|\alpha_s\| + \|\beta_s\| \le cs^{1/2}$. Further, by Lemma 5.3–5.5 and (5.25), $\|f(L[\gamma_s])\| \le \|cL[\gamma_s]^2\| \le c\|\alpha_s\|^2 + c\|\beta_s\|^2 \le cs$. Hence, to show (5.27), it remains to bound the last summand in (5.29) by $cs^{3/2}$. By the definition (5.10) of f,

$$f(L[\gamma_s]) - \frac{1}{d^2}\binom{d}{2}L[\alpha_s]^2 = 2\frac{1}{d^2}\binom{d}{2}L[\alpha_s]L[\beta_s] + \frac{1}{d^2}\binom{d}{2}L[\beta_s]^2 + \sum_{k=3}^{d} c_k L[\gamma_s]^k \tag{5.30}$$

for some $c_k \in (0,\infty)$. Hence, after taking the norm,

$$\left\|f(L[\gamma_s]) - \frac{1}{d^2}\binom{d}{2}L[\alpha_s]^2\right\| \le c\|L[\alpha_s]L[\beta_s]\| + c\|L[\beta_s]^2\| + c\sum_{k=3}^{d}\|L[\gamma_s]^k\|, \tag{5.31}$$

for some constant $c > 0$. By the Cauchy-Schwarz inequality, Proposition 3.2 and Lemmas 5.4, 5.5,

$$\begin{aligned}\|L[\alpha_s]L[\beta_s]\|^2 &= \langle L[\alpha_s]L[\beta_s], L[\alpha_s]L[\beta_s]\rangle = \langle L[\alpha_s]^2, L[\beta_s]^2\rangle \\ &\le \|L[\alpha_s]^2\|\|L[\beta_s]^2\| \le c\|\alpha_s\|^2\|\beta_s\|^2 \le cs^{3/2}, \\ \|L[\beta_s]^2\| &\le c\|\beta_s\|^2 \le cs^2,\end{aligned} \tag{5.32}$$

and for $3 \le k \le d$, by the same arguments,

$$\|L[\gamma_s]^k\| \le c\|\gamma_s\|^k \le c(\|\alpha_s\| + \|\beta_s\|)^k \le c2^{k-1}(\|\alpha_s\|^k + \|\beta_s\|^k) \le cs^{3/2}. \tag{5.33}$$

This proves that the third summand on the right-hand side of (5.29) is bounded by $cs^{3/2}$ and thus completes the proof of (5.27).

We can now show the lemma. From (5.27), using $\lim_{s\downarrow 0} s^{-1}(1-e^{-s}) = 1$, it easily follows that

$$\lim_{s\downarrow 0}\left(\frac{1}{s}(\gamma_s - L[\gamma_s]) + \frac{1}{d^2}\binom{d}{2}\frac{L[\alpha_s]^2}{s}\right) = 1_{[h^*,\infty)} \quad \text{in } L^2(\nu). \tag{5.34}$$

Since $\chi = 0$ on $(-\infty, h)$ and $L[\alpha_s] = \alpha_s = a_1(s)\chi$, this implies that

$$\begin{aligned}\langle 1,\chi\rangle &= \left\langle \lim_{s\downarrow 0}\left(\frac{1}{s}(\gamma_s - L[\gamma_s]) + \frac{1}{d^2}\binom{d}{2}\frac{\alpha_s^2}{s}\right),\chi\right\rangle \\ &= \lim_{s\downarrow 0}\left(\left\langle\frac{1}{s}(\gamma_s - L[\gamma_s]),\chi\right\rangle + \frac{1}{d^2}\left\langle\binom{d}{2}\frac{\alpha_s^2}{s},\chi\right\rangle\right) \\ &= \frac{d-1}{2d}\langle\chi^2,\chi\rangle \lim_{s\downarrow 0}\frac{a_1(s)^2}{s},\end{aligned} \tag{5.35}$$

where in the last equality we used $\langle \gamma_s - L[\gamma_s],\chi\rangle = 0$, by (5.15). The claim of the lemma then follows. ☐

We now have all ingredients to give the proof of Proposition 5.2, directly followed by the proof of Theorem 2.3.

Proof of Proposition 5.2. By the Lemmas 5.5 and 5.6,

$$\lim_{s\downarrow 0}\frac{\gamma_s}{\sqrt{s}} = \lim_{s\downarrow 0}\left(\frac{a_1(s)\chi}{\sqrt{s}} + \frac{\beta_s}{\sqrt{s}}\right) = C_1\Gamma(1/2)\chi \qquad \text{in } L^2(\nu). \tag{5.36}$$

By the stochastic domination (3.4), the function $a \mapsto \gamma_s(a)$ is increasing for any $s > 0$, and by Proposition 3.1, the limit function $C_1\Gamma(1/2)\chi$ is continuous on $[h^*,\infty)$. This implies that the convergence in (5.36) is pointwise as well. ☐

Proof of Theorem 2.3. Claim (2.11) follows directly from Propositions 5.1 and 5.2.

To prove (2.12), let $\tilde{\mathcal{L}}_a$ be the Laplace transform of $|\mathcal{C}_o^{h^*}|$ under P_a,

$$\tilde{\mathcal{L}}_a(s) := E_a\big[e^{-s|\mathcal{C}_o^{h^*}|}\big], \qquad a\in\mathbb{R}, s\ge 0. \tag{5.37}$$

Using the same arguments as in the proof of the recursion property (5.5), it follows that

$$\tilde{\mathcal{L}}_a(s) = e^{-s}\big(E_Y[\mathcal{L}_{\frac{a}{d}+Y}(s)]\big)^{d+1}, \qquad \text{for } s>0, a\ge h^*, \tag{5.38}$$

which together with (5.5) yields

$$\tilde{\mathcal{L}}_a(s) = e^{s/d}\mathcal{L}_a(s)^{(d+1)/d} \qquad \text{for } s>0, a\ge h^*. \tag{5.39}$$

Using Proposition 5.2, this implies that, as $s\downarrow 0$,

$$\begin{aligned}\tilde{\mathcal{L}}_a(s) &= \Big(1-\frac{s}{d}+o(s)\Big)\Big(1-C_1\Gamma(1/2)\chi(a)s^{1/2}+o(s^{1/2})\Big)^{(d+1)/d} \\ &= 1-\frac{d+1}{d}C_1\Gamma(1/2)\chi(a)s^{1/2}+o(s^{1/2}).\end{aligned} \tag{5.40}$$

Claim (2.12) then follows by another application of Proposition 5.1. □

6. Behaviour of the connectivity for near-critical level set percolation

In this section we prove Theorem 2.6 which describes the asymptotic behaviour of the percolation probabilities $\eta(h,a)$ and $\eta^+(h,a)$ for fixed $a \in \mathbb{R}$ as h approaches h^* from below.

The proof is based on a careful analysis of the functional equation for η^+ that was proved in [AČ20] and that we recall in the next proposition.

Proposition 6.1 ([AČ20] Theorem 4.1)**.** *For every $h \in \mathbb{R}$, the forward percolation probability $\eta_h^+ := \eta^+(h,\cdot)$ solves the functional equation*

$$f(a) = 1_{[h,\infty)}(a)\Big(1 - \big(1 - d^{-1}L_h[f]\big)^d\Big), \quad a \in \mathbb{R}. \tag{6.1}$$

In addition, the only two solutions of (6.1) in the set

$$\mathcal{S}_h := \{f \in L^2(\nu) : 0 \le f \le 1 \text{ and } f = 0 \text{ on } (-\infty,h)\} \tag{6.2}$$

are the constant function $f = 0$ and η_h^+. For $h \ge h^*$ these two solutions coincide and for $h < h^*$ they are distinct.

The last claim of this proposition together with the continuity of the percolation functions (cf. Corollary 2.2) implies that the solution set to (6.1) has a bifurcation at the critical point h^*. Therefore, in order to describe the behaviour of η_h^+ as $h \uparrow h_*$, we will analyse the solution set around this bifurcation.

Our main tool will be the theorem on transcritical bifurcations on general Banach spaces, stated as Proposition 6.2 below. To introduce this theorem we need more notation. For Banach spaces X, Y, let $B(X,Y)$ be the space of bounded linear operators from X to Y. For a function $F : X \to Y$, we use $DF : X \to B(X,Y)$ to denote its Fréchet derivative and $DF(x) : X \to Y$ its Fréchet derivative evaluated at point $x \in X$. If T is an open interval in $\mathbb{R}$ and $G : T \times X \to Y$, then we use $D_xG(t,x) : X \to Y$ and $D_tG(t,x) \in Y$ to denote the partial Fréchet derivative in the x and t direction, evaluated at point (t,x). Similarly, $D_{xx}G$ or $D_{xt}G$ denote the respective second partial Fréchet derivatives. Finally, we use $N(F)$ and $R(F)$ to denote the kernel and the range of a linear functional F.

Proposition 6.2 ([CR71], Theorems 1.7 and 1.18)**.** *Let X, Y be Banach spaces, V a neighbourhood of 0 in X, $I = (t_0 - 1, t_0 + 1)$ for some $t_0 \in \mathbb{R}$. Assume that a (non-linear) functional $F : I \times V \to Y$ satisfies:*

- *(a) $F(t,0) = 0$ for $t \in I$,*
- *(b) The partial derivatives D_tF, D_xF and $D_{tx}F$ exist and are continuous,*
- *(c) $N(D_xF(t_0,0)) = \operatorname{span}\{x_0\}$ for a $x_0 \in X$, and $\operatorname{codim} R(D_xF(t_0,0)) = 1$,*
- *(d) $D_{tx}F(t_0,0)(x_0) \notin R(D_xF(t_0,0))$, where x_0 is given in (c).*

Then for any complement Z of x_0 in X (i.e., for any subspace Z of X with $Z \oplus$ span$\{x_0\} = X$*) there is a neighbourhood U of $(t_0, 0)$ in $\mathbb{R} \times X$, an interval $(-a, a)$, and continuous functions $\varphi : (-a, a) \to I$, $\psi : (-a, a) \to Z$ such that $\varphi(0) = t_0$, $\psi(0) = 0$ and*

$$F^{-1}(0) \cap U = \{(\varphi(\alpha), \alpha x_0 + \alpha\psi(\alpha)) : |\alpha| < a\} \cup \{(t, 0) : (t, 0) \in U\}. \tag{6.3}$$

If, in addition to (a)–(d), $D_{xx}F$ is continuous, then the functions φ and ψ have a continuous derivative with respect to α and

$$\frac{1}{2} D_{xx}F(t_0, 0)(x_0, x_0) + D_xF(t_0, 0)(\psi'(0)) + \varphi'(0)D_{tx}F(t_0, 0)(x_0) = 0. \tag{6.4}$$

One of the main difficulties in applying this proposition to our situation is to choose suitable spaces X and Y where its conditions can be verified. We start by shifting the functions η_h^+ so that they have a common zero set. To this end, let θ_a be the usual shift operator acting on $f : \mathbb{R} \to \mathbb{R}$ by $\theta_a f(x) = f(x + a)$, and define $\widetilde{\eta}_h := \theta_h \eta_h^+$. Note that, for $h < h^*$, $\widetilde{\eta}_h(a) > 0$ iff $a \in [0, \infty)$. For $h \in \mathbb{R}$, let H_h be an operator defined by

$$H_h[f] = d^{-1}\theta_h L_h[\theta_h^{-1} f]. \tag{6.5}$$

Using the definition (3.6) of L_h, after an easy computation, this operator can be written more explicitly:

$$H_h[f](a) = \begin{cases} \int_0^\infty f(x)\rho_Y\big(x - \frac{a}{d} + \frac{d-1}{d}\, h\big)\, \mathrm{d}x, & \text{when } a \geq 0, \\ 0, & \text{otherwise,} \end{cases} \tag{6.6}$$

where ρ_Y denotes the centred Gaussian density with variance σ_Y^2. This notation allows to rewrite equation (6.1) in terms of $\widetilde{\eta}_h$ as

$$0 = -\widetilde{\eta}_h(a) + 1_{[0,\infty)}(a)\Big(1 - \big(1 - H_h[\widetilde{\eta}_h](a)\big)^d\Big), \qquad a \in \mathbb{R}. \tag{6.7}$$

Finally, let ν^* be a Gaussian measure obtained from the Gaussian measure ν (see above (3.6)) by shifting it by h^*, that is the corresponding densities satisfy $\rho_{\nu^*} = \theta_{h^*}\rho_\nu$.

In view of (6.7), to prove Theorem 2.6, we will show that Proposition 6.2 is applicable to

$$F(h, f) = -f + 1_{[0,\infty)}\Big(1 - \big(1 - H_h[f]\big)^d\Big), \tag{6.8}$$

viewed as a map from $I \times L^2(\nu^*)$ to $L^2(\nu^*)$, with $I = (h^* - 1, h^* + 1)$. Showing the applicability of Proposition 6.2 is divided into multiple steps. First, we prove that F is indeed a map from $I \times L^2(\nu^*)$ to $L^2(\nu^*)$. Then we compute the necessary partial Fréchet derivatives, and finally we verify the remaining assumptions of the proposition. Before starting with this programme, we state a simple estimate that will later be useful several times.

Lemma 6.3. *Let ρ be any centred Gaussian density. Then there exist constants $r, c \in (0,\infty)$ such that for all $x \in \mathbb{R}$ and $|s| \in (-1,1) \setminus \{0\}$,*

$$\left|\frac{\rho(x+s)-\rho(x)}{s}\right| \le c\big(\rho(x+r)+\rho(x-r)\big). \tag{6.9}$$

Proof. By Taylor's theorem, $\rho(x+s) = \rho(x) + \rho'(x)s + \frac{1}{2}\rho''(\xi_{x,s})s^2$, for some $\xi_{x,s}$ between x and $x+s$. Therefore, for $|s| \in (0,1)$,

$$\left|\frac{\rho(x+s)-\rho(x)}{s}\right| \le |\rho'(x)| + |\rho''(\xi_{x,s})|. \tag{6.10}$$

Since $\rho'(x) = P_1(x)\rho(x)$ and $\rho''(x) = P_2(x)\rho(x)$ for some polynomials P_1, P_2, it follows easily that there is $x_0 < \infty$ such that $|\rho'(x)|+|\rho''(\xi_{x,s})| \le (\rho(x+2)+\rho(x-2))$ for all $x \notin [-x_0, x_0]$ and $s \in [-1,1]$. Finally, since $\rho > 0$ on $\mathbb{R}$, we can made the last inequality valid on whole $\mathbb{R}$ by multiplying the right-hand side by a sufficiently large constant c. ☐

We now show that F maps $I \times L^2(\nu^*)$ to $L^2(\nu^*)$. To this end it is enough to prove that for $f \in L^2(\nu^*)$ and $h \in \mathbb{R}$, $H_h[f] \in L^{2d}(\nu^*)$. The following lemma shows a little bit more, as it will be needed in the later proofs, and also proves the continuity of $h \mapsto H_h$.

Lemma 6.4. *(a) If $h, s \in \mathbb{R}$ and $f \in L^2(\nu^*)$, then*

$$\|\theta_s H_h[f]\|_{L^{2d}(\nu^*)} \le C_{h,s}\|f\|_{L^2(\nu^*)}. \tag{6.11}$$

In particular, $\theta_s H_h[f] \in L^{2d}(\nu^*)$.

(b) The function $h \mapsto H_h$ from I to $B(L^2(\nu^*), L^{2d}(\nu^*))$ is (strongly) continuous.

Proof. (a) For $f \in L^2(\nu^*)$, let $g = \theta_{h^*}^{-1} f \in L^2(\nu)$. Then, for $a \ge 0$, by (6.6),

$$\begin{aligned} H_h[f](a) &= \int_0^\infty f(x)\rho_Y\Big(x - \frac{a}{d} + \frac{d-1}{d}h\Big)\,\mathrm{d}x \\ &= \int_{h^*}^\infty g(x)\rho_Y\Big(x - \frac{a+dh^*-(d-1)h}{d}\Big)\,\mathrm{d}x \\ &= G[g]\big(a + dh^* - (d-1)h\big), \end{aligned} \tag{6.12}$$

where we defined $G[g](a) := \int_{h^*}^\infty g(x)\rho_Y(x - \frac{a}{d})\,\mathrm{d}x$. Note that G is strongly related to L_{h^*} (see (3.6)) which suggests that we eventually should apply Proposition 3.2. We therefore write

$$\begin{aligned} \big\|\theta_s H_h[f]\big\|_{L^{2d}(\nu^*)}^{2d} &\le \int_{\mathbb{R}} \big(G[g](a+s+dh^*-(d-1)h)\big)^{2d}\rho_\nu(a+h^*)\,\mathrm{d}a \\ &= \int_{\mathbb{R}} \big(G[g](a)\big)^{2d}\rho_\nu\big(a-s-(d-1)h^*+(d-1)h\big)\,\mathrm{d}a \\ &= \int_{\mathbb{R}} \big(G[g](a)\big)^{2d}\frac{\rho_\nu\big(a-s-(d-1)(h^*-h)\big)}{\rho_\nu(a)}\rho_\nu(a)\,\mathrm{d}a. \end{aligned} \tag{6.13}$$

Hölder's inequality with $p = 5/4$ and $q = 5$ then yields

$$\big\|\theta_s H_h[f]\big\|^{2d}_{L^{2d}(\nu^*)} \le \big\|G[g]\big\|^{2d}_{L^{5d/2}(\nu)}\bigg\|\frac{\rho_\nu(\cdot - s - (d-1)(h^* - h))}{\rho_\nu(\cdot)}\bigg\|_{L^5(\nu)}. \tag{6.14}$$

The function inside the second norm on the right-hand side grows at most exponentially at infinity and thus its $L^5(\nu)$-norm is finite, we denote it $\tilde{C}_{h,s}$. The first norm satisfies $\|G[g]\|^{2d}_{L^{5d/2}(\nu)} \le \|G[g]\|^{2d}_{L^{d^2+1}(\nu)}$ since $5d/2 \le d^2 + 1$ for every $d \ge 2$. Moreover, since $G[f](a) = E_Y\big[1_{[h^*,\infty)}(Y + \frac{a}{d})f(Y + \frac{a}{d})\big]$, the hypercontractivity (3.10) implies that $\|G[g]\|^{2d}_{L^{d^2+1}(\nu)} \le d\,\|g\|^{2d}_{L^2(\nu)}$ which is finite because $g \in L^2(\nu)$. This together implies (6.11) with with $C_{h,s} = (d\tilde{C}_{h,s})^{1/(2d)}$.

(b) Let $h, h' \in I$ be such that $|h - h'| < 1$, and let $f \in L^2(\nu^*)$ and $g = \theta^{-1}_{h^*} f$. Then, analogously to (6.12), using then Lemma 6.3,

$$\begin{aligned}
&\big|H_h[f](a) - H_{h'}[f](a)\big| \\
&\le \int_{h^*}^{\infty} |g(x)|\bigg|\rho_Y\Big(x - \frac{a + dh^* - (d-1)h}{d}\Big) - \rho_Y\Big(x - \frac{a + dh^* - (d-1)h'}{d}\Big)\bigg|\,\mathrm{d}x \\
&\le c|h - h'| \sum_{u=\pm1} \int_{h^*}^{\infty} |g(x)|\rho_Y\Big(x - \frac{a + dh^* - (d-1)h}{d} + ur\Big)\,\mathrm{d}x \\
&\le c|h - h'| \sum_{u=\pm1} G[|g|]\big(a + dh^* - (d-1)h + urd\big).
\end{aligned} \tag{6.15}$$

Therefore, using exactly the same arguments as in the proof of (a) and the triangle inequality,

$$\big\|H_h[f] - H_{h'}[f]\big\|_{L^{2d}(\nu^*)} \le c|h - h'|\|f\|_{L^2(\nu^*)}, \tag{6.16}$$

which proves the stated continuity. ☐

We can now compute the first partial Fréchet derivatives of the function F defined in (6.8).

Lemma 6.5. *The partial derivative $D_f F$ at point (h, f) is given by*

$$D_f F(h, f)(g) = -g + 1_{[0,\infty)} d(1 - H_h[f])^{d-1} H_h[g]. \tag{6.17}$$

In particular, $D_f F(h, f)$ is a bounded linear operator on $L^2(\nu^*)$ and it depends continuously on $h \in I$ and $f \in L^2(\nu^*)$.

Proof. We recall that for every $k \le d$ the Fréchet derivative of the power function $f \mapsto f^k$, viewed as a map from $L^{2d}(\nu^*)$ to $L^{2d/k}(\nu^*)$ is a continuous function of f and is given by

$$D_f(f^k)(g) = kf^{k-1}g, \tag{6.18}$$

(see, e.g., [Zei95, Chap. 4.3]), and that the Fréchet derivatives satisfy the chain rule (e.g., Corollary to Theorem 4.D in [Zei95]). Therefore, using also that $H_h[f] \in$

$L^{2d}(\nu^*)$ by Lemma 6.4(a),

$$\begin{aligned} D_f\big((1-H_h[f])^d\big)(g) &= -d(1-H_h[f])^{d-1}D_fH_h[f](g) \\ &= -d(1-H_h[f])^{d-1}H_h[g], \end{aligned} \tag{6.19}$$

where in the last step we used the fact that H_h is a linear operator and thus $D_fH_h[f](g) = H_h[g]$. Recalling the definition (6.8) of F, formula (6.17) directly follows. The fact that $D_fF(h,f)$ is a bounded linear operator on $L^2(\nu^*)$ then follows by Lemma 6.4(a) and Hölder's inequality.

To prove the continuity of $D_fF(h,f)$, let $h,h' \in I$ and $f,f',g \in L^2(\nu^*)$. Ignoring the non-essential prefactor $d1_{[0,\infty)}$, $D_fF(h,f)(g) - D_fF(h',f')(g)$, can be written as

$$\begin{aligned} &(1-H_h[f])^{d-1}H_h[g] - (1-H_{h'}[f'])^{d-1}H_{h'}[g] \\ &= \big((1-H_h[f])^{d-1} - (1-H_{h'}[f'])^{d-1}\big)H_h[g] + (1-H_{h'}[f'])^{d-1}\big(H_h[g]-H_{h'}[g]\big). \end{aligned} \tag{6.20}$$

By Hölder's inequality and (6.16), the $L^2(\nu^*)$-norm of the second summand is bounded by $C(1+\|f'\|_{L^2(\nu^*)})^{d-1}\|g\|_{L^2(\nu^*)}|h-h'|$. The first summand can be rewritten using the formula $a^k - b^k = (a-b)\sum_{i=0}^{k-1} a^i b^{k-1-i}$, and the so arising term $H_h[f] - H_{h'}[f']$ can be expanded as $(H_h[f] - H_{h'}[f]) + (H_{h'}[f] - H_{h'}[f'])$. Therefore, using the linearity of H_h, Lemma 6.4(a,b), and Hölder's inequality again, the $L^2(\nu^*)$-norm of the first summand is bounded by

$$C\|g\|\big[|h-h'|\big(1+\|f\|+\|f'\|\big)^{d-1} + \|f-f'\|\big(1+\|f\|+\|f'\|\big)^{d-2}\big], \tag{6.21}$$

where all norms are in $L^2(\nu^*)$. The continuity of $(f,h) \mapsto D_f(f,h)$ then directly follows from these estimates. □

Lemma 6.6. *The partial derivative D_hF at point (h,f) is given by*

$$D_hF(h,f) = 1_{[0,\infty)}d\big(1-H_h[f]\big)^{d-1}H'_h[f], \tag{6.22}$$

where, for $h \in I$ and $f \in L^2(\nu^*)$, $H'_h[f] : \mathbb{R} \to L^{2d}(\nu^*)$ is given by

$$H'_h[f](a) = \frac{d-1}{d}\int_0^\infty f(x)\rho'_Y\Big(x - \frac{a}{d} + \frac{d-1}{d}h\Big)\,\mathrm{d}x, \tag{6.23}$$

with ρ'_Y being the derivative of ρ_Y. In particular, $D_hF(h,f) \in L^2(\nu^*)$ and it is a continuous function of $h \in I$ and $f \in L^2(\nu^*)$.

Proof. We start by showing that for any $f \in L^2(\nu^*)$ the h-derivative of $H_h[f]$ is given by $D_hH_h[f] = H'_h[f] \in L^{2d}(\nu^*)$. To see this we have to check that

$$\lim_{s\to 0}\frac{1}{s}\big(H_{h+s}[f] - H_h[f]\big) = H'_h[f] \quad \text{in } L^{2d}(\nu^*). \tag{6.24}$$

We first show the pointwise convergence. By (6.6), for fixed $a \in \mathbb{R}$, it holds

$$\begin{aligned}&\frac{1}{s}\big(H_{h+s}[f](a) - H_h[f](a)\big)\\ &\quad= \int_0^\infty f(x)\frac{1}{s}\Big(\rho_Y\Big(x - \frac{a}{d} + \frac{d-1}{d}(h+s)\Big) - \rho_Y\Big(x - \frac{a}{d} + \frac{d-1}{d}h\Big)\Big)\,\mathrm{d}x\\ &\quad=: \int_0^\infty f(x)\Phi_s\Big(x - \frac{a}{d} + \frac{d-1}{d}h\Big)\,\mathrm{d}x.\end{aligned} \tag{6.25}$$

Obviously, $\lim_{s\to 0}\Phi_s(x - \frac{a}{d} + \frac{d-1}{d}h) = \frac{d-1}{d}\rho'_Y(x - \frac{a}{d} + \frac{d-1}{d}h)$, and, by Lemma 6.3, $\max_{|s|<1}|\Phi_s(y)| \le c(\rho_Y(y+r) + \rho_Y(y-r)) =: \bar\Phi(y)$. Since $f \in L^2(\nu^*) \subset L^1(\nu^*)$, it holds that $f\rho_{\nu^*} \in L^1(\mathrm{d}x)$. Moreover, since the variance of ν^* is larger than the variance of Y, that is $\sigma_\nu^2 > \sigma_Y^2$, the ratio $\rho_Y(y+c)/\rho_{\nu^*}(y)$ is bounded for any $c \in \mathbb{R}$. As consequence, $f(\cdot)\bar\Phi(\cdot - \frac{a}{d} + \frac{d-1}{d}h) \in L^1([0,\infty), \mathrm{d}x)$, and thus, by the dominated convergence theorem,

$$\begin{aligned}&\lim_{s\to 0}\frac{1}{s}\big(H_{h+s}[f](a) - H_h[f](a)\big) = \int_0^\infty f(x)\lim_{s\to\infty}\Phi_s\Big(x - \frac{a}{d} + \frac{d-1}{d}h\Big)\,\mathrm{d}x\\ &= \frac{d-1}{d}\int_0^\infty f(x)\rho'_Y\Big(x - \frac{a}{d} + \frac{d-1}{d}h\Big)\,\mathrm{d}x = H'_h[f](a),\end{aligned} \tag{6.26}$$

which establishes the pointwise convergence in (6.24).

To show the convergence in $L^{2d}(\nu^*)$ we observe, by Lemma 6.3 again, that the function

$$\bar H_h[f](a) := \int_0^\infty |f(x)|\bar\Phi\Big(x - \frac{a}{d} + \frac{d-1}{d}h\Big)\,\mathrm{d}x = c\big(H_h[f](a+r) + H_h[f](a-r)\big) \tag{6.27}$$

dominates $\left|\frac{1}{s}(H(h+s,f) - H(h,s))\right|$ for all small $|s|$. Moreover, by Lemma 6.4, $\theta_{\pm r}H_h[f] \in L^{2d}(\nu^*)$ and thus also $\bar H_h[f] \in L^{2d}(\nu^*)$. The $L^{2d}(\nu^*)$ convergence in (6.24) thus follows by another application of the dominated convergence theorem.

Claim (6.22) then follows from (6.24) and the definition (6.8) of F by the chain rule:

$$D_hF(h,f) = -D_h\Big(1_{[0,\infty)}\big(1 - H_h[f]\big)^d\Big) = 1_{[0,\infty)}d\big(1 - H_h[f]\big)^{d-1}H'_h[f], \tag{6.28}$$

as required.

Finally, we show that $(h,f) \mapsto D_hF(h,f)$ is continuous. We first observe that by similar arguments as in the proof of Lemma 6.4, one can show that $H'_h[f]$ satisfies analogous estimates as $H_h[f]$, namely, for $h, h' \in \mathbb{R}$ and $f \in L^2(\nu^*)$,

$$\begin{aligned}\|H'_h[f]\|_{L^{2d}(\nu^*)} &\le C_h\|f\|_{L^2(\nu^*)},\\ \|H'_h[f] - H'_{h'}[f]\|_{L^{2d}(\nu^*)} &\le c|h-h'|\|f\|_{L^2(\nu^*)}.\end{aligned} \tag{6.29}$$

Then, again similarly to the proof of the continuity of $D_f F(h,f)$, ignoring the non-essential prefactor $d1_{[0,\infty)}$, $D_h F(h,f) - D_h F(h',f')$ can be written as

$$\begin{aligned}&(1-H_h[f])^{d-1}H'_h[f] - (1-H_{h'}[f'])^{d-1}H'_{h'}[f'] \\ &= \big((1-H_h[f])^{d-1} - (1-H_{h'}[f'])^{d-1}\big)H'_h[f] + (1-H_{h'}[f'])^{d-1}\big(H'_h[f] - H'_{h'}[f']\big).\end{aligned} \tag{6.30}$$

From this the continuity of $D_h F(h,f)$ follows by the same arguments as before, replacing some of the estimates on $H_h[f]$ by analogous estimates (6.29) when needed. □

Lemma 6.7. *The relevant second partial Fréchet derivatives of F are continuous and given by*

$$D_{ff}F(h,f)(g_1,g_2) = -1_{[0,\infty)}d(d-1)\big(1-H_h[f]\big)^{d-2}H_h[g_1]H_h[g_2]. \tag{6.31}$$

and

$$\begin{aligned}D_{hf}F(h,f)(g) = &-1_{[0,\infty)}d(d-1)\big(1-H_h[f]\big)^{d-2}H_h[g]H'_h[f] \\ &+ 1_{[0,\infty)}d\big(1-H_h[f]\big)^{d-1}H'_h[g].\end{aligned} \tag{6.32}$$

Proof. For fixed $h \in \mathbb{R}$, $D_f F(h,\cdot)$ can be written as a composition of functions $D_f F(h,\cdot) = F_2 \circ F_1$ with $F_1 : L^2(\nu^*) \to L^{2d/(d-1)}(\nu^*)$, $F_1(f) = 1_{[0,\infty)}d(1-H_h[f])^{d-1}$ and $F_2 : L^{2d/(d-1)}(\nu^*) \to B(L^2(\nu^*), L^2(\nu^*))$, $F_2(l)(g) = -g + l \cdot H_h[g]$. By (6.18), F_1 is C^1 with $DF_1(f)(g) = -1_{[0,\infty)}d(d-1)(1-H_h[f])^{d-2}H_h[g]$. Further, via calculating the term $F_2(l+u) - F_2(l)$, one gets $DF_2(l)(u)(g) = u \cdot H_h[g]$. Using the chain rule gives $D_{ff}F(h,f)$ as stated in (6.31).

To compute $D_{hf}F(h,f)$, we fix $h \in \mathbb{R}$ and write $D_h F(h,\cdot)$ as a multiplication of functions $D_h F(h,\cdot) = F_1(\cdot)H'_h[\cdot]$, where F_1 is as described above. Noting that due to linearity, $D_f H'_h[f](g) = H'[g]$, and using the product rule for Fréchet derivatives (see, e.g., Standard Example 3 to Theorem 4.D in [Zei95]) gives (6.32).

To see the continuity of $D_{ff}F(h,f)$, let $h, h' \in I$ and $f, f', g \in L^2(\nu^*)$. The difference $D_{ff}F(h,f)(g_1,g_2) - D_{ff}F(h',f')(g_1,g_2)$ can be written as (again ignoring the prefactor $d1_{[0,\infty)}$)

$$\begin{aligned}(1-&H_h[f])^{d-2}H_h[g_1]H_h[g_2] - (1-H_{h'}[f'])^{d-2}H_{h'}[g_1]H_{h'}[g_2] \\ &= \big((1-H_h[f])^{d-2} - (1-H_{h'}[f'])^{d-1}\big)H_h[g_1]H_h[g_2] \\ &\quad + \frac{1}{2}(1-H_{h'}[f'])^{d-2}\big(H_h[g_1] - H_{h'}[g_1]\big)\big(H_h[g_2] + H_{h'}[g_2]\big) \\ &\quad + \frac{1}{2}(1-H_{h'}[f'])^{d-2}\big(H_h[g_1] + H_{h'}[g_1]\big)\big(H_h[g_2] - H_{h'}[g_2]\big).\end{aligned} \tag{6.33}$$

Using the Hölder's inequality for three functions on every summand together with similar arguments as in the proofs of the Lemmas 6.5 and 6.6, then implies the continuity of $(f,h) \mapsto D_{ff}F(f,h)$. Similarly, for $D_{fh}F(h,f)$, the analogous decomposition of $D_{fh}F(h,f) - D_{fh}F(h',f')$ together with previous arguments gives the continuity. □

It is left to check that properties (c) and (d) of Proposition 6.2 are satisfied.

Lemma 6.8. *Let $\chi^* = \theta_{h^*}\chi_{h^*}$ be a shift of χ_{h^*}. It holds that*

$$N(D_fF(h^*,0)) = R(D_fF(h^*,0))^\perp = \operatorname{span}\{\chi^*\}, \tag{6.34}$$

where $^\perp$ stands for the orthogonal complement in $L^2(\nu^*)$. In particular,

$$\dim N(D_fF(h^*,0)) = \operatorname{codim} R(D_fF(h^*,0)) = 1. \tag{6.35}$$

Proof. By Lemma 6.5 and (6.5) it holds

$$D_fF(h^*,0)(g) = -g + 1_{[0,\infty)}dH_{h^*}[g] = -g + \theta_{h^*}L_h[\theta_{h^*}^{-1}g]. \tag{6.36}$$

Therefore $g \in N(D_fF(h^*,0))$ if and only if $\theta_{h^*}^{-1}g$ is an eigenfunction of L_{h^*} corresponding to eigenvalue 1. By Proposition 3.1, χ is the only such eigenfunction, and thus

$$N(D_fF(h^*,0)) = \operatorname{span}\{\theta_{h^*}\chi_{h^*}\} = \operatorname{span}\{\chi^*\} \tag{6.37}$$

and its dimension is equal to one.

The property that $l \in R(D_fF(h^*,0))^\perp$ is equivalent to $\langle l, D_fF(h^*,0)(g)\rangle_{\nu^*} = 0$ for all $g \in L^2(\nu^*)$. However, since by Proposition 3.1 L_h is self-adjoint on $L^2(\nu)$, (6.36) implies that $D_fF(h^*,0)$ is self-adjoint on $L^2(\nu^*)$. Therefore, this is equivalent to $0 = \langle l, D_fF(h^*,0)(g)\rangle_{\nu^*} = \langle D_fF(h^*,0)(l), g\rangle_{\nu^*}$ for all $g \in L^2(\nu^*)$. However, this is true iff $l \in N(D_fF(h^*,0)) = \operatorname{span}\{\chi^*\}$. □

Lemma 6.9. *It holds*

$$D_{hf}F(h^*,0)(\chi^*) \notin R(D_fF(h^*,0)). \tag{6.38}$$

Proof. By Lemma 6.8, $g \in R(D_fF(h^*,0))$ iff g orthogonal to $\operatorname{span}\{\chi^*\}$. We thus only need to show that $\langle \chi^*, D_{hf}F(h^*,0)(\chi^*)\rangle_{\nu^*} \neq 0$.

Recall that $\sigma_Y^2 = \frac{d+1}{d}$, and thus $\rho_Y'(x) = -\frac{d}{d+1}x\rho_Y(x)$. By Lemma 6.7, for $a, s \in \mathbb{R}$, using that $H_{h^*}[0] = 0$ and the definition (6.23) of H_h',

$$\begin{aligned}
&D_{hf}F(h^*,0)(\chi^*)(a) = 1_{[0,\infty)}(a)dH_{h^*}'[\chi^*](a) \\
&= 1_{[0,\infty)}(a)(d-1)\int_0^\infty \chi^*(x)\rho_Y'\Big(x - \frac{a}{d} + \frac{d-1}{d}h^*\Big)\,\mathrm{d}x \\
&= -1_{[0,\infty)}(a)\frac{d(d-1)}{d+1}\int_0^\infty \chi^*(x)\Big(x - \frac{a}{d} + \frac{d-1}{d}h^*\Big)\rho_Y\Big(x - \frac{a}{d} + \frac{d-1}{d}h^*\Big)\,\mathrm{d}x.
\end{aligned} \tag{6.39}$$

Writing $x - \frac{a}{d} + \frac{d-1}{d}h^* = x + h^* - \frac{1}{d}(a + h^*)$, and observing that $(\cdot + h^*) = \theta_{h^*}\mathrm{Id}$ with Id being the identity map on $\mathbb{R}$, this can be written as

$$\begin{aligned} &= 1_{[0,\infty)}(a)\frac{d-1}{d+1}\Big((a+h^*)H_{h^*}[\chi^*](a) - dH_{h^*}[\chi^*\theta_{h^*}\mathrm{Id}](a)\Big) \\ &= 1_{[0,\infty)}(a)\frac{d-1}{d+1}\Big(\frac{1}{d}(\chi^*\theta_{h^*}\mathrm{Id})(a) - dH_{h^*}[\chi^*\theta_{h^*}\mathrm{Id}](a)\Big), \end{aligned} \tag{6.40}$$

where in the last equality we used that χ^* is an eigenfunction of H_{h^*} with eigenvalue d^{-1}, by (6.5). Therefore, using $\chi^* = 1_{[0,\infty)}\chi^*$ and the self-adjointness of H_{h^*} on $L^2(\nu^2)$,

$$\begin{aligned} &\big\langle \chi^*, D_{hf}F(h^*,0)(\chi^*)\big\rangle_{\nu^*} \\ &\quad= \frac{d-1}{d+1}\Big(\frac{1}{d}\big\langle\chi^*,\chi^*\theta_{h^*}\mathrm{Id}\big\rangle_{\nu^*} - d\big\langle\chi^*, H_{h^*}[\chi^*\theta_{h^*}\mathrm{Id}]\big\rangle_{\nu^*}\Big) \\ &\quad= \frac{d-1}{d+1}\Big(\frac{1}{d}\big\langle\chi^*,\chi^*\theta_{h^*}\mathrm{Id}\big\rangle_{\nu^*} - d\big\langle H_{h^*}[\chi^*], \chi^*\theta_{h^*}\mathrm{Id}\big\rangle_{\nu^*}\Big) \\ &\quad= \frac{d-1}{d+1}\Big(\frac{1}{d}\big\langle\chi^*,\chi^*\theta_{h^*}\mathrm{Id}\big\rangle_{\nu^*} - \big\langle\chi^*,\chi^*\theta_{h^*}\mathrm{Id}\big\rangle_{\nu^*}\Big) \\ &\quad= -\frac{(d-1)^2}{d(d+1)}\langle\chi,\chi\mathrm{Id}\rangle_\nu \neq 0, \end{aligned} \tag{6.41}$$

where in the last step we first applied $\theta_{h^*}^{-1}$ and then used the fact that $\chi(a) > 0$ iff $a \in [h^*, \infty)$, and thus $\langle\chi,\chi\mathrm{Id}\rangle_\nu > 0$. ☐

We can now prove Theorem 2.6.

Proof of Theorem 2.6. As already explained, we apply Proposition 6.2 to the function F defined in (6.8), with $X = Y = L^2(\nu^*)$, h and f playing the role of t and x, respectively, and with h^* corresponding to t_0. By Lemmas 6.5–6.9, the requirements (a)–(d) of this proposition are satisfied with $x_0 = \chi^*$, and $D_{ff}F$ is continuous. Taking for $Z = (\chi^*)^\perp$ for sake of concreteness, there is thus a neighbourhood $U \subset \mathbb{R} \times L^2(\nu^*)$ of $(h^*, 0)$ such that all non-trivial (that is non-zero) solutions of $F(h,f) = 0$ in U can be written as $\{(\varphi(\alpha), \alpha\chi^* + \alpha\psi(\alpha)) : |\alpha| < a\}$ for some $a > 0$ and C^1 functions $\varphi : (-a,a) \to \mathbb{R}$, $\psi : (-a,a) \to (\chi^*)^\perp \subset L^2(\nu^*)$ with $\varphi(0) = h^*$, $\psi(0) = 0$. Further, by (6.4),

$$\frac{1}{2}D_{ff}F(h^*,0)(\chi^*,\chi^*) + D_fF(h^*,0)(\psi'(0)) + \varphi'(0)D_{hf}F(h^*,0)(\chi^*) = 0. \tag{6.42}$$

Since, $\varphi(\alpha) = h^* + \alpha\varphi'(0) + o(\alpha)$ as $\alpha \to 0$, and thus $\varphi^{-1}(h) = \frac{h-h^*}{\varphi'(0)} + o(h - h^*)$ as $h \to h^*$, the non-trivial solution $\alpha\chi^* + \alpha\psi(\alpha)$ can be expanded as a function of h, in a neighbourhood of h^*,

$$\varphi^{-1}(h)\chi^* + \varphi^{-1}(h)\psi(\varphi^{-1}(h)) = \frac{h-h^*}{\varphi'(0)}(\chi^* + r_h^*), \tag{6.43}$$

where the reminder function r_h^* satisfies $\lim_{h\to h^*}\|r_h^*\|_{L^2(\nu^*)} = 0$.

Recall now from (6.7), (6.8) that $F(h, f) = 0$ is the equation for the shifted forward percolation probability $\widetilde{\eta}_h$. Shifting everything back by θ_h^{-1}, using the fact that by the continuity of the percolation probabilities (Corollary 2.2), the solution obtained from the bifurcation analysis must agree with $\eta^+(h, a)$, we obtain from (6.43) that

$$\eta^+(h, \cdot) = \frac{h - h^*}{\varphi'(0)} \theta_h^{-1}(\chi^* + r_h^*) = \frac{h - h^*}{\varphi'(0)} \big(\chi + (\theta_{h^*-h}\chi - \chi) + \theta_{-h} r_h^*\big). \tag{6.44}$$

We first show that

$$r_h^+ := (\theta_{h-h^*}\chi - \chi) + \theta_{-h} r_h^* \to 0 \qquad \text{as } h \to h^* \text{ in } L^{2-\varepsilon}(\nu). \tag{6.45}$$

For the first summand $\theta_{h-h^*}\chi - \chi$, this follows easily from the continuity of χ (Proposition 3.1), growth estimates on χ (Proposition 3.3) and the dominated convergence theorem. For the second summand, it holds that

$$\begin{aligned} \|\theta_{-h} r_h^*\|_{L^{2-\varepsilon}(\nu)}^{2-\varepsilon} &= \int |r_h^*(a - h)|^{2-\varepsilon} \rho_\nu(a) \, \mathrm{d}a \\ &= \int |r_h^*(a)|^{2-\varepsilon} \frac{\rho_\nu(a + h)}{\rho_{\nu^*}(a)} \rho_{\nu^*}(a) \, \mathrm{d}a \\ &\le \|r_h^*\|_{L^2(\nu^*)}^{2-\varepsilon} \Big\| \frac{\rho_\nu(a + h)}{\rho_{\nu^*}(a)} \Big\|_{L^{2/\varepsilon}(\nu^*)}, \end{aligned} \tag{6.46}$$

where we applied Hölder's inequality on the last line. The first factor on the right-hand side converges to 0 as $h \to h^*$, and the second factor remains bounded, which completes the proof of (6.45).

We proceed by computing $\varphi'(0)$. To this end we project (6.42) to the χ^* direction by applying $\langle \chi^*, \cdot \rangle_{\nu^*}$ on both sides. Note that, by Lemma 6.8, the range of $D_f F(h^*, 0)$ is orthogonal to χ^*, and thus $\langle \chi^*, D_f F(h^*, 0)(\psi'(0)) \rangle_{\nu^*} = 0$. Hence,

$$\varphi'(0) = -\frac{\langle \chi^*, D_{ff} F(h^*, 0)(\chi^*, \chi^*) \rangle_{\nu^*}}{2 \langle \chi^*, D_{hf} F(h^*, 0)(\chi^*) \rangle_{\nu^*}}. \tag{6.47}$$

The scalar product in the denominator was computed in (6.41). For the numerator, by Lemma 6.7,

$$D_{ff} F(h^*, 0)(\chi^*, \chi^*) = -d(d-1) H_{h^*}[\chi^*] H_{h^*}[\chi^*] = -\frac{d-1}{d} (\chi^*)^2, \tag{6.48}$$

and therefore,

$$\langle \chi^*, D_{ff} F(h^*, 0)(\chi^*, \chi^*) \rangle_{\nu^*} = -\frac{d-1}{d} \langle \chi^*, (\chi^*)^2 \rangle_{\nu^*} = -\frac{d-1}{d} \langle \chi, \chi^2 \rangle_\nu. \tag{6.49}$$

As consequence,

$$\varphi'(0) = -\frac{1}{2} \frac{d+1}{d-1} \frac{\langle \chi, \chi^2 \rangle_\nu}{\langle \chi^2, \mathrm{Id} \rangle_\nu}. \tag{6.50}$$

Claim (2.15) of the theorem then follows directly from (6.44), (6.45) and (6.50).

To obtain the asymptotics (2.16) of η, note that for any $a \geq h^*$, using similar arguments in the proof of the recursion property (5.5),

$$\begin{aligned}\eta(h,a) &= P_a[|\mathcal{C}_o^h| = \infty] = 1 - P_a[|\mathcal{C}_o^h| < \infty] \\ &= 1 - \Big(E_Y\Big[1 - \eta^+\Big(h, \frac{a}{d} + Y\Big)\Big]\Big)^{d+1} \\ &= (d+1)E_Y\Big[\eta^+\Big(h, \frac{a}{d} + Y\Big)\Big]\big(1 + o(1)\big)\end{aligned} \tag{6.51}$$

as $h \uparrow h^*$. The same argument applied to η^+ implies

$$\eta^*(h,a) = dE_Y\Big[\eta^+\Big(h, \frac{a}{d} + Y\Big)\Big]\big(1 + o(1)\big) \tag{6.52}$$

and thus, for every $a \in \mathbb{R}$,

$$\lim_{h \uparrow h^*} \frac{\eta(h,a)}{\eta^+(h,a)} = \frac{d+1}{d}. \tag{6.53}$$

which together with (2.15) shows (2.16) and completes the proof. □

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