MOMENTS AND DISTRIBUTION OF THE LOCAL TIME OF A TWO-DIMENSIONAL RANDOM WALK

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ABSTRACT. Let $\ell(n,x)$ be the local time of a random walk on $\mathbb{Z}^2$. We prove a strong law of large numbers for the quantity $L_n(\alpha) = \sum_{x\in\mathbb{Z}^2} \ell(n,x)^\alpha$ for all $\alpha \geq 0$. We use this result to describe the distribution of the local time of a typical point in the range of the random walk.

1. Introduction

Let X_i, $i \in \mathbb{N}$, be a sequence of i.i.d. random vectors on some probability space $(\Omega, \mathbb{P})$, which have values in $\mathbb{Z}^2$, mean 0, and a finite non-singular covariance matrix Σ. We write

$$S_0 := 0, \qquad S_n := \sum_{i=1}^{n} X_i, \quad n \geq 1, \tag{1}$$

for a $\mathbb{Z}^2$-valued random walk. Let $\ell(n,x)$ be its local time,

$$\ell(n,x) := \sum_{i=0}^{n} \mathbb{1}\{S_i = x\}, \qquad x \in \mathbb{Z}^2. \tag{2}$$

We will always assume that the characteristic function of X_i,

$$\chi(k) := \mathbb{E}\exp\big(i\langle k, X_1\rangle\big), \qquad k \in J := [-\pi,\pi)^2, \tag{3}$$

satisfies $\chi(k) = 1 \Leftrightarrow k = 0$. Here $\langle\cdot,\cdot\rangle$ stands for the standard scalar product in $\mathbb{R}^2$.

In this paper we prove the following strong law of large numbers for random variables

$$L_n(\alpha) := \sum_{x\in\mathbb{Z}^2} \ell(n,x)^\alpha, \qquad \alpha \geq 0, n \in \mathbb{N}. \tag{4}$$

Theorem 1. *For all $\alpha \geq 0$, $\mathbb{P}$-a.s.,*

$$\lim_{n\to\infty} \frac{L_n(\alpha)}{n(\log n)^{\alpha-1}} = \frac{\Gamma(\alpha+1)}{(2\pi\sqrt{\det\Sigma})^{\alpha-1}}. \tag{5}$$

Date: September 14, 2006.

Key words and phrases. Local time, Random walk, Strong law of large numbers.

* Work supported by the DFG Research Center "Matheon".

Remark. This result is trivial for $\alpha = 1$ and well known for $\alpha = 0$. In the second case, $L_n(0) = \sum_x \mathbb{1}\{\ell(n,x) \geq 1\} =: R(n)$ is the size of the range of the random walk. For the simple random walk it was proved in [DE51] that the range satisfies

$$\lim_{n\to\infty} \frac{\log n}{n} R(n) = \pi, \qquad \mathbb{P}\text{-a.s.} \tag{6}$$

For a non-simple walk with a covariance matrix Σ the right hand side of (6) must be multiplied by $2\sqrt{\det \Sigma}$.

There are at least two reasons why the quantity $L_n(\alpha)$ is worth to study. First, if α is an integer, then $L_n(\alpha)$ is related to the number of α-fold self-intersections of the random walk (see also (11) below). This is of much importance, mainly with $\alpha = 2$ or $\alpha = 0$, for the so-called self-interacting random walk, see e.g. [BS95]. In this paper, however, we do not require α being integer. $L_n(\alpha)$ can be then considered as a possible candidate for a definition of the number of α-fold self-intersections for all real positive α.

The second related subject, which was the original motivation for studying $L_n(\alpha)$, is so-called *random walk in random scenery* and with it closely connected problem of *aging in trap models*. We describe this problem briefly. Let τ_x, $x \in \mathbb{Z}^2$, be a collection of i.i.d. random variables independent of X_i. Define

$$Z_n := \sum_{i=0}^{n} \tau_{S_i}. \tag{7}$$

This process (called usually random walk in random scenery) was first time considered for one-dimensional random walks in [KS79]. Two-dimensional walks were studied in [Bol89], where the random scenery τ_x was required to have mean zero and a finite variance σ^2. It was proved there that the process $Z_{\lfloor nt \rfloor}/\sqrt{n \log n}$ converges to the standard Brownian motion with a variance depending on σ and Σ.

In [BČM06] we needed to control the behaviour of Z_n for a scenery τ_x in the domain of attraction of a non-negative, α-stable, $\alpha \in (0,1)$, law. The interest in this kind of scenery originated in the study of aging in so called Bouchaud's trap model. This model was proposed by [Bou92] in physics literature to explain basic mechanisms that can be responsible for peculiar dynamical properties (like aging) of complex disordered systems. The α-stable sceneries with small α correspond to the low-temperature regime in these systems that is particularly interesting. In the simplest case, Bouchaud's trap model is a Markov process $\mathcal{X}(t)$ on $\mathbb{Z}^2$ (or some other graph) which is defined as a random time change of the random walk, $\mathcal{X}(t) := S_{Z^{-1}(t)}$ (here Z^{-1} denotes the right-continuous inverse of Z_n). To show aging behaviour in this model entails, e.g., to prove that the probability of the event $\mathcal{X}(\theta t) = \mathcal{X}(t)$, $\theta > 0$, converges to some non-trivial value as $t \to \infty$. Since $\mathcal{X}(t)$ is a time change of the random walk, the first step in proving such a claim should be logically the behaviour of the time-change process Z_n.

What is the connection of Z_n with $L_n(\alpha)$? Consider for simplicity τ_x to be α-stable with $\mathbb{E}\exp(-\lambda\tau_x) = \exp(-c\lambda^\alpha)$. Then the Laplace transformation of Z_n can be rewritten as

$$\mathbb{E}_{\tau,X}e^{-\lambda Z_n} = \mathbb{E}_X \exp\Big(-c\lambda^\alpha \sum_x \ell(n,x)^\alpha\Big) = \mathbb{E}_X e^{-c\lambda^\alpha L_n(\alpha)}. \tag{8}$$

Here the first expectation is over both τ_x and X_i. When we started to investigate aging on $\mathbb{Z}^2$, we did not find any useful result about $L_n(\alpha)$ in the literature. Therefore in [BČM06] we used methods which do not rely on formula (8) to show that for α-stable τ_x, the process $Z_{\lfloor nt\rfloor}/\sqrt{n(\log n)^{\alpha-1}}$ converges to an α-stable subordinator for a.e. random environment. Going back, this result together with (8) allows to deduce a weak law of large numbers for $L_n(\alpha)$, $\alpha \in (0,1)$. It is however not possible without a major effort to use the techniques of [BČM06] to show a strong law. This consequently induces complications when one tries to extend the convergence to $\alpha > 1$. That is why different methods are used here.

To close the introduction it should be remarked that even knowing the behaviour of $L_n(\alpha)$, the proof of aging would be not completely straightforward. The methods used in [BČM06] describe more precisely the process $\mathcal{X}(t)$ and not only the time change Z_n.

The proof of Theorem 1 for $\alpha \in \mathbb{N}$ is relatively standard, as will be seen later. The main question is how to extend it to all $\alpha \geq 0$. This extension is made possible by the following theorem that describes the distribution of the local time of a "typical" point in the range of the random walk.

Theorem 2. *Given $X := \{X_1, X_2, \dots\}$ let Y_n be a point chosen uniformly in the range of the random walk up to time n, that is*

$$\mathbb{P}[Y_n = x|X] = R(n)^{-1}\mathbb{1}\{\ell(n,x) \geq 1\}. \tag{9}$$

Then for $\mathbb{P}$-a.e. X, the normalised random variable $\ell(n,Y_n)$ is asymptotically exponentially distributed, namely

$$\mathbb{P}\Big[2\pi\sqrt{\det\Sigma}\,\frac{\ell(n,Y_n)}{\log n} \geq u\Big|X\Big] \xrightarrow{n\to\infty} e^{-u}. \tag{10}$$

Remark. This result is, to a certain extent, related to the fact that the distribution of the normalised local time of the origin, $(\log n)^{-1}\ell(n,0)$, converges to the exponential distribution with mean π, which was proved for the simple random walk in [ET60]. A possible interpretation of Theorem 2 is then: "The origin becomes asymptotically typical."

The following strategy will be used in the proofs. We first prove Theorem 1 for $\alpha \in \mathbb{N}$. This will allow us to show Theorem 2 and then extend Theorem 1 to $\alpha \ge 0$.

2. Proofs of the theorems

We first prove Theorem 1 for $\alpha \in \mathbb{N}$. We compute the expected value, $\mathbb{E}L_n(\alpha)$, and bound from above the variance, $\operatorname{Var} L_n(\alpha)$, using relatively standard techniques (see e.g. [Bol89] which we follow closely). We then use these estimates to prove a strong law of large numbers along sufficiently fast increasing sequences, and finally we fill the gaps in these sequences.

Expected value. For $\alpha \in \mathbb{N}$ the random variable $L_n(\alpha)$ can be written as

$$L_n(\alpha) = \sum_{x\in\mathbb{Z}^2} \Big(\sum_{i=0}^n \mathbb{1}\{S_i = x\}\Big)^\alpha = \sum_{k_1,\dots,k_\alpha=0}^n \mathbb{1}\{S_{k_1} = \dots = S_{k_\alpha}\}. \quad (11)$$

Therefore,

$$\begin{aligned}\mathbb{E}L_n(\alpha) &= \sum_{k_1,\dots,k_\alpha=0}^n \mathbb{P}[S_{k_1} = \dots = S_{k_\alpha}] \\ &= \sum_{\beta=1}^\alpha C(\alpha,\beta) \sum_{0\le k_1<\dots<k_\beta\le n} \mathbb{P}[S_{k_1} = \dots = S_{k_\beta}],\end{aligned} \quad (12)$$

where $C(\alpha,\beta)$ are combinatorial factors depending only on α and on β, which is the number of different values in sequence $k_1,\dots,k_\alpha$. In particular $C(\alpha,\alpha) = \alpha! = \Gamma(\alpha+1)$. Values of all others $C(\alpha,\beta)$ are irrelevant, as we will see. Using the Markov property we get

$$a_\beta(n) := \sum_{0\le k_1<\dots<k_\beta\le n} \mathbb{P}[S_{k_1} = \dots = S_{k_\beta}] = \sum_{m\in M_n} \prod_{i=1}^{\beta-1} \mathbb{P}[S_{m_i} = 0], \quad (13)$$

where

$$M_n = \Big\{m = (m_0,\dots,m_\beta) \in \mathbb{N}_0^{\beta+1}, m_1,\dots,m_{\beta-1} \ge 1, \sum m_i = n\Big\}. \quad (14)$$

We set $\rho_\beta(\lambda) = \sum_{n=0}^\infty \lambda^n a_\beta(n)$ and use the fact that

$$\mathbb{P}(S_j = x) = (2\pi)^{-2} \int_J \chi(k)^j \exp(-i\langle k,x\rangle)\, dk. \quad (15)$$

An easy computation yields

$$\rho_\beta(\lambda) = (1-\lambda)^{-2}\Big(\int_J \frac{dk}{(2\pi)^2}\,\frac{\lambda\chi(k)}{1-\lambda\chi(k)}\Big)^{\beta-1}. \tag{16}$$

As in [Bol89], for two positive functions $f_\delta(\lambda)$ and $g_\delta(\lambda)$, $\delta > 0$, $\lambda \in (0,1)$, which diverge for $\lambda \to 1$ we write

$$f_\delta(\lambda) \underset{\delta\to 0}{\sim} g_\delta(\lambda) \tag{17}$$

if

$$\lim_{\delta\to 0}\liminf_{\lambda\to 1} f_\delta(\lambda)/g_\delta(\lambda) = \lim_{\delta\to 0}\limsup_{\lambda\to 1} f_\delta(\lambda)/g_\delta(\lambda) = 1. \tag{18}$$

Let $U_\delta \subset J$, $k \in U_\delta \Leftrightarrow \langle k, \Sigma k\rangle \le \delta$. It is easy to see that

$$\Big|\int_{J\setminus U_\delta} \frac{dk}{(2\pi)^2}\,\frac{\lambda\chi(k)}{1-\lambda\chi(k)}\Big| \le \text{const.}\,\delta^{-1} \qquad \text{for all } \lambda \le 1. \tag{19}$$

To treat the integral over U_δ, we observe first that the characteristic function of X_i, $\chi(k)$, has the following expansion around 0:

$$\chi(k) = 1 - \tfrac{1}{2}\langle k, \Sigma k\rangle + R(k), \qquad \text{where } |R(k)| = o(|k|^2) \text{ for } k \to 0. \tag{20}$$

Using this expansion it can be shown that

$$\int_{U_\delta} \frac{dk}{(2\pi)^2}\,\frac{\lambda\chi(k)}{1-\lambda\chi(k)} \underset{\delta\to 0}{\sim} \big(2\pi\sqrt{\det\Sigma}\big)^{-1}\log\frac{1}{1-\lambda}. \tag{21}$$

Inserting this back into (16) it follows from the Tauberian theorem for sequences (see [Fel71], Theorem XIII 5.5), and the fact that $a_\beta(n)$ are monotone that

$$a_\beta(n) = n\Big(\frac{\log n}{2\pi\sqrt{\det\Sigma}}\Big)^{\beta-1}(1+o(1)), \qquad \text{as } n\to\infty. \tag{22}$$

In particular $a_\alpha(n) \gg a_\beta(n)$ for all $\beta < \alpha$. Therefore, using also (12), for all $\alpha \in \mathbb{N}$

$$\mathbb{E}L_n(\alpha) = \frac{\Gamma(\alpha+1)}{(2\pi\sqrt{\det\Sigma})^{\alpha-1}}\, n(\log n)^{\alpha-1}(1+o(1)), \qquad \text{as } n\to\infty. \tag{23}$$

Variance. The computation of the variance is similar but relatively complicated. We will show that

$$\operatorname{Var} L_n(\alpha) = O\big(n^2(\log n)^{2\alpha-4}\big). \tag{24}$$

We first rewrite Var $L_n(\alpha)$ in spirit of (11),

$$\begin{aligned}\mathrm{Var} L_n(\alpha) = \sum_{k_1,\dots,k_\alpha}\sum_{l_1,\dots,l_\alpha} &\mathbb{P}[S_{k_1}=\dots=S_{k_\alpha}, S_{l_1}=\dots=S_{l_\alpha}]\\ &- \mathbb{P}[S_{k_1}=\dots=S_{k_\alpha}]\mathbb{P}[S_{l_1}=\dots=S_{l_\alpha}]\\ = \sum_{\beta,\gamma=1}^{\alpha} C(\alpha,\beta,\gamma) &\sum_{\substack{0\le k_1<\dots<k_\beta\le n\\ 0\le l_1<\dots<l_\gamma\le n}} \mathbb{P}[S_{k_1}=\dots=S_{k_\beta}, S_{l_1}=\dots=S_{l_\gamma}]\\ &- \mathbb{P}[S_{k_1}=\dots=S_{k_\beta}]\mathbb{P}[S_{l_1}=\dots=S_{l_\gamma}]\\ =: \sum_{\beta,\gamma=1}^{\alpha} C(\alpha,\beta,\gamma) &a_{\beta,\gamma}(n).\end{aligned} \tag{25}$$

Here again the precise values of the combinatorial factors $C(\alpha,\beta,\gamma)$ are irrelevant.

We want to compute $a_{\beta,\gamma}(n)$ using the same methods as for the expectation. To this end we need several definitions. Given two ordered sequences $k_1,\dots,k_\beta$ and $l_1,\dots,l_\gamma$ we define a sequence of pairs

$$(j_i,\kappa_i), \qquad i\in\{1,\dots,\beta+\gamma\}, \tag{26}$$

which satisfies $j_i\in\{0,\dots,n\}$, $\kappa_i\in\{0,1\}$, $j_i\le j_{i+i}$ for all $i\le\beta+\gamma-1$ and

$$\{j_i:\kappa_i=0\}=\{k_1,\dots,k_\beta\}, \qquad \{j_i:\kappa_i=1\}=\{l_1,\dots,l_\gamma\}. \tag{27}$$

To rule out possible ties we require: if $j_i=j_{i+1}$, then $\kappa_i<\kappa_{i+1}$. We then set $m_0=j_1$, $m_{\beta+\gamma}=n-j_{\beta+\gamma}$, and

$$\varepsilon_i=\kappa_{i+1}-\kappa_i, \qquad m_i=j_{i+1}-j_i, \qquad \text{for } i=1,\dots,\beta+\gamma-1. \tag{28}$$

Let $E(\beta,\gamma)\subset\{-1,0,1\}^{\beta+\gamma-1}$ be the set of all possible sequences $\varepsilon=\{\varepsilon_i, i=1,\dots,\beta+\gamma-1\}$ that can be produced using this construction. This set is obviously finite. Let further $M_{\beta,\gamma}(\varepsilon,n)$ be the set of all $m=(m_0,\dots,m_{\beta+\gamma})$ such that $m_i\in\mathbb{N}_0$, $\sum m_i=n$, and m is compatible with ε. To be compatible with ε imposes $m_i\ge 1$ for some ε-dependent i's. Since we are looking for an upper bound we will generally ignore these restrictions.

We can now compute $a_{\beta,\gamma}(n)$. Observe first that if there is only one $\varepsilon_i\ne 0$, then $k_\beta\le l_1$ or $l_\gamma\le k_1$, and by Markov property the positive and negative term of $a_{\beta,\gamma}(n)$ in definition (25) exactly cancel each other. Therefore we can consider only $\varepsilon\in E'(\beta,\gamma):=\{\varepsilon:\sum|\varepsilon_i|\ge 2\}$. For these ε we *first* completely ignore the negative term. Therefore, again by Markov property,

$$a_{\beta,\gamma}(n) \leq \sum_{\varepsilon\in E'(\beta,\gamma)} \sum_{z\in\mathbb{Z}^2} \sum_{m\in M_{\beta,\gamma}(\varepsilon,n)} \prod_{i=1}^{\beta+\gamma-1} \mathbb{P}[S_{m_i} = \varepsilon_i z] =: \sum_{\varepsilon\in E'} a(\varepsilon, n). \tag{29}$$

Taking $\rho_\varepsilon(\lambda) = \sum_{n=0}^{\infty} a(\varepsilon,n)\lambda^n$ and setting $M_{\beta,\gamma}(\varepsilon) = \bigcup_n M_{\beta,\gamma}(\varepsilon,n)$ we get

$$\rho_\varepsilon(\lambda) = \sum_{z\in\mathbb{Z}^2} \sum_{m\in M_{\beta,\gamma}(\varepsilon)} \lambda^{m_0+m_{\beta+\gamma}} \prod_{j=1}^{\beta+\gamma-1} \int_J \frac{dk_j}{(2\pi)^2}(\lambda\chi(k_j))^{m_j} e^{-i\langle k_j, z\varepsilon_j\rangle}. \tag{30}$$

Since $\varepsilon \in E'$, there are at least two j's such that $\varepsilon_j \neq 0$. Suppose, for simplicity, that ε_1 is one of them. Using the substitution $k_1' = \sum_{i=1}^{\beta+\gamma-1} \varepsilon_i k_i$, $k_j' = k_j$ for $j \geq 2$ and applying the Fourier inversion we get

$$\rho_\varepsilon(\lambda) \leq \text{const.}(1-\lambda)^{-2} \int_{J^{\beta+\gamma-2}} \prod_{i=2}^{\beta+\gamma-1} \frac{dk_i}{1-\lambda\chi(k_i)} \frac{1}{1-\lambda\chi(f(\boldsymbol{k}))}, \tag{31}$$

where $f(\boldsymbol{k}) = \varepsilon_1 \sum_{i=2}^{\beta+\gamma-1} \varepsilon_i k_i$. Let $\delta > 0$ and let $U_\delta = \{\langle k_i, \Sigma k_i\rangle \leq \delta, \forall i = 2,\dots,\beta+\gamma-1\}$. The integral over U_δ can be rewritten using again the expansion (20) and several easy substitutions as

$$\int_{U_\delta} \prod_{i=2}^{\beta+\gamma-1} \frac{dk_i}{1-\lambda\chi(k_i)} \frac{1}{1-\lambda\chi(f(\boldsymbol{k}))} \underset{\delta\to 0}{\sim} \text{const.}(1-\lambda)^{-1} \int_{B^{\beta+\gamma-2}_{\delta/\sqrt{1-\lambda}}} \prod_{i=2}^{\beta+\gamma-1} \frac{dk_i}{1+k_i^2} \frac{1}{1+(f(\boldsymbol{k}))^2}, \tag{32}$$

where B_r is the ball in $\mathbb{R}^2$ with radius r centered at the origin. Integrating over all k_i that are not contained in $f(\boldsymbol{k})$, that means over all k_i such that $\varepsilon_i = 0$, say there is ω_ε of them, we get a factor $(\log 1/(1-\lambda))^{\omega_\varepsilon}$. The integral over the remaining k_i's stays bounded as $\lambda \to 1$. Therefore, the last expression is

$$\underset{\delta\to 0}{\sim} \text{const.}(1-\lambda)^{-1}(\log 1/(1-\lambda))^{\omega_\varepsilon}, \tag{33}$$

It can be seen easily that the integral over the set $J^{\beta+\gamma-2} \setminus U_\delta$ diverges at most as fast as the integral over U_δ. The equations (31) and (33) yield

$$\rho_\varepsilon(\lambda) \underset{\delta\to 0}{\sim} \text{const.}(1-\lambda)^{-3}\big(\log 1/(1-\lambda)\big)^{\omega_\varepsilon}. \tag{34}$$

The Tauberian theorem then implies that $a(\varepsilon,n) = O(n^2(\log n)^{\omega_\varepsilon})$.

If $\omega_\varepsilon \leq 2\alpha - 4$, this bound would be strong enough to imply (24). This is however not always the case. There is one exception: $\beta = \gamma = \alpha$ and $\varepsilon_i \neq 0$ only for two values of i, call them u, v. In this case $\omega_\varepsilon = 2\alpha - 3$. So that we cannot ignore the negative term in (25), and the computation must be refined. For simplicity we assume that $u < v$ and $\varepsilon_u = 1$, then $\varepsilon_v = -1$. Using again the Markov property we get for the contribution of this ε

$$\sum_{m \in M_{\alpha,\alpha}(\varepsilon,n)} \sum_{z \in \mathbb{Z}^2} \mathbb{P}[S_{m_u} = z] \mathbb{P}[S_{m_v} = -z] \prod_{\substack{i=1 \\ i \notin \{u,v\}}}^{2\alpha-1} \mathbb{P}[S_{m_i} = 0] \\ - \mathbb{P}[S_{m_u + \dots + m_v} = 0] \prod_{\substack{i=1 \\ i \notin \{u,v\}}}^{2\alpha-1} \mathbb{P}[S_{m_i} = 0] =: b_{u,v}(n). \tag{35}$$

Setting $\rho_{u,v}(\lambda) = \sum_{n=0}^{\infty} \lambda^n b_{u,v}(n)$, after a standard computation we get

$$\rho_{u,v}(\lambda) = \text{const.}(1-\lambda)^{-2} \Big(\log \frac{1}{1-\lambda} \Big)^{u-2+2\alpha-v} \\ \Big\{ \int \frac{1}{1-\lambda\chi(-k_u)} \prod_{i=u}^{v-1} \frac{dk_i}{1-\lambda\chi(k_i)} \\ - \int \frac{dk_u}{(1-\lambda\chi(k_u))^2} \prod_{i=u+1}^{v-1} \frac{dk_i}{1-\lambda\chi(k_i)\chi(k_u)} \Big\}. \tag{36}$$

Here, the logarithmic factor on the first line comes from those terms in (35) where $i < u$ or $i > v$. Narrowing the domain of integration to a δ-neighbourhood of the origin (which gives as always a leading divergence), using again (20) and some obvious substitutions, we get that the difference in the braces is of the order of

$$(1-\lambda)^{-1} \int_{B^{v-u}_{\delta/\sqrt{1-\lambda}}} \frac{1}{1+k_u^2} \prod_{j=u}^{v-1} \frac{dk_j}{1+k_j^2} \Big[1 - \prod_{i=u+1}^{v-1} \frac{1+k_i^2}{1+k_i^2+k_u^2} \Big]. \tag{37}$$

The difference in the brackets can be telescoped as $1 - abc = (1-a) + a(1-b) + ab(1-c)$, giving a sum of several integrals. All of them can be shown to be at most $O((\log 1/(1-\lambda))^{v-u-2})$. That is the power smaller by one than if the difference in the brackets was replaced by one. This is exactly what we needed. The usual reasoning then gives that $b_{u,v}(n) = O(n^2 (\log n)^{2\alpha-4})$ and since there is only finitely many u's and v's the proof of (24) is finished.

Strong law of large numbers for $\alpha \in \mathbb{N}$. The result for $\alpha = 1$ is trivial, therefore we consider $\alpha \geq 2$. Let $n_k = \exp k^\theta$, $1/2 < \theta < 1$. Then by Chebyshev inequality

$$\sum_{k=0}^{\infty} \mathbb{P}\big[\big(L_{n_k}(\alpha) - \mathbb{E}L_{n_k}(\alpha)\big) \geq \varepsilon \mathbb{E}L_{n_k}(\alpha)\big] \leq C(\varepsilon) \sum_{k=0}^{\infty} (\log n_k)^2 < \infty. \tag{38}$$

Therefore $L_{n_k}(\alpha)/\mathbb{E}L_{n_k}(\alpha) \to 1$ a.s. as $k \to \infty$. Let now $n_k \leq n < n_{k+1}$. Then

$$L_{n_k}(\alpha) - \mathbb{E}L_{n_{k+1}}(\alpha) \leq L_n(\alpha) - \mathbb{E}L_n(\alpha) \leq L_{n_{k+1}}(\alpha) - \mathbb{E}L_{n_k}(\alpha). \tag{39}$$

The absolute value of the two extremal terms is a.s. for all n large enough bounded by

$$\varepsilon L_{n_{k+1}}(\alpha) + \mathbb{E}L_{n_{k+1}}(\alpha) - \mathbb{E}L_{n_k}(\alpha) \leq 3\varepsilon \mathbb{E}L_n(\alpha). \tag{40}$$

This finishes the proof of Theorem 1 for $\alpha \in \mathbb{N}$.

Proof of Theorem 2. We want to show that the distribution of

$$Z_n := 2\pi\sqrt{\det \Sigma}\, \frac{\ell(n, Y_n)}{\log n} \tag{41}$$

converges a.s to the exponential distribution. We compute integer moments of Z_n.

$$\begin{aligned} \mathbb{E}[Z_n^\alpha | X] &= (2\pi\sqrt{\det \Sigma})^\alpha R(n)^{-1} \sum_{x \in \mathbb{Z}^2} \frac{\ell(n,x)^\alpha}{(\log n)^\alpha} \\ &= \frac{(2\pi\sqrt{\det \Sigma})^{\alpha-1} \sum_x \ell(n,x)^\alpha}{n(\log n)^{\alpha-1}}\, \frac{2\pi n (\log n)^{-1}\sqrt{\det \Sigma}}{R(n)}. \end{aligned} \tag{42}$$

By Theorem 1 and (6) the last expression converges a.s. to $\Gamma(\alpha+1)$. Since the α-th moment of the exponential distribution with mean one is $\Gamma(1+\alpha)$, and this distribution is determined by its integer moments, Theorem 2 is proved.

Proof of Theorem 1 for $\alpha \geq 0$. This proof is now trivial. It is sufficient to read (42) from right to left and use the fact that by Theorem 2 and by the convergence of integer moments for all integers larger than α,

$$\lim_{n \to \infty} \mathbb{E}[Z_n^\alpha | X] = \Gamma(\alpha + 1) \tag{43}$$

a.s. for all $\alpha \geq 0$. □

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