Quenched invariance principles for the maximal particle in branching random walk in random environment and the parabolic Anderson model

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July 29, 2018

Abstract

We consider branching random walk in spatial random branching environment (BRWRE) in dimension one, as well as related differential equations: the Fisher-KPP equation with random branching and its linearized version, the parabolic Anderson model (PAM). When the random environment is bounded, we show that after recentering and scaling, the position of the maximal particle of the BRWRE, the front of the solution of the PAM, as well as the front of the solution of the randomized Fisher-KPP equation fulfill quenched invariance principles. In addition, we prove that at time t the distance between the median of the maximal particle of the BRWRE and the front of the solution of the PAM is in $O(\ln t)$. This partially transfers results from [Bra78] to the setting of BRWRE.

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1 Introduction

Branching random walk and branching Brownian motion, and in particular the position of their maximally displaced particles, have been the subject of highly intensive research during the last couple of decades, see the monographs [Shi15, Bov16] as well as the references in these sources.

Indeed, in [Ham74], [Kin75], and [Big76] it has successively been shown that under suitable assumptions the position of the maximal or rightmost particle $M(n)$ of the branching random walk at time n satisfies a law of large numbers; i.e., almost surely

$$\lim_{n\to\infty} \frac{M(n)}{n} = v_0 \tag{1.1}$$

for some non-random $v_0 \in \mathbb{R}$. Subsequently, concentration results for $M(n)$ around its median $m(n)$, cf. [McD95, Ree03, Drm03, CD06], as well as corresponding results on the distributional convergence have been obtained, see [Bac00, BZ09, BZ12, Aïd13]. In particular, in [AR09, HS09] the law of large numbers of (1.1) has been improved in that for a wide class of branching random walks the position of the maximal particle $M(n)$ at time n satisfies

$$M(n) = v_0 n - \frac{3}{2} c \ln n + O(1), \tag{1.2}$$

where $c > 0$ is a parameter depending on the specifics of the branching and displacement mechanisms.

In the continuum setting of branching Brownian motion (BBM) with binary branching, replacing n by t in a suggestive way for the respective quantities, even more precise asymptotics, namely

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \ln t + o(1), \tag{1.3}$$

has been proved much earlier in seminal works by Bramson [Bra78, Bra83] already. Bramson made use of the fact that the function $w^{\mathsf{BBM}}(t,x) := \mathbb{P}(M(t) \ge x)$, $t \ge 0$, $x \in \mathbb{R}$, solves the Fisher-KPP equation

$$\begin{aligned} &\frac{\partial w^{\mathsf{BBM}}}{\partial t}(t,x) = \frac{1}{2}\Delta w^{\mathsf{BBM}}(t,x) + w^{\mathsf{BBM}}(t,x)(1 - w^{\mathsf{BBM}}(t,x)), \quad (t,x) \in [0,\infty) \times \mathbb{R}, \\ &w^{\mathsf{BBM}}(0,\cdot) = \mathbf{1}_{(-\infty,0]}. \end{aligned} \tag{1.4}$$

He then investigated the solution to this equation through an impressively refined analysis of its Feynman-Kac representation.

While the above results for branching random walk have been derived in the context of homogeneous branching mechanisms, there has recently been an increased activity in the investigation of branching random walk with non-homogeneous branching rates that depend on either time or space in special *deterministic* ways, see [LS88, LS89, FZ12a, FZ12b, BBH+15, NRR15, MZ16, Mal15, BH14, BH15]. Among other things, as an interesting consequence of the inhomogeneous branching rates, in these sources second order terms that differ from the logarithmic correction of [Bra78] and (1.2) have been obtained.

While the above sources focus on the case of deterministic branching environments, there are very compelling reasons for trying to achieve a better understanding of the case of spatially random branching environments. On the one hand, this is already interesting from a purely mathematical point of view. On the other hand, when it comes to modeling real world applications, though branching environments are not random, they oftentimes are locally irregular but exhibit certain spatial averaging properties. One natural approach is then to model the environment as random and try to understand the evolution of the process either conditionally on a realization of the branching environment or averaged over all such environments. In this setting, notable research has been conducted over the past decades on a variety of aspects such as survival and growth properties, transience vs. recurrence, diffusivity, as well as localization properties (see e.g. [GdH92, MP00, CP07a, CP07b, HY09, GMPV10, HNY11, Nak11, OR16] for a non-exhaustive list).

To the best of our knowledge, the only source that in some sense focuses on the maximal particle is Comets and Popov [CP07b]. They prove a shape theorem for a BRWRE on $\mathbb{Z}^d$, $d \geq 1$, from which, as a corollary, one can infer that the maximal particle has an asymptotic velocity, that is (1.1) holds.

Finally, branching random walk in an environment that is randomly changing in *time* was studied in [HL14, MM15b] recently. Among other results, Huang and Liu [HL14] proved a law of large numbers for the maximal particle. Mallein and Miłoś [MM15b] considered the backlog of the maximal particle behind what can be interpreted as the breakpoint in their setting (cf. (2.5) below) and proved that it is strictly larger than in the setting of constant branching rates. As a corollary, their results yield a central limit theorem for the position of the maximally displaced particle. It should be noted here that the time-dependent random environment seems to be easier to handle since certain techniques of the theory of multi-type branching processes apply in this case. We were not able to use them for the model considered in this paper.

2 Definition of the model and main results

Let us now introduce the model of branching random walk in random (branching) environment considered in this paper. The random environment is given by a family $(\xi(x))_{x\in\mathbb{Z}}$ of random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that the environment is i.i.d. and bounded:

$$\begin{aligned} &(\xi(x))_{x\in\mathbb{Z}} \text{ are i.i.d. under } \mathbb{P}, \\ &0 < \mathsf{ei} := \operatorname{ess\,inf} \xi(0) < \operatorname{ess\,sup} \xi(0) =: \mathsf{es} < \infty, \end{aligned} \tag{POT}$$

The i.i.d. property can be relaxed with some additional technical effort, but we prefer to work in this context for the sake of simplicity. Some form of boundedness is essential for our investigations.

We furthermore assume, again for reasons of simplicity, that the initial configuration $u_0 : \mathbb{Z} \to \mathbb{N}_0$ is such that

$$C\mathbf{1}_{-\mathbb{N}_0} \geq u_0 \geq \mathbf{1}_{\{0\}} \quad \text{for some } C \in [1, \infty). \tag{INI}$$

In particular, $u_0 = \mathbf{1}_{\{0\}}$ and $u_0 = \mathbf{1}_{-\mathbb{N}_0}$ fulfill (INI). Later, as a consequence of Lemmas 4.13 and 5.1 below, we show that any initial configuration satisfying (INI) is comparable for our purposes to $u_0 = \mathbf{1}_{\{0\}}$ in the results that follow. Hence, the Reader may assume $u_0 = \mathbf{1}_{\{0\}}$ from now onwards without loss of generality.

Let us now describe the dynamics of the BRWRE in detail. Given a realization of $(\xi(x))_{x\in\mathbb{Z}}$ and an initial condition $u_0 : \mathbb{Z} \to \mathbb{N}_0$, at each $x \in \mathbb{Z}$ we place $u_0(x)$ particles at time 0. As time evolves, all particles move independently according to continuous time simple random walk with jump rate 1. In addition, and independently of everything else, while at a site x, a particle splits into two at rate $\xi(x)$, and if it does so, the two new particles evolve independently according to the same diffusion mechanism as the remaining particles. This defines *branching random walk in the branching environment* ξ with binary branching, where again the latter is for simplicity but not essential. Given a realization of $(\xi(x))_{x\in\mathbb{Z}}$, we write $\mathtt{P}^{\xi}_{u_0}$ for the quenched law of the process conditional on starting with a particle configuration u_0 at time 0, and $\mathtt{E}^{\xi}_{u_0}$ for the corresponding expectation. We use $\mathbb{P}\times\mathtt{P}^{\xi}_{u_0}$ to denote the averaged law of the process. To simplify notation, we abbreviate $\mathtt{P}^{\xi}_{x}=\mathtt{P}^{\xi}_{\mathbf{1}_{\{x\}}}$.

We use $N(t)$ to denote the set of particles alive at time t in this BRWRE. For any particle $Y\in N(t)$, we denote by $(Y_s)_{s\in[0,t]}$ the trajectory of itself and its ancestors up to time t. We will also call $(Y_s)_{s\in[0,t]}$ the *genealogy* of Y. For $t\geq 0$ and $x\in\mathbb{Z}$, we define

$$N(t,x):=\big|\{Y\in N(t)\,:\,Y_t=x\}\big|\quad\text{ and}$$
$$N^{\geq}(t,x):=\big|\{Y\in N(t)\,:\,Y_t\geq x\}\big|=\sum_{y\geq x}N(t,y)\tag{2.1}$$

as the number of particles in the process at time t which are located at or to the right of x.

To state our last assumption, we recall that it is well known from the studies on the parabolic Anderson model (cf. Section 2.2 below) that there is a deterministic function $\lambda:\mathbb{R}\to\mathbb{R}$, the *Lyapunov exponent*, such that for a.e. realization of ξ the quenched expectation of $N(t,x)$ satisfies

$$\lambda(v)=\lim_{t\to\infty}\frac{1}{t}\ln\mathtt{E}^{\xi}_{0}\big[N(t,\lfloor tv\rfloor)\big],\qquad v\in\mathbb{R}.\tag{2.2}$$

Under (POT), one can show that λ is even, concave everywhere and strictly concave whenever $v\geq v_c$, for some critical velocity $v_c\in[0,\infty)$, cf. Lemma A.3 below. Furthermore, the asymptotic velocity of the maximally displaced particle (cf. (1.1)) is given by the unique $v_0\in(0,\infty)$ such that

$$\lambda(v_0)=0.\tag{2.3}$$

Throughout the paper we assume that the maximally displaced particle is faster than v_c, that is

$$v_0>v_c.\tag{VEL}$$

This assumption will ensure that there is certain tilted Gibbs measure related to BRWRE (cf. (4.4) and below) under which the particles have the speed v_0. It is not hard to see that there are potentials ξ fulfilling (POT) but not (VEL), as well as there are potentials that fulfill both. Heuristically, if the branching rates are in some sense 'large enough', then (VEL) is fulfilled.

2.1 Behavior of the maximally displaced particle

From a probabilistic point of view, in this article we are mainly interested in the behavior of the position of the maximally displaced particle at time t,

$$M(t):=\max\{Y_t\,:\,Y\in N(t)\}.$$

We use $m(t)$ to denote the quenched median of the distribution of $M(t)$,

$$m(t):=\sup\{x\in\mathbb{Z}:\mathtt{P}^{\xi}_{u_0}[M(t)\geq x]\geq 1/2\}.\tag{2.4}$$

Note that $m(t)$ is a random variable on $(\Omega,\mathcal{F},\mathbb{P})$.

To describe the behavior of $M(t)$, it is useful to introduce a quantity $\overline{m}(t)$ which is sometimes referred to as the *breakpoint* in the case of homogeneous branching rates (in our setting it is also instructive to interpret it as the front of the solution to the parabolic Anderson model, cf. Section 2.2 below). It is defined as

$$\overline{m}(t):=\sup\Big\{x\in\mathbb{Z}\,:\,\mathtt{E}^{\xi}_{u_0}\big[N^{\geq}(t,x)\big]\geq\frac{1}{2}\Big\}=\sup\Big\{x\in\mathbb{Z}\,:\,\sum_{i\leq 0}u_0(i)\mathtt{E}^{\xi}_{i}\big[N^{\geq}(t,x)\big]\geq\frac{1}{2}\Big\},\tag{2.5}$$

where the second equality follows from the independence of particles under $\mathtt{P}^{\xi}_{u_0}$.

A substantial step on our way to prove a functional central limit theorem for $M(t)$ and related quantities is the following approximation result, which is interesting in its own right.

Theorem 2.1. *Assume* (POT)*,* (INI) *and* (VEL) *to hold. Then* $m(t) \le \overline{m}(t)$*, and there exists a constant* $C \in (0,\infty)$ *such that for* $\mathbb{P}$*-a.e. realization of* ξ*,*

$$\limsup_{t\to\infty} \frac{\overline{m}(t) - m(t)}{\ln t} \le C. \tag{2.6}$$

Remark 2.2. This result should be compared to the classical results of Bramson [Bra78, Bra83] for homogeneous BBM (and to corresponding results for branching random walk [AR09, HS09]). In the case of BBM the breakpoint satisfies $\overline{m}(t) = \sqrt{2}t - \frac{1}{2\sqrt{2}} \ln t + O(1)$ which can be proved easily using Gaussian tail estimates. Together with (1.3), this yields, in the case of the BBM,

$$\lim_{t\to\infty} \frac{\overline{m}(t) - m(t)}{\ln t} = \frac{1}{\sqrt{2}}.$$

Our result thus shows that in the case of random branching rates we can recover an upper bound whose order matches that of the homogeneous branching setting. The question of whether there is a limit in (2.6) remains open.

Bearing in mind Theorem 2.1, we aim at proving a functional central limit theorem for a suitably centered and rescaled version of the process $\overline{m}(t)$ first, and then take advantage of Theorem 2.1 in order to extend it to $m(t)$. In fact, we start with proving a slightly more general statement: As a generalization to (2.5), we define

$$\overline{m}_v(t) := \sup\Big\{x \in \mathbb{N} \,:\, \mathtt{E}^{\xi}_{u_0}\big[N^{\ge}(t,x)\big] \ge \frac{1}{2} e^{t\lambda(v)}\Big\}, \qquad v > 0, t > 0, \tag{2.7}$$

where λ is the Lyapunov exponent defined in (2.2). Note that, due to the definition (2.3) of v_0, we have $\overline{m}(t) = \overline{m}_{v_0}(t)$. One of the main technical results of this paper is the following functional central limit theorem for $\overline{m}_v(t)$ with arbitrary $v > v_c$.

Theorem 2.3. *Under assumptions* (POT) *and* (INI)*, for every* $v > v_c$*, the sequence of processes*

$$[0,\infty) \ni t \mapsto \frac{\overline{m}_v(nt) - vnt}{\overline{\sigma}_v \sqrt{n}}, \quad n \in \mathbb{N},$$

converges as $n \to \infty$ *in* $\mathbb{P}$*-distribution to standard Brownian motion. The value of* $\overline{\sigma}_v \in (0,\infty)$ *is given in* (5.21) *below.*

Remark 2.4. Without further mentioning, in the functional central limit theorems we prove, we consider the space of càdlàg functions endowed with the Skorokhod topology as the underlying space.

Combining Theorems 2.1 and 2.3 we obtain the following functional limit theorem for the median $m(t)$.

Corollary 2.5. *Assuming* (POT)*,* (INI) *and* (VEL)*, with* $\overline{\sigma}_{v_0}$ *as in Theorem 2.3, the sequence of processes*

$$[0,\infty) \ni t \mapsto \frac{m(nt) - v_0 nt}{\overline{\sigma}_{v_0} \sqrt{n}}, \quad n \in \mathbb{N},$$

converges as $n \to \infty$ *in* $\mathbb{P}$*-distribution to standard Brownian motion.*

We further show that the position of the maximum $M(t)$ is close to the median.

Proposition 2.6. *Under assumptions* (POT)*,* (INI) *and* (VEL)*, there is* $C < \infty$ *such that* $\mathbb{P} \times \mathtt{P}^{\xi}_{u_0}$*-a.s.*

$$\limsup_{t\to\infty} \frac{|M(t) - m(t)|}{\ln t} \le C.$$

Finally, combining Corollary 2.5 and Proposition 2.6 we immediately obtain a functional central limit theorem for the position of the maximally displaced particle:

Theorem 2.7. *Assuming* (POT)*,* (INI) *and* (VEL)*, the sequence of processes*

$$[0,\infty) \ni t \mapsto \frac{M(nt) - v_0 nt}{\overline{\sigma}_{v_0} \sqrt{n}}, \quad n \in \mathbb{N},$$

converges as $n \to \infty$ *in* $\mathbb{P} \times \mathtt{P}^{\xi}_{u_0}$*-distribution to standard Brownian motion.*

2.2 Implications for the PAM and randomized Fisher-KPP equation

As we have touched upon previously in the introduction, there is a close connection between certain partial differential equations and branching processes: In the case of BBM, it is easy to see that the density $u(t,x)$ of the expected number of particles satisfies

$$\frac{\partial}{\partial t}u^{\mathsf{BBM}}(t,x) = \frac{1}{2}\Delta u^{\mathsf{BBM}}(t,x) + u^{\mathsf{BBM}}(t,x). \tag{2.8}$$

As this equation is essentially the heat equation (write $u^{\mathsf{BBM}} = e^t v$), this allows to estimate the corresponding breakpoint $\sup\{x > 0 : u(t,x) \geq 1/2\}$ with high accuracy using Gaussian tail estimates. Moreover, as we already mentioned, $w^{\mathsf{BBM}}(t,x) = \mathbb{P}(M(t) \geq x)$ satisfies the Fisher-KPP equation (1.4). In particular, the front of the solution to (1.4), defined as $\sup\{x \in \mathbb{R} : w^{\mathsf{BBM}}(t,x) \geq 1/2\}$, coincides with the median $m(t)$ of the distribution of the maximal particle of the BBM. Therefore, Bramson's result (1.3) immediately gives equally precise information on the position of the front of the solution to (1.4) as well.

In our setting of inhomogeneous branching rates the situation is both more complicated but also more interesting. The breakpoint in the case of heterogeneous branching rates corresponds to the front of the solution to the *parabolic Anderson model* (PAM), a discrete randomized version of (2.8),

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \Delta_d u(t,x) + \xi(x)u(t,x), \qquad t \geq 0, x \in \mathbb{Z}, \\ u(0,x) &= u_0(x), \qquad x \in \mathbb{Z}, \end{aligned} \tag{2.9}$$

Here, Δ_d denotes the discrete Laplace operator

$$\Delta_d f(x) = \frac{1}{2} \sum_{y \in \mathbb{Z} : |y-x|=1} (f(y) - f(x)), \qquad x \in \mathbb{Z}.$$

It is well-known that conditionally on ξ, the expected number of particles at time t and position x

$$u(t,x) := \mathtt{E}^{\xi}_{u_0}[N(t,x)] \tag{2.10}$$

solves (2.9) (cf. the original source [GM90] as well as [GK05] and [Kön16] for more recent surveys). Hence, due to (2.5) and (2.10), the process $\overline{m}(t)$ can be viewed as the *front* of the solution to the PAM, which, according to Theorem 2.3, fulfills a corresponding functional central limit theorem.

This functional central limit theorem can be supplied with another one, for the logarithm of the function $u(t,x)$ itself: Since statement (2.2) can be read as a law of large numbers for $t^{-1}\ln u(t, \lfloor tv \rfloor)$, it is natural to inquire about the fluctuations. Our investigations lead to a corresponding invariance principle which is of independent interest.

Theorem 2.8. *Under assumptions* (POT) *and* (INI)*, for every $v > v_c$ there exists $\sigma_v \in (0,\infty)$ given explicitly in* (4.24) *below, such that the sequence of processes*

$$[0,\infty) \ni t \mapsto \frac{1}{\sigma_v\sqrt{vn}}\big(\ln u(nt, \lfloor vnt \rfloor) - nt\lambda(v)\big), \qquad n \in \mathbb{N},$$

converges as $n \to \infty$ in $\mathbb{P}$-distribution to standard Brownian motion.

While this result for the front of the solution to the PAM is interesting in its own right, the question naturally arises of what one can say about the front of the solution to its non-linear version, the randomized discrete Fisher-KPP equation

$$\frac{\partial w}{\partial t}(t,x) = \Delta_d w(t,x) + \xi(x)w(t,x)(1 - w(t,x)), \qquad (t,x) \in [0,\infty) \times \mathbb{Z}, \tag{2.11}$$

Previous results (in continuum space) on the front of the solution to (2.11) have been obtained in [GF79] (see also [Fre85]), [Nol11], and [HNRR16]. First, under suitable regularity and mixing assumptions, and a Heaviside type initial condition, $w(0,\cdot) = \mathbf{1}_{-\mathbb{N}_0}$, as in (1.4), the existence of the speed of the front

$$\widehat{m}(t) := \sup\{x \in \mathbb{R} : w(t,x) = 1/2\} \tag{2.12}$$

of the solution to the randomized Fisher-KPP equation (2.11) is known [Fre85, Theorem 7.6.1]: For $\mathbb{P}$-a.e. realization of ξ,

$$\lim_{t\to\infty} \frac{\widehat{m}(t)}{t} = v_0, \tag{2.13}$$

where v_0 is non-random and corresponds to the speed of the front of the linearized equation, which is a 'continuum space PAM'. Here, as in Bramson's work [Bra83], a precise analysis of the Feynman-Kac formula plays an important role in the proofs.

In the case of ξ periodic instead of random, in [HNRR16] it has been shown that there is a logarithmic correction term between $m(t)$ and $\widehat{m}(t)$, and the authors were able to characterize the constant in front of the logarithmic correction as a certain minimizer.

To our knowledge, nothing is known about the fluctuations of $\widehat{m}(t)$ for the Heaviside-type initial conditions in the case of random branching rates. For a different, for technical reasons restricted, set of initial conditions, Nolen [Nol11] has derived a central limit theorem for the position of the front of the solution to (2.11) by analytic means. To put our results into context, let us describe the assumptions of [Nol11] more precisely: The initial condition $w_0(x,\xi)$ of [Nol11] is required to depend on the randomness of the environment. It should satisfy $\lim_{x\to-\infty} w_0(x,\xi) = 1$ (which roughly corresponds to our assumption (INI)), and, more importantly,

$$c(\xi)\widetilde{w}(x,\xi,\gamma) \le w_0(x,\xi) \le C(\xi)\widetilde{w}(x,\xi,\gamma) \quad \text{for all } x > 0. \tag{2.14}$$

Here $\widetilde{w} = \widetilde{w}(x,\xi,\gamma)$, $t \ge 0$, $x \ge 0$, is a non-negative solution to the ordinary differential equation $\frac{1}{2}\Delta\widetilde{w} = (\xi-\gamma)\widetilde{w}$ satisfying $\widetilde{w}(0,\xi,\gamma) = 1$ and which decays to 0 as $x \to \infty$. It was known previously that $\widetilde{w}$ exists whenever γ is larger than a certain $\overline{\gamma}$. In addition, there is another $\gamma^* > \overline{\gamma}$ such that whenever $\gamma \ge \gamma^*$ and the initial condition satisfies (2.14), then the law of large numbers for the velocity of the traveling wave, that is (2.13), holds with the same speed v_0. In order to prove his central limit theorem, Nolen needs to assume that $\gamma \in (\overline{\gamma}, \gamma^*)$, which leads to traveling waves with a larger velocity $v(\gamma) > v_0$. The initial conditions corresponding to such γ decays to 0 exponentially as $x \to \infty$, but the rate of decay is small.

It is worthwhile to remark that such a distinction between the waves with the minimal (or 'critical') velocity, and the waves with strictly larger velocity is present already in the paper of Bramson [Bra83]. Already there it turns out that the 'supercritical' is easier to handle.

One of our main motivations for writing this paper was to understand the behavior of the front of the traveling wave solution to randomized Fisher-KPP equation in the 'critical' case, in particular for initial conditions of the form $w_0 = \mathbf{1}_{-\mathbb{N}_0}$, that are, from the point of view of the BRWRE as well as of the PAM, more natural.

Theorem 2.9. *Let $\widehat{m}(t)$ be the front of the solution to discrete randomized Fisher-KPP equation (2.11) with initial condition $w_0 = \mathbf{1}_{-\mathbb{N}_0}$ defined similarly as in (2.12) by*

$$\widehat{m}(t) := \sup\{x \in \mathbb{Z} : w(t,x) \ge 1/2\}. \tag{2.15}$$

Then, assuming (POT), (VEL),

$$\frac{\widehat{m}(t) - v_0 t}{\overline{\sigma}_{v_0}\sqrt{t}}$$

converges as $t \to \infty$ in $\mathbb{P}$-distribution to a standard normal random variable.

The previous theorem is a *non-functional* central limit theorem only, which might look surprising in view of our previous results. The reason for this is the fact that the connection between the BRWRE and the corresponding randomized Fisher-KPP equation is slightly more complicated than in the homogeneous case, due to the fact that the BRWRE is not translation and reflection invariant (given ξ): We will prove in Proposition 7.1 that

$$w(t,x) = \mathtt{P}_x^{\xi}(M(t) \ge 0)$$

solves the randomized Fisher-KPP equation with initial condition $w(0,\cdot) = \mathbf{1}_{\mathbb{N}_0}$. This should be contrasted with the definition of $w^{\mathsf{BBM}}(t,x) = \mathbb{P}(M(t) \geq x)$ used in (1.4).

Acknowledgment The authors would like to thank the Columbia University Mathematics Department as well as the Mathematics Department of the University of Vienna for their hospitality and financial support. Part of this work has been completed during a stay at the 'Random Geometry' semester programme of the Isaac Newton Institute for Mathematical Sciences in Cambridge, UK, in 2015.

3 Strategy of the proof

We now roughly explain the strategy of the proof of our main results, and describe the organization of the paper. As it is common in the branching random walk literature, a first moment method is used to provide an upper bound on the maximum of the BRWRE; a complementary truncated second moment computation gives a lower bound.

Luckily, similarly to the homogeneous case, the moments of the number of particles in the BRWRE (possibly satisfying certain additional restrictions) have an explicit representation. This representation, in terms of expectations of certain functionals of simple random walk, is called Feynman-Kac formula, 'many-to-one lemma' or 'many-to-few lemma', depending on the source and context. To introduce it, for $x \in \mathbb{Z}$, let P_x denote the law of the continuous-time simple random walk $(X_t)_{t\geq 0}$ on $\mathbb{Z}$ with jump rate 1 and denote by E_x the corresponding expectation. The following proposition, which is an adaptation of Section 4.2 of [HR17] or Theorem 2.1 of [GKS13], gives the representation for first and second moments. Its proof is an easy modification of the proofs of these results, and it is therefore omitted.

Proposition 3.1 (Feynman-Kac formula)**.** *Let $\varphi_1, \varphi_2 : [0,t] \to [-\infty,\infty]$ be càdlàg functions with $\varphi_1 \leq \varphi_2$. Then the first and second moments of the number of particles in $N(t)$ whose genealogy stays between φ_1 and φ_2 are given by*

$$\begin{aligned} &\mathtt{E}_0^\xi\big[\big|\big\{Y \in N(t) : \varphi_1(r) \leq Y_r \leq \varphi_2(r) \ \forall r \in [0,t]\big\}\big|\big] \\ &\quad = E_0\Big[\exp\Big\{\int_0^t \xi(X_r)\,\mathrm{d}r\Big\}; \varphi_1(r) \leq X_r \leq \varphi_2(r) \ \forall r \in [0,t]\Big] \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\mathtt{E}_0^\xi\big[\big|\big\{Y \in N(t) : \varphi_1(r) \leq Y_r \leq \varphi_2(r) \ \forall r \in [0,t]\big\}\big|^2\big] \\ &\quad = E_0\Big[\exp\Big\{\int_0^t \xi(X_r)\,\mathrm{d}r\Big\}; \varphi_1(r) \leq X_r \leq \varphi_2(r) \ \forall r \in [0,t]\Big] \\ &\qquad + 2\int_0^t E_0\Big[\exp\Big\{\int_0^s \xi(X_r)\,\mathrm{d}r\Big\}\xi(X_s)\mathbf{1}_{\varphi_1(r)\leq X_r\leq\varphi_2(r) \ \forall r\in[0,s]} \\ &\qquad\quad \times \Big(E_{X_s}\Big[\exp\Big\{\int_0^{t-s} \xi(X_r)\,\mathrm{d}r\Big\}; \varphi_1(r+s) \leq X_r \leq \varphi_2(r+s) \ \forall r \in [0,t-s]\Big]\Big)^2\Big]\mathrm{d}s. \end{aligned} \tag{3.2}$$

In particular, (3.1) implies that

$$\mathtt{E}_{u_0}^\xi[N^{\geq}(t,n)] = \sum_{i\in\mathbb{Z}} u_0(i)E_i\Big[\exp\Big\{\int_0^t \xi(X_r)\,\mathrm{d}r\Big\}; X_t \geq n\Big]. \tag{3.3}$$

In the first principal step of the proof we analyze the first moment formula (3.3) for $n = vt$ with $v > 0$. To understand this analysis, it is useful to recall the corresponding representation from the homogeneous case (cf. [AR09]). In that setting it is almost trivial that $\mathtt{E}_0^{\xi\equiv 1}[N^{\geq}(t,vt)] = e^t P_0(X_t \geq vt)$. The probability on the right-hand side can be then analyzed using exact large deviation results (see e.g. [DZ98], Theorem 3.7.4) to obtain a precise asymptotic formula.

While (3.3) has a different structure, its asymptotics can be understood, at least at a *heuristic* level, using the ingredients that are usually used in the proof of exact large deviation theorems: a tilting and a local central limit theorem. Slightly more in detail, by introducing a tilting parameter η, using (3.3), we can write

$$\mathtt{E}_0^\xi[N(t,vt)] = e^{-t\eta} E_0\Big[\exp\Big\{\int_0^t (\xi(X_r)+\eta)\,\mathrm{d}r\Big\}; X_t = vt\Big]. \tag{3.4}$$

This suggests to introduce new 'Gibbs measures' on the space of random walk trajectories, whose density with respect to simple random walk is the exponential factor in (3.4) (cf. Section 4.1). We then adjust η so that the event $X_t = vt$ is typical under a such Gibbs measure. Next, using a suitable local central limit theorem, the right-hand side of (3.4) can be approximated by (cf. Proposition 4.9)

$$\sim \frac{c}{\sqrt{t}} e^{-t\eta} E_0\Big[\exp\Big\{\int_0^t (\xi(X_r)+\eta)\,\mathrm{d}r\Big\}; H_{vt} \le t\Big],$$

where H_x stands for the hitting time of x by the simple random walk X, see (4.1) below. This can further be rewritten as

$$\sim \frac{c}{\sqrt{t}} e^{-t\eta} E_0\Big[\prod_{x=1}^{vt} \exp\Big\{\int_{H_{x-1}}^{H_x} (\xi(X_r)+\eta)\,\mathrm{d}r\Big\} \times \exp\Big\{\int_{H_{vt}}^{t} (\xi(X_r)+\eta)\,\mathrm{d}r\Big\}\Big].$$

If one ignores the last factor in the expectation, which can be done due to a concentration of the hitting times of the random walk under the Gibbs measure, see Section 4.4, the Markov property yields that we can continue to estimate

$$\begin{aligned}
&\sim \frac{c}{\sqrt{t}} e^{-t\eta} \prod_{x=1}^{vt} E_{x-1}\Big[\exp\Big\{\int_0^{H_x} (\xi(X_r)+\eta)\,\mathrm{d}r\Big\}\Big] \\
&= \frac{c}{\sqrt{t}} e^{-t\eta} \exp\Big\{\sum_{x=1}^{vt} \ln E_{x-1}\Big[\exp\Big\{\int_0^{H_x} (\xi(X_r)+\eta)\,\mathrm{d}r\Big\}\Big]\Big\}.
\end{aligned}$$

The application of a suitable central limit theorem to the above sum then suggests the central limit theorem behavior of the PAM, Theorem 2.8.

Making the above heuristics rigorous requires a non-negligible effort. In particular, it turns out that the tilting parameter η making the event $X_t = vt$ typical under the Gibbs measure is *random* (i.e., ξ-dependent). This disallows a straightforward application of a central limit theorem in the last formula above. Section 4.3 deals with this problem, building on a preparatory Section 4.2. Other approximations appearing in the previous heuristic computation are treated in Section 4.4; Section 4.5 controls the influence of the initial conditions. The functional central limit theorem for the PAM, Theorem 2.8, then follows easily, cf. Section 4.6.

In order to show the functional central limit theorem for the breakpoint, Theorem 2.3, we essentially need to find the largest root of the function $x \mapsto \ln \mathtt{E}_0^\xi[N^\ge(t,x)]$, which requires, in a certain sense, to invert the functional central limit theorem for the PAM, cf. Section 5.2. In oder to perform this inversion, we study how sensitive $\mathtt{E}_0^\xi[N^\ge(t,x)]$ is to perturbations in the space and time direction, cf. Section 5.1.

Let us now comment on the second moment computation required to prove the remaining main results of this paper. Similarly to the homogeneous case, the second moment of $N^\ge(t,vt)$ explodes too quickly to yield any useful estimates. This explosion is, essentially, due to particles that are much faster than the breakpoint at times in the bulk of the interval $[0,t]$. In the homogeneous case this is solved by a truncation which involves considering only so-called *leading* particles, that is the particles that are slower than the breakpoint, $X_s \le v_0 s$ for all $s \in [0,t]$ (here, $v_0 t$ is a first order breakpoint asymptotics). The principal ingredient for the computation of the moments for the leading particles is then a 'ballot theorem' for the random walk bridge, which gives the probability that a random walk bridge from $(0,0)$ to $(t,0)$ stays positive for all intermediate times.

Following the above strategy in the case of BRWRE suggests to call a particle $Y \in N(t)$ *leading at time* t if (a) Y_t is close to the breakpoint $\overline{m}(t)$, and (b) Y is slower than breakpoint at intermediate times, $Y_s \le \overline{m}(s)$ for all $s \in [0,t]$[1]. Since $\overline{m}(t)$ satisfies a functional central limit theorem itself, it naturally leads to a ballot estimate of the following form: Let B, W be two independent Brownian motions (or centered random walks, possibly not identically distributed). What is the behavior of

$$\mathbb{P}\big(B(t) \ge W(t), B(s) \le 1 + W(s) \forall s \in [0,t] \,\big|\, \sigma(W)\big)?$$

Observe that the process W is 'quenched' in this computation as we condition on the σ-field $\sigma(W)$ generated by W. This modified ballot problem was recently studied by Mallein and Miłoś [MM15a]. We were however not able to use their results directly due to the lack of the independence that we encounter in our model.

The first and second moment of the number of leading particles is computed in Section 6.1. In particular, a lengthy proof of a (relatively week version of) a ballot estimate can be found in Section 6.1.1. Theorem 2.1 and thus Theorem 2.7 are then shown in Section 6.2. Section 7 proves the functional central limit theorem for the Fisher-KPP equation, Theorem 2.9. Finally, Section 8 discusses some open problems.

Notational conventions. For two functions $f, g : [0,\infty) \to (0,\infty)$, we write $f \sim g$ when $\lim_{t\to\infty} f(t)/g(t) = 1$, and $f \asymp g$ when $0 < \inf_{t\in[0,\infty)} f(t)/g(t) \le \sup_{t\in[0,\infty)} f(t)/g(t) < \infty$. We use c and C to denote positive finite constants whose value may change during computations. Indexed constants such as c_1 keep their value from their first time of occurrence. We use $E[f; A]$ as an abbreviation for $E[f\mathbf{1}_A]$.

For $x \in \mathbb{R} \setminus \mathbb{Z}$ we define P_x by linear interpolation. More precisely, for $x = \lfloor x \rfloor + \lambda$ we define $P_x := (1-\lambda)P_{\lfloor x \rfloor} + \lambda P_{\lfloor x \rfloor + 1}$. Similarly, other quantities which are only defined for integers a priori are to be interpreted as the linear interpolation of the evaluations at their integer neighbors, which will usually be clear from the context.

While we have stated the precise assumptions needed in the main results given above,

we will from now on assume (POT), (INI) and (VEL) to be fulfilled

as standing assumptions without further mentioning. This helps in keeping notation lighter compared to mentioning a suitable subset of these assumptions at each of the numerous subsequent auxiliary results.

4 Expected number of particles of given velocity

In this section we study the asymptotic behavior of $\mathtt{E}^{\xi}_{u_0}[N(t, vt)]$ and related quantities, following the strategy described in Section 3.

4.1 Tilted random walk measures

We introduce the tilted distributions of random walk in random potential, and show that one can tilt the random walk in a suitable way to make the extremal behavior typical.

Recall that $(X_t)_{t\ge0}$ denotes continuous-time simple random walk on $\mathbb{Z}$ with jump rate 1. For $i \in \mathbb{Z}$ we define the hitting time of i as

$$H_i := \inf\{s \in [0,\infty) : X_s = i\}, \tag{4.1}$$

and set $\tau_i := H_i - H_{i-1}$. Recalling (POT) and writing

$$\zeta(x) := \xi(x) - \mathsf{es}, \qquad x \in \mathbb{Z}, \tag{4.2}$$

we infer

$$-\infty < \operatorname{ess\,inf} \zeta < \operatorname{ess\,sup} \zeta = 0. \tag{4.3}$$

[1]The actual definition of the leading particles in Section 6 is slightly different for technical reasons.

For $n \geq 1$, $A \in \sigma(X_{s\wedge H_n},\, s \in [0,\infty))$ and $\eta \in \mathbb{R}$, we define

$$P^{\zeta,\eta}_{(n)}(A) := (Z^{\zeta,\eta}_{(n)})^{-1} E_0\Big[\exp\Big\{\int_0^{H_n} (\zeta(X_s)+\eta)\,\mathrm{d}s\Big\}; A\Big], \tag{4.4}$$

where

$$Z^{\zeta,\eta}_{(n)} := E_0\Big[\exp\Big\{\int_0^{H_n} (\zeta(X_s)+\eta)\,\mathrm{d}s\Big\}\Big].$$

We will see below, cf. Lemma 4.1, that these quantities are finite iff $\eta \leq 0$.

It can be seen easily, using the strong Markov property, that $P^{\zeta,\eta}_{(n)}(A) = P^{\zeta,\eta}_{(m)}(A)$ for every $m > n$ and $A \in \sigma(X_{s\wedge H_n},\, s \in [0,\infty))$. We may thus use Kolmogorov's extension theorem to extend $P^{\zeta,\eta}_{(n)}$ to a measure $P^{\zeta,\eta}$ on $\sigma(X_s, s \geq 0)$. We write P^{ζ} for $P^{\zeta,0}$.

It will be suitable to introduce the following logarithmic moment generating functions

$$L^{\zeta}_i(\eta) := \ln E_{i-1}\Big[\exp\Big\{\int_0^{H_i} (\zeta(X_s)+\eta)\,\mathrm{d}s\Big\}\Big], \tag{4.5}$$

$$\overline{L}^{\zeta}_n(\eta) := \frac{1}{n}\sum_{i=1}^{n} L^{\zeta}_i(\eta), \tag{4.6}$$

$$L(\eta) := \mathbb{E}\big[L^{\zeta}_1(\eta)\big]. \tag{4.7}$$

By the strong Markov property again,

$$Z^{\zeta,\eta}_{(n)} = \exp\Big\{\sum_{i=1}^{n} L^{\zeta}_i(\eta)\Big\} = \exp\{n\overline{L}^{\zeta}_n(\eta)\}. \tag{4.8}$$

We now discuss the finiteness of the above objects.

Lemma 4.1. *Under* (POT) *the quantities defined in* (4.5)–(4.7) *are finite if and only if* $\eta \leq 0$.

Proof. Since $\operatorname{ess\,sup} \zeta(x) \leq 0$, the 'if' part of the lemma is trivial. In the case of random walk with a drift, the 'only if' statement is a direct consequence of Proposition 3.1 in [Dre08]. Its lengthy proof, however, can directly be transferred to the case of simple random walk without drift. As we do not need the only if part in this paper, we omit its proof here. □

Recalling that $\tau_i = H_i - H_{i-1}$, as an easy corollary of Lemma 4.1 we obtain

$$\ln E^{\zeta,\eta}\big[e^{\lambda\tau_i}\big] = L^{\zeta}_i(\eta+\lambda) - L^{\zeta}_i(\eta), \quad \eta \leq 0, \lambda \in \mathbb{R},$$

as well as

$$E^{\zeta,\eta}[e^{\lambda\tau_i}] < \infty \quad \text{for every } \eta \leq 0 \text{ and } \lambda \leq |\eta|. \tag{4.9}$$

Finally, Birkhoff's ergodic theorem implies that

$$L(\eta) \overset{\mathbb{P}\text{-a.s.}}{=} \lim_{n\to\infty} \overline{L}^{\zeta}_n(\eta). \tag{4.10}$$

Other simple properties of functions L^{ζ} and L are given in the Appendix.

We will primarily be interested in those values $\eta = \overline{\eta}^{\zeta}_n(v)$ which make certain large deviations events typical, more precisely for which

$$E^{\zeta,\overline{\eta}^{\zeta}_n(v)}[H_n] = \frac{n}{v}, \qquad v > 0. \tag{4.11}$$

In order to discuss the existence of such η we recall that v_c, which we will refer to as the *critical velocity*, had been introduced below (2.2). It will be shown in Lemma A.3, partially using the results of this paper, that the identity

$$v_c = \frac{1}{L'(0)} \tag{4.12}$$

holds true, where the derivative is taken from the left only. Throughout the paper we use (4.12) as the primary definition.

We further define $\overline{\eta}(v) \in (-\infty, 0)$ as the unique solution to

$$L^*(1/v) = \sup_{\eta\in\mathbb{R}}(\eta/v - L(\eta)) = \overline{\eta}(v)/v - L(\overline{\eta}(v)). \tag{4.13}$$

Due to Lemma A.1, L is smooth, strictly increasing and strictly convex on $(-\infty, 0)$. In addition, it can be seen easily that $\lim_{\eta\to-\infty} L'(\eta) = 0$ (see [Dre08, Lemma 3.5] for the corresponding statement in the case of a random walk with drift; the proof for simple random walk proceeds in the same way and is omitted here). Therefore, recalling also (4.12), we see that the solution to (4.13) exists for every $v > v_c$ and, due to usual properties of Legendre transform, is characterized by

$$L'(\overline{\eta}(v)) = \frac{1}{v}. \tag{4.14}$$

Moreover, $\overline{\eta}(v)$ is a smooth, strictly decreasing function of $v \in (v_c, \infty)$.

We now show that $\overline{\eta}_n^\zeta(v)$, as defined in (4.11), exists $\mathbb{P}$-a.s. for $v > v_c$ and n large enough and, in fact, concentrates around $\overline{\eta}(v)$.

Proposition 4.2. *For each $v > v_c$ there exists a $\mathbb{P}$-a.s. finite random variable $\mathcal{N} = \mathcal{N}(v)$ such that for all $n \geq \mathcal{N}$ there exists $\overline{\eta}_n^\zeta(v) \in (-\infty, 0)$ satisfying (4.11). Moreover, for every $q \in \mathbb{N}$ and $V \subset (v_c, \infty)$ compact there exists a constant $C = C(q, V) < \infty$ such that for all $n \in \mathbb{N}$,*

$$\mathbb{P}\bigg(\sup_{v\in V} |\overline{\eta}(v) - \overline{\eta}_n^\zeta(v)| \geq C\sqrt{\frac{\ln n}{n}}\bigg) \leq Cn^{-q} \tag{4.15}$$

(defining, arbitrarily, $\overline{\eta}_n^\zeta(v) = 0$ whenever the solution to (4.11) does not exist).

Proof. By Lemma A.1, for $\eta < 0$,

$$E^{\zeta,\eta}[H_n] = n\frac{\mathrm{d}}{\mathrm{d}\eta}\overline{L}_n^\zeta(\eta).$$

Hence, we should show that the solution $\overline{\eta}_n^\zeta(v)$ to

$$\big(\overline{L}_n^\zeta\big)'(\overline{\eta}_n^\zeta(v)) = 1/v \tag{4.16}$$

satisfies (4.15). In combination with the fact that $\sup_{v\in V}\overline{\eta}(v) < 0$, the first claim of the lemma then follows by a Borel-Cantelli argument.

Comparing (4.14) and (4.16), we see that we need to understand the concentration properties of $(\overline{L}_n^\zeta)'$ first. We claim that

Claim 4.3. *For every $q \in \mathbb{N}$ and $\Delta \subset (-\infty, 0)$ compact, there exists $C = C(q, \Delta) < \infty$ such that for all $n \in \mathbb{N}$*

$$\mathbb{P}\bigg(\sup_{\eta\in\Delta}\Big|(\overline{L}_n^\zeta)'(\eta) - L'(\eta)\Big| \geq C\sqrt{\frac{\ln n}{n}}\bigg) \leq Cn^{-q}.$$

Proof. We apply a Hoeffding type bound for mixing sequences recalled in Lemma A.4. Define the σ-algebras $\mathcal{F}_k := \sigma(\xi(i) : i \leq k)$, $k \in \mathbb{Z}$. Due to Lemma A.1, $\big((L_i^\zeta)'(\eta) - L'(\eta)\big)_{i\in\mathbb{Z}}$ is a stationary sequence of bounded random variables, and, by Lemma A.2, there is $c < \infty$ such that $\big|\mathbb{E}\big[(L_i^\zeta)'(\eta)|\mathcal{F}_k\big] - L'(\eta)\big| \leq ce^{-(i-k)/c}$ for all $i \geq k$ and $\eta \in \Delta$. Therefore, the assumptions of Lemma A.4 are satisfied with $m_i = c$, and thus uniformly over $\eta \in \Delta$, for C large enough,

$$\mathbb{P}\bigg(\Big|(\overline{L}_n^\zeta)'(\eta) - L'(\eta)\Big| \geq C\sqrt{\frac{\ln n}{n}}\bigg) \leq Ce^{-C\ln n} \leq Cn^{-q-1}.$$

Hence, by a union bound,

$$\mathbb{P}\bigg(\sup_{\eta\in\frac{1}{n}\mathbb{Z}\cap\Delta}\Big|(\overline{L}_n^\zeta)'(\eta) - L'(\eta)\Big| \geq C\sqrt{\frac{\ln n}{n}}\bigg) \leq Cn^{-q}. \tag{4.17}$$

Moreover, by Lemma A.1, L' is increasing on $(-\infty, 0)$ with a continuous derivative $L'' > 0$. Hence, for any $\Delta \subset (-\infty, 0)$ compact, there is $c < \infty$ such that

$$c^{-1} < \inf_{\Delta} L'' \le \sup_{\Delta} L'' < c. \tag{4.18}$$

Together with (4.17) this implies the claim. □

To prove (4.15), fix a compact $\Delta \subset (-\infty, 0)$ such that the image $\overline{\eta}(V)$ is contained in the interior of Δ, which is possible by the discussion surrounding (4.14). Recalling (4.14) and (4.16), the claim (4.15) then follows directly from Claim 4.3 and (4.18). □

For future reference we recall that whenever $\overline{\eta}_n^\zeta(v)$ exists, then it is characterized, due to the usual properties of the Legendre transform, by

$$(\overline{L}_n^\zeta)^*(1/v) := \sup_{\eta \in \mathbb{R}} \Big(\frac{\eta}{v} - \overline{L}_n^\zeta(\eta) \Big) = \frac{\overline{\eta}_n^\zeta(v)}{v} - \overline{L}_n^\zeta(\overline{\eta}_n^\zeta(v)). \tag{4.19}$$

Technical assumption. In order to keep the constants in the paper independent of the velocity v, from now on and for the rest of this paper, we assume that

$$\text{the velocities } v \text{ that we are considering are contained in a fixed compact interval } V \subset (v_c, \infty) \text{ which has } v_0 \text{ in its interior.} \tag{4.20}$$

Such V exists due to (VEL). The constants appearing in the results below may depend on V. Using Proposition 4.2 and the monotonicity of $\overline{\eta}$ and $\overline{\eta}_n^\zeta$ in v and ζ, it is then possible to fix a compact interval $\Delta \subset (-\infty, 0)$ such that there is a $\mathbb{P}$-a.s. finite random variable $\mathcal{N}_1$ such that the event

$$\mathcal{H}_n := \mathcal{H}_n(V) := \{\overline{\eta}_n^\zeta(v) \in \Delta \text{ for all } v \in V\} \quad \text{occurs for all } n \ge \mathcal{N}_1. \tag{4.21}$$

We also recall that we arbitrarily set $\overline{\eta}_n^\zeta(v) = 0$ in the case when (4.11) does not have any solution. This occurs on $\mathcal{H}_n^c$ only.

For future use we state the following easy estimate.

Lemma 4.4. *For each $\delta \in (0, 1)$ there exists a constant $C = C(\delta)$ such that $\mathbb{P}$-a.s. for all n large enough, uniformly for $v \in V$ and $h \le n^{1-\delta}$,*

$$\big|\overline{\eta}_n^\zeta(v) - \overline{\eta}_{n+h}^\zeta(v)\big| \le \frac{Ch}{n}.$$

Proof. Let Δ be as in (4.21). We claim that there exists a constant $C < \infty$ such that for all $n \ge 1$, $h \le n$ and $\eta \in \Delta$,

$$\big|(\overline{L}_{n+h}^\zeta)'(\eta) - (\overline{L}_n^\zeta)'(\eta)\big| \le \frac{Ch}{n}. \tag{4.22}$$

Indeed, plugging in the definitions we obtain

$$(\overline{L}_{n+h}^\zeta)'(\eta) - (\overline{L}_n^\zeta)'(\eta) = -\frac{h}{n(n+h)} \sum_{i=1}^{n} (L_i^\zeta)'(\eta) + \Big(\frac{1}{n+h}\Big) \sum_{i=n+1}^{n+h} (L_i^\zeta)'(\eta),$$

from which we can then deduce (4.22) by observing that $(L_i^\zeta)'(\eta)$ can be bounded uniformly over $\mathbb{P}$-a.a. realizations of ζ and $\eta \in \Delta$, by Lemma A.1. The claim of the lemma then follows from (4.14), (4.16), (4.18) and (4.22) by the same arguments as at the end of the proof of Proposition 4.2. □

4.2 An invariance principle for the empirical Legendre transforms

In this section we show an invariance principle for the suitably centered and rescaled Legendre transforms of the functions $\overline{L}_n^\zeta$ defined in (4.6).

We start by introducing some quantities used in those invariance principles. Let

$$V_i^{\zeta,v}(\eta) := \eta/v - L_i^\zeta(\eta). \tag{4.23}$$

and set

$$\sigma_v^2 := \mathrm{Var}_{\mathbb{P}}\left(V_1^{\zeta,v}(\overline{\eta}(v))\right) + 2\sum_{j\geq 2}\mathrm{Cov}_{\mathbb{P}}\left(V_1^{\zeta,v}(\overline{\eta}(v)), V_j^{\zeta,v}(\overline{\eta}(v))\right). \tag{4.24}$$

Using the non-degeneracy part of assumption (POT), and the exponential decay of correlations of the L_i^ζ proved in Lemma A.2, it is easy to see that $\sigma_v^2 \in (0,\infty)$.

We can now state the promised invariance principle. Recall the convention that for non-integer n the L_n and other quantities are defined by linear interpolation

Proposition 4.5. *For each $v \in V$, the sequence of processes*

$$t \mapsto W_n(t) := \frac{1}{\sigma_v} t\sqrt{n}\big((\overline{L}_{nt}^\zeta)^*(1/v) - L^*(1/v)\big), \qquad n \in \mathbb{N}, \tag{4.25}$$

converges as $n \to \infty$, in $\mathbb{P}$-distribution to standard Brownian motion.

Heuristically, the proof of this proposition is based on the fact that the fluctuations of the Legendre transforms $(\overline{L}_n^\zeta)^*$ are essentially given by the fluctuations of the functions $\overline{L}_n^\zeta$, whereas the influence of the fluctuations of the maximizing argument at which the supremum is attained in the definition (4.19) of the Legendre transform is negligible.

Proof of Proposition 4.5. Recall that, due to (4.19), on $\mathcal{H}_n$,

$$(\overline{L}_n^\zeta)^*(1/v) = \frac{\overline{\eta}_n^\zeta(v)}{v} - \overline{L}_n^\zeta(\overline{\eta}_n^\zeta(v)) = \frac{1}{n}\sum_{i=1}^n V_i^{\zeta,v}(\overline{\eta}_n^\zeta(v)) = \frac{1}{n} S_n^{\zeta,v}(\overline{\eta}_n^\zeta(v)), \tag{4.26}$$

where we set

$$S_n^{\zeta,v}(\eta) := \sum_{i=1}^n V_i^{\zeta,v}(\eta) \tag{4.27}$$

as a shorthand. Using this notation, we expand the quantity of interest as

$$\begin{aligned} tn\big((\overline{L}_{tn}^\zeta)^*(1/v) - L^*(1/v)\big) = &\big(tn(\overline{L}_{tn}^\zeta)^*(1/v) - S_{tn}^{\zeta,v}(\overline{\eta}(v))\big) \\ &+ \big(S_{tn}^{\zeta,v}(\overline{\eta}(v)) - \mathbb{E}[S_{tn}^{\zeta,v}(\overline{\eta}(v))]\big) \\ &+ \big(\mathbb{E}[S_{tn}^{\zeta,v}(\overline{\eta}(v))] - tnL^*(1/v)\big). \end{aligned} \tag{4.28}$$

We will show that the first and the third summand on the right-hand side are negligible in a suitable sense, and that the second summand converges in distribution after rescaling by $\sigma_v\sqrt{n}$ to standard Brownian motion under $\mathbb{P}$.

The third summand in (4.28) is the easiest since it vanishes. Indeed, by (4.7) and (4.13),

$$tnL^*(1/v) = tn\Big(\frac{\overline{\eta}(v)}{v} - L(\overline{\eta}(v))\Big) = tn\mathbb{E}\Big[\frac{\overline{\eta}(v)}{v} - \overline{L}_{tn}^\zeta(\overline{\eta}(v))\Big] = \mathbb{E}\big[S_{tn}^{\zeta,v}(\overline{\eta}(v))\big].$$

The next lemma deals with the second summand in (4.28).

Lemma 4.6. *The sequence of processes*

$$[0,\infty) \ni t \mapsto \widetilde{W}_n(t) := \frac{1}{\sigma_v\sqrt{n}}\big(S_{tn}^{\zeta,v}(\overline{\eta}(v)) - \mathbb{E}[S_{tn}^{\zeta,v}(\overline{\eta}(v))]\big), \qquad n \in \mathbb{N},$$

converges as $n \to \infty$ in $\mathbb{P}$-distribution to standard Brownian motion.

Proof. By the definition of $S_n^{\zeta,v}$,

$$\frac{1}{\sigma_v\sqrt{n}}\big(S_{tn}^{\zeta,v}(\overline{\eta}(v)) - \mathbb{E}[S_{tn}^{\zeta,v}(\overline{\eta}(v))]\big) = \frac{1}{\sigma_v\sqrt{n}}\sum_{i=1}^{tn} V_i^{\zeta,v}(\overline{\eta}(v)) - \mathbb{E}[V_i^{\zeta,v}(\overline{\eta}(v))].$$

The random variables $V_i^{\zeta,v}(\overline{\eta}(v))$ form a non-degenerate stationary sequence which are coordinatewise decreasing in the ζ's. Therefore, by the FKG-inequality, they also form an associated sequence in the sense that any two coordinatewise decreasing functions of the $V_i^{\zeta,v}(\overline{\eta}(v))$'s of finite variance are non-negatively correlated. Hence, the functional central limit theorem for associated random variables proved in [NW81, Theorem 3] supplies us with convergence in $C([0,M])$ for each $M \in (0,\infty)$, and the result is then extended to $C([0,\infty))$ in the standard fashion. □

Finally, for the first summand in (4.28), we have the following estimate.

Lemma 4.7. *There is $C<\infty$ such that $\mathbb{P}$-a.s. for every $M \in (1,\infty)$ and $v \in V$*

$$\limsup_{n\to\infty}\frac{1}{\ln n}\sup_{t\in[0,M]}\big|tn(\overline{L}_{tn}^{\zeta})^*(1/v) - S_{tn}^{\zeta,v}(\overline{\eta}(v))\big| \le C.$$

Proof. By Proposition 4.2 and (4.21), the representation (4.26) holds for all $n \ge \mathcal{N}_1$, with $\mathcal{N}_1$ a $\mathbb{P}$-a.s. finite random variable. As a consequence, it is sufficient to show that $\mathbb{P}$-a.s.,

$$\limsup_{n\to\infty}\frac{1}{\ln n}\max_{\mathcal{N}_1\le k\le Mn}\Big|S_k^{\zeta,v}(\overline{\eta}_k^{\zeta}(v)) - S_k^{\zeta,v}(\overline{\eta}(v))\Big| \le C. \tag{4.29}$$

Assuming $k \ge \mathcal{N}_1$ in what follows, using a Taylor expansion of the smooth function $S_k^{\zeta,v}$ around $\overline{\eta}_k^{\zeta}(v)$ we get

$$S_k^{\zeta,v}(\overline{\eta}(v)) - S_k^{\zeta,v}(\overline{\eta}_k^{\zeta}(v)) = (S_k^{\zeta,v})'(\overline{\eta}_k^{\zeta}(v))(\overline{\eta}(v)-\overline{\eta}_k^{\zeta}(v)) + (S_k^{\zeta,v})''(\widetilde{\eta}_k^{\zeta})\frac{(\overline{\eta}(v)-\overline{\eta}_k^{\zeta}(v))^2}{2}, \tag{4.30}$$

for some $\widetilde{\eta}_k^{\zeta} \in \Delta$ with

$$|\widetilde{\eta}_k^{\zeta} - \overline{\eta}_k^{\zeta}(v)| \le |\overline{\eta}(v) - \overline{\eta}_k^{\zeta}(v)|. \tag{4.31}$$

By (4.19), $S_k^{\zeta,v}(\eta)$ is maximized for $\eta = \overline{\eta}_k^{\zeta}(v)$, so $(S_k^{\zeta,v})'(\overline{\eta}_k^{\zeta}(v)) = 0$ and the first term on the right-hand side of (4.30) vanishes.

To bound the second term, observe that $(S_k^{\zeta,v})''(\widetilde{\eta}_k^{\zeta}) = k(\overline{L}_k^{\zeta})''(\widetilde{\eta}_k^{\zeta})$. By Lemma A.1, $\mathbb{P}$-a.s., $(L_1^{\zeta})''(\eta)$ is bounded from above, uniformly over $\eta \in \Delta$ (cf. (4.21)). Hence, $\mathbb{P}$-a.s.,

$$(S_k^{\zeta,v})''(\widetilde{\eta}_k^{\zeta}) \le Ck, \qquad \text{for all } k \ge \mathcal{N}_1,\ v \in V. \tag{4.32}$$

Going back to (4.30), $\mathbb{P}$-a.s. for all $k \ge \mathcal{N}_1$,

$$\big|S_k^{\zeta,v}(\overline{\eta}(v)) - S_k^{\zeta,v}(\overline{\eta}_k^{\zeta}(v))\big| \le ck\big|\overline{\eta}(v) - \overline{\eta}_k^{\zeta}(v)\big|^2.$$

Using the concentration estimates for $\overline{\eta}_k^{\zeta}(v)$ from Proposition 4.2, it is possible to fix a constant $C<\infty$ and a $\mathbb{P}$-a.s. finite random variable $\mathcal{N}_2 \ge \mathcal{N}_1$ such that for all $k \ge \mathcal{N}_2$,

$$\big|\overline{\eta}_k^{\zeta}(v) - \overline{\eta}(v)\big| \le C\sqrt{\frac{\ln k}{k}}.$$

Putting all together, this implies that $\mathbb{P}$-a.s. the left-hand side in (4.29) is bounded by

$$\limsup_{n\to\infty}\frac{1}{\ln n}\Big\{\max_{\mathcal{N}_1\le k\le\mathcal{N}_2}|S_k^{\zeta,v}(\overline{\eta}_k^{\zeta}(v)) - S_k^{\zeta,v}(\overline{\eta}(v))| + \max_{\mathcal{N}_2\le k\le Mn} C\ln k\Big\} \le C.$$

This completes the proof. □

Proposition 4.5 now follows from (4.28) and Lemmas 4.6 and 4.7. □

The proof of Proposition 4.5 has the following corollary which provides a useful explicit approximation to $W_n(t)$.

Corollary 4.8. *There is a constant $C < \infty$ such that $\mathbb{P}$-a.s. for every $M \in (0, \infty)$ and $v \in V$,*

$$\limsup_{n\to\infty} \frac{1}{\ln n} \sup_{t\in[0,M]} \Big|\sigma_v \sqrt{n} W_n(t) - \sum_{i=1}^{nt} \big(L(\overline{\eta}(v)) - L_i^\zeta(\overline{\eta}(v))\big)\Big| \le C.$$

Proof. It suffices to use the definition (4.25) of $W_n(t)$ together with Lemma 4.7. The claim then follows after a straightforward computation by inserting the definition of $S_{tn}^{\zeta,v}(\overline{\eta}(v))$ and using that $L^*(1/v) = \overline{\eta}(v)/v - L(\overline{\eta}(v))$. □

4.3 An auxiliary invariance principle

We now prove an invariance principle for the logarithm of the auxiliary process

$$Y_v(n) := E_0\Big[\exp\Big\{\int_0^{H_n} \zeta(X_s)\,\mathrm{d}s\Big\}; H_n \le \frac{n}{v}\Big], \qquad n \in \mathbb{N}, v \in V,$$

which we will relate to quantities considered in the Feynman-Kac representation (3.1) later on. Observe that this invariance principle can be seen as a first step to exact large deviation estimates, as explained in Section 3 above.

For convenience we split the process Y_v into the two summands

$$\begin{aligned} Y_v^{\approx}(n) &:= E_0\Big[\exp\Big\{\int_0^{H_n} \zeta(X_s)\,\mathrm{d}s\Big\}; H_n \in \Big[\frac{n}{v} - K, \frac{n}{v}\Big]\Big] \quad \text{and} \\ Y_v^{<}(n) &:= E_0\Big[\exp\Big\{\int_0^{H_n} \zeta(X_s)\,\mathrm{d}s\Big\}; H_n < \frac{n}{v} - K\Big], \end{aligned} \tag{4.33}$$

where $K > 0$ is a large constant which will be fixed later.

For $n \in \mathbb{N}$ and $v \in V$ we define random variables $\sigma_n^\zeta(v)$

$$\sigma_n^\zeta(v) := \begin{cases} |\overline{\eta}_n^\zeta(v)| \sqrt{\operatorname{Var}_{P^{\zeta,\overline{\eta}_n^\zeta(v)}}[H_n]}, & \text{on } \mathcal{H}_n, \\ \max \Delta \sqrt{\operatorname{Var}_{P^{\zeta,\max\Delta}}[H_n]}, & \text{on } \mathcal{H}_n^c. \end{cases} \tag{4.34}$$

Under every $P^{\zeta,\eta}$ we can write $H_n = \sum_{i=1}^n \tau_i$ as a sum of independent random variables (see (4.1) and below). Moreover, by Lemma A.1, there is a constant $c < \infty$ such that $c^{-1} \le \operatorname{Var}_{P^{\zeta,\eta}}[\tau_i] \le c$ for all $n \in \mathbb{N}$, $\eta \in \Delta$ and $\mathbb{P}$-a.e. ζ, and thus

$$c^{-1}\sqrt{n} \le \sigma_n^\zeta(v) \le c\sqrt{n} \qquad \text{for all } n \in \mathbb{N},\ v \in V \text{ and } \mathbb{P}\text{-a.e. } \zeta. \tag{4.35}$$

Proposition 4.9. *Let V be as in (4.20), and let K from (4.33) be a large enough fixed constant. Then there exists a constant $C < \infty$ such that*

$$Y_v^{\approx}(n)\sigma_n^\zeta(v) \exp\big\{nL^*(1/v) + \sigma_v\sqrt{n}W_n(1)\big\} \in [C^{-1}, C] \quad \text{for all } v \in V, n \in \mathbb{N} \text{ on } \mathcal{H}_n, \tag{4.36}$$

where the process W_n is given in (4.25) of Proposition 4.5 and $\sigma_v \in (0,\infty)$ is as in (4.24). In addition, for some $\widetilde{C} < \infty$,

$$\frac{Y_v^{\approx}(n)}{Y_v^{<}(n)} \in [\widetilde{C}^{-1}, \widetilde{C}] \quad \text{for all } v \in V, n \in \mathbb{N}, \text{ on } \mathcal{H}_n. \tag{4.37}$$

In particular, each of the three sequences of processes

$$\begin{aligned} t &\mapsto \frac{1}{\sigma_v\sqrt{n}}\big(\ln Y_v^{\approx}(tn) + tnL^*(1/v)\big), \quad n \in \mathbb{N}, \\ t &\mapsto \frac{1}{\sigma_v\sqrt{n}}\big(\ln Y_v^{<}(tn) + tnL^*(1/v)\big), \quad n \in \mathbb{N}, \\ t &\mapsto \frac{1}{\sigma_v\sqrt{n}}\big(\ln Y_v(tn) + tnL^*(1/v)\big), \quad n \in \mathbb{N}, \end{aligned} \tag{4.38}$$

converges as $n \to \infty$ in $\mathbb{P}$-distribution to standard Brownian motion.

Proof. Throughout the proof we assume that n is large enough so that $\mathcal{H}_n$ occurs. To simplify the notation, we also omit the dependence of $\overline{\eta}_n^\zeta$ and σ_n^ζ on the parameter v.

Let $\widehat{\tau}_i := \tau_i - E^{\zeta,\overline{\eta}_n^\zeta}[\tau_i]$. Using the definition of the tilted measure $P^{\zeta,\eta}$ (see (4.4) and below) with (4.8), and the fact that $\sum_{i=1}^n E^{\zeta,\overline{\eta}_n^\zeta}[\tau_i] = E^{\zeta,\overline{\eta}_n^\zeta}[H_n] = n/v$, we can rewrite $Y_v^{\approx}(n)$ as

$$\begin{aligned} Y_v^{\approx}(n) &= E^{\zeta,\overline{\eta}_n^\zeta}\Big[\exp\Big\{-\overline{\eta}_n^\zeta\sum_{i=1}^n \widehat{\tau}_i\Big\}; \sum_{i=1}^n \tau_i \in \Big[\frac{n}{v}-K,\frac{n}{v}\Big]\Big]\exp\Big\{-n\big(v^{-1}\overline{\eta}_n^\zeta - \overline{L}_n^\zeta(\overline{\eta}_n^\zeta)\big)\Big\} \\ &= E^{\zeta,\overline{\eta}_n^\zeta}\Big[\exp\Big\{-\sigma_n^\zeta\frac{\overline{\eta}_n^\zeta}{\sigma_n^\zeta}\sum_{i=1}^n \widehat{\tau}_i\Big\}; \frac{\overline{\eta}_n^\zeta}{\sigma_n^\zeta}\sum_{i=1}^n \widehat{\tau}_i \in \Big[0,-\frac{K\overline{\eta}_n^\zeta}{\sigma_n^\zeta}\Big]\Big]\exp\Big\{-n(\overline{L}_n^\zeta)^*\Big(\frac{1}{v}\Big)\Big\}. \end{aligned} \tag{4.39}$$

Writing μ_n^ζ for the distribution of $\frac{\overline{\eta}_n^\zeta}{\sigma_n^\zeta}\sum_{i=1}^n \widehat{\tau}_i$ under $P^{\zeta,\overline{\eta}_n^\zeta}$ (depending implicitly on v), we obtain

$$Y_v^{\approx}(n) = \exp\big\{-n(\overline{L}_n^\zeta)^*(1/v)\big\}\int_0^{-K\overline{\eta}_n^\zeta/\sigma_n^\zeta} e^{-\sigma_n^\zeta x}\,\mathrm{d}\mu_n^\zeta(x), \tag{4.40}$$

and, in a similar vein,

$$Y_v^{<}(n) = \exp\big\{-n(\overline{L}_n^\zeta)^*(1/v)\big\}\int_{-K\overline{\eta}_n^\zeta/\sigma_n^\zeta}^{\infty} e^{-\sigma_n^\zeta x}\,\mathrm{d}\mu_n^\zeta(x). \tag{4.41}$$

The first factor in (4.40) and (4.41) can be controlled by Proposition 4.5 and Corollary 4.8. The following lemma gives estimates for the second factors.

Lemma 4.10. *Let V and K be as in Proposition 4.9. Then there exists $C \in (1,\infty)$ such that on $\mathcal{H}_n$,*

$$\sigma_n^\zeta\int_0^{-K\overline{\eta}_n^\zeta/\sigma_n^\zeta} e^{-\sigma_n^\zeta x}\,\mathrm{d}\mu_n^\zeta(x) \in [C^{-1},C] \qquad \text{for all } v\in V, \tag{4.42}$$

and

$$\sigma_n^\zeta\int_{-K\overline{\eta}_n^\zeta/\sigma_n^\zeta}^{\infty} e^{-\sigma_n^\zeta x}\,\mathrm{d}\mu_n^\zeta(x) \in [C^{-1},C] \qquad \text{for all } v\in V. \tag{4.43}$$

In order not to hinder the flow of reading, we finish the proof of Proposition 4.9 first. Using Lemma 4.10, (4.40), and recalling the definition (4.25) of W_n directly yields (4.36). From (4.41), (4.40), and Lemma 4.10 we deduce (4.37). Finally, replacing n by nt in (4.36), observing that $\sqrt{t}W_{nt}(1) = W_n(t)$, and using (4.37), the fact that $\mathcal{H}_n$ occurs $\mathbb{P}$-a.s. for n large, in combination with and Proposition 4.5, yields the convergence of the three sequences in (4.38) to standard Brownian motion. □

We now show Lemma 4.10 which was used in the last proof.

Proof of Lemma 4.10. We start with proving (4.42). Throughout the proof we assume that $\mathcal{H}_n$ occurs. Observe that the $\widehat{\tau}_i$, $1\le i\le n$, are independent under $P_n^{\zeta,\overline{\eta}_n^\zeta}$ and have small exponential moments uniformly in n (cf. (4.9)). Moreover, recalling (4.34), the variance of the distribution μ_n^ζ is one by definition. A local central limit theorem for such independent normalized sequences, Theorem 13.3 (or formula (13.43)) of [BR76], thus yields

$$\sup_A |\mu_n^\zeta(A) - \Phi(A)| \le Cn^{-1/2}, \tag{4.44}$$

where the supremum runs over all intervals in $\mathbb{R}$, and Φ denotes the standard Gaussian measure. Applying (4.44) to $A = [0,-K\overline{\eta}_n^\zeta/\sigma_n^\zeta]$ and bearing in mind (4.35), this implies that for all K large enough, uniformly in $v\in V$,

$$c^{-1}n^{-1/2} < \mu_n^\zeta([0,-K\overline{\eta}_n^\zeta/\sigma_n^\zeta]) < cn^{-1/2}.$$

Since the function $e^{-\sigma_n^\zeta x}$ is uniformly bounded from above and below in this interval, (4.42) follows by another application of (4.35).

In order to show (4.43), we observe that uniformly in $v \in V$,

$$\sigma_n^\zeta \int_{-K\overline{\eta}_n^\zeta/\sigma_n^\zeta}^{\infty} e^{-\sigma_n^\zeta x} \, \mathrm{d}\mu_n^\zeta(x) \geq \sigma_n^\zeta \int_{-K\overline{\eta}_n^\zeta/\sigma_n^\zeta}^{-2K\overline{\eta}_n^\zeta/\sigma_n^\zeta} e^{-\sigma_n^\zeta x} \, \mathrm{d}\mu_n^\zeta(x) \geq C^{-1}$$

by the same arguments as in the proof of (4.42). On the other hand, using (4.35) and (4.44) again,

$$\begin{aligned} \sigma_n^\zeta \int_{-K\overline{\eta}_n^\zeta/\sigma_n^\zeta}^{\infty} e^{-\sigma_n^\zeta x} \, \mathrm{d}\mu_n^\zeta(x) &\leq \sigma_n^\zeta \sum_{j=1}^{\infty} \Big(\mu_n^\zeta\big(\big[-jK\overline{\eta}_n^\zeta/\sigma_n^\zeta, -(j+1)K\overline{\eta}_n^\zeta/\sigma_n^\zeta \big] \big) \Big) e^{-jK|\overline{\eta}_n^\zeta|} \\ &\leq c\sigma_n^\zeta \sum_{j=1}^{\infty} n^{-1/2} e^{-jK|\overline{\eta}_n^\zeta|} \leq C, \end{aligned} \tag{4.45}$$

uniformly in $v \in V$. This completes the proof of the lemma. □

Remark 4.11. The previous proof is the only place where the random tilting by $\overline{\eta}_n^\zeta(v)$ is really necessary. The reason for this is the application of the local central limit theorem-like estimate (4.44), which is useful only for events of sufficiently large probability. Deterministic tilting by $\overline{\eta}(v)$, which would simplify the remaining parts of the paper, unfortunately requires dealing with events of much smaller probability.

4.4 The walk lingers in the bulk

We now show that the invariance principles of Proposition 4.9 are useful in order to analyze the Feynman-Kac representation (3.3) of $\mathtt{E}_{u_0}^\xi[N^{\geq}(t, vt)]$. We explore the fact that, under the considered distributions, conditioning on $X_{n/v} = n$ (as in the Feynman-Kac representation) implies that with high probability H_n is close to n/v, that is the 'walk lingers in the bulk'.

Lemma 4.12. *Let $K > 0$ and V be as in Proposition 4.9. Then there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$ and $v \in V$, on $\mathcal{H}_n$,*

$$\begin{aligned} cY_v^{\approx}(n) &\leq E_0\Big[\exp\Big\{ \int_0^{n/v} \zeta(X_s) \, \mathrm{d}s \Big\}; X_{n/v} = n \Big] \\ &\leq E_0\Big[\exp\Big\{ \int_0^{n/v} \zeta(X_s) \, \mathrm{d}s \Big\}; X_{n/v} \geq n \Big] \leq c^{-1} Y_v^{\approx}(n). \end{aligned} \tag{4.46}$$

In particular,

$$c e^{\mathtt{es}\, n/v} Y_v^{\approx}(n) \leq \mathtt{E}_0^\xi[N(n/v, n)] \leq \mathtt{E}_0^\xi[N^{\geq}(n/v, n)] \leq c^{-1} e^{\mathtt{es}\, n/v} Y_v^{\approx}(n). \tag{4.47}$$

Proof. The second claim of the lemma follows directly from the first one. Indeed, recalling $\xi(x) = \zeta(x) + \mathtt{es}$ and (3.3), we obtain $\mathtt{E}_0^\xi[N(n/v, n)] = e^{\mathtt{es}\, n/v} E_0[\exp\{\int_0^{n/v} \zeta(X_s) \, \mathrm{d}s\}; X_{n/v} = n]$ as well as $\mathtt{E}_0^\xi[N^{\geq}(n/v, n)] = e^{\mathtt{es}\, n/v} E_0[\exp\{\int_0^{n/v} \zeta(X_s) \, \mathrm{d}s\}; X_{n/v} \geq n]$.

To prove the first claim we define

$$p_n^\zeta(s) := E_n\Big[\exp\Big\{ \int_0^s \zeta(X_r) \, \mathrm{d}r \Big\}; X_s = n \Big], \qquad n \in \mathbb{Z}, t \geq 0,$$

and set $t = n/v$, to simplify notation. Using the strong Markov property, we obtain

$$\begin{aligned} E_0\Big[\exp\Big\{ \int_0^t \zeta(X_s) \, \mathrm{d}s \Big\}; X_t = n \Big] &= E_0\Big[\exp\Big\{ \int_0^{H_n} \zeta(X_s) \, \mathrm{d}s \Big\} p_n^\zeta(t - H_n); H_n \leq t \Big] \\ &\geq E_0\Big[\exp\Big\{ \int_0^{H_n} \zeta(X_s) \, \mathrm{d}s \Big\}; H_n \in [t-K, t] \Big] \inf_{s \leq K} p_n^\zeta(s). \end{aligned}$$

Since the $\zeta(x)$'s are bounded from below by assumption (POT), the infimum on the right-hand side can be bounded from below by a deterministic constant $c = c(K) > 0$, which completes the proof of the first inequality in (4.46).

The second inequality of (4.46) is obvious. For the third one, observe that $\{X_t \geq n\} \subset \{H_n \leq t\}$. Therefore, decomposing the integral according to the value of H_n and using the fact that $\zeta \leq 0$, we obtain

$$\begin{aligned} E_0\Big[\exp\Big\{\int_0^t \zeta(X_s)\,\mathrm{d}s\Big\}; X_t \geq n\Big] &= E_0\Big[\exp\Big\{\int_0^{H_n} \zeta(X_s)\,\mathrm{d}s\Big\}\exp\Big\{\int_{H_n}^t \zeta(X_s)\,\mathrm{d}s\Big\}; X_t \geq n\Big] \\ &\leq E_0\Big[\exp\Big\{\int_0^{H_n} \zeta(X_s)\,\mathrm{d}s\Big\}; H_n \leq t\Big] = Y_v(n). \end{aligned}$$

By Proposition 4.9, $Y_v(n)$ and $Y_v^{\approx}(n)$ are comparable on $\mathcal{H}_n$, which proves the third inequality. □

4.5 Initial condition stability

The following lemma shows that initial conditions u_0 satisfying assumption (INI) are comparable to the 'one-particle' initial condition $u_0 = \mathbf{1}_{\{0\}}$.

Lemma 4.13. *Let V be as in (4.20). There exists a finite constant C such that for all u_0 as in (INI) and for all $n \in \mathbb{N}$, $t \geq 0$ such that $n/t \in V$, on $\mathcal{H}_n$,*

$$1 \leq \frac{\mathtt{E}^{\xi}_{u_0}\big[N^{\geq}(t,n)\big]}{\mathtt{E}^{\xi}_{0}\big[N(t,n)\big]} \leq C. \tag{4.48}$$

Proof. The first inequality in (4.48) is obvious, so we proceed to the second one. Moreover, since $\mathtt{E}^{\xi}_{u_0}\big[N^{\geq}(t,n)\big]$ is an increasing function of $u_0(x)$ for every $x \in -\mathbb{N}_0$, we can assume that $u_0 = c\mathbf{1}_{-\mathbb{N}_0}$. Using the Feynman-Kac representation (3.3), and replacing ξ by ζ, we see that

$$\frac{\mathtt{E}^{\xi}_{c\mathbf{1}_{-\mathbb{N}_0}}\big[N^{\geq}(t,n)\big]}{\mathtt{E}^{\xi}_{0}\big[N(t,n)\big]} = \frac{c\sum_{x\leq 0} E_x\Big[\exp\Big\{\int_0^t \zeta(X_s)\,\mathrm{d}s\Big\}; X_t \geq n\Big]}{E_0\Big[\exp\Big\{\int_0^t \zeta(X_s)\,\mathrm{d}s\Big\}; X_t = n\Big]}. \tag{4.49}$$

Applying the strong Markov property on the numerator of the right-hand side, we obtain

$$\begin{aligned} \sum_{x\leq 0} E_x\Big[\exp\Big\{\int_0^t \zeta(X_s)\,\mathrm{d}s\Big\}; X_t \geq n\Big] &\leq \sum_{x\leq 0} E_x\Big[\exp\Big\{\int_0^{H_n} \zeta(X_s)\,\mathrm{d}s\Big\}; H_n \leq t\Big] \\ = \sum_{x\leq 0}\int_0^t E_x\Big[\exp\Big\{\int_0^{H_0} \zeta(X_s)\,\mathrm{d}s\Big\}; H_0 \in \mathrm{d}a\Big] & E_0\Big[\exp\Big\{\int_0^{H_n} \zeta(X_s)\,\mathrm{d}s\Big\}; H_n \leq t-a\Big]. \end{aligned}$$

Using the same techniques as in the proof of Proposition 4.9, in particular using (4.41) with $v = n/t$ and $K = a$, and recalling the estimate (4.45) from the proof of Lemma 4.10, it can be seen that on $\mathcal{H}_n$ the second factor on the right-hand side can be bounded from above by $Ce^{-ca}Y^{\approx}_{n/t}(n)$ for all n with $n/t \in V$ and $a \in [0,t]$. This implies that the right-hand side of the last display is bounded from above by

$$CY^{\approx}_{n/t}(n)\sum_{x\leq 0}\int_0^t P_x(H_0 \in \mathrm{d}a)e^{-ca} \leq CY^{\approx}_{n/t}(n),$$

where for the last inequality we used the fact that $\sum_{x\leq 0} P_x(H_0 \in \mathrm{d}a) = \sum_{x\geq 0} P_0(H_x \in \mathrm{d}a)$ which is the probability that an arbitrary point $x \geq 0$ is visited for the first time at time $\mathrm{d}a$, so it is bounded by $\mathrm{d}a$. By Lemma 4.12, on $\mathcal{H}_n$, $Y^{\approx}_{n/t}(n)$ is comparable to the denominator of the right-hand side in (4.49), which completes the proof. □

4.6 Functional central limit theorem for the PAM

We have all ingredients to show our first main result, the invariance principle for the PAM, Theorem 2.8.

Proof of Theorem 2.8. Recall that we have to show that $t \mapsto \big(\ln u(nt, \lfloor vnt \rfloor) - nt\lambda(v)\big)/(\sigma_v\sqrt{vn})$ converges to standard Brownian motion in $\mathbb{P}$-distribution as $n \to \infty$, where $u(t,x) = \mathtt{E}^{\xi}_{u_0}[N(t,x)]$ was defined in (2.10).

By Lemma 4.13 we can assume without loss of generality that $u_0 = \mathbf{1}_{\{0\}}$. Moreover, by Lemma 4.12, $\mathbb{P}$-a.s. for all large t,

$$cY_v^{\approx}(vt)e^{t\mathsf{es}} \leq u(t, \lfloor vt \rfloor) \leq c^{-1}Y_v^{\approx}(vt)e^{t\mathsf{es}}. \tag{4.50}$$

Replacing t by vt in Proposition 4.9, we see that

$$t \mapsto \frac{1}{\sigma_v\sqrt{nv}}\big(\ln Y_v^{\approx}(tvn) + tvnL^*(1/v)\big)$$

converges as $n \to \infty$ to standard Brownian motion. Combining this with (4.50) easily implies the theorem and incidentally also shows that the Lyapunov exponent $\lambda(v)$ defined in (2.2) satisfies $\lambda(v) = \mathsf{es} - vL^*(1/v)$ for $v > v_c$, as claimed in (A.5) of Lemma A.3. This completes the proof of Theorem 2.8. □

Remark 4.14. Theorem 2.8 remains valid when the function $u(t,x)$ is replaced by $\mathtt{E}^{\xi}_{u_0}[N^{\geq}(t,x)]$ with u_0 as in (INI). This is a consequence of Lemma 4.13 again.

Remark 4.15. It will be useful to have a more explicit formula for $\ln \mathtt{E}^{\xi}_0[N^{\geq}(n/v_0, n)]$. Combining (4.47) with Corollary 4.8, Proposition 4.9 and (A.5) yields the existence of a constant $C < \infty$ and a $\mathbb{P}$-a.s. finite random variable $\mathcal{N}_3$ such that $\mathbb{P}$-a.s. for all $n \geq \mathcal{N}_3$,

$$\Big|\ln \mathtt{E}^{\xi}_0[N^{\geq}(n/v_0, n)] - \sum_{i=1}^{n} L_i^{\zeta}(\overline{\eta}(v_0)) + nL(\overline{\eta}(v_0))\Big| \leq C\ln n.$$

5 Breakpoint behavior

The goal of this section is to prove the functional central limit theorem for the breakpoint, Theorem 2.3. This is done in Section 5.2 after some additional preparations.

5.1 Perturbation estimates

The results of Section 4 give a reasonably precise description of the behavior of expectations of $N^{\geq}(t, vt)$. We are now interested in how sensitive the expectation of $N^{\geq}(\cdot,\cdot)$ is to perturbations in the space and time coordinate. The first lemma deals with space perturbations:

Lemma 5.1. *(a) For each $\varepsilon > 0$ there exists a constant $C_0 = C_0(\varepsilon) < \infty$ such that $\mathbb{P}$-a.s.,*

$$\limsup_{t\to\infty} \sup\Big\{\Big|\frac{1}{h}\ln\frac{\mathtt{E}^{\xi}_{u_0}[N^{\geq}(t, vt+h)]}{\mathtt{E}^{\xi}_{u_0}[N^{\geq}(t, vt)]} - L(\overline{\eta}(v))\Big| : C_0\ln t \leq |h| \leq t\widetilde{\varepsilon}(t),\ v, v' \in V\Big\} \leq \varepsilon, \tag{5.1}$$

where $\widetilde{\varepsilon}(t)$ is an arbitrary positive function satisfying $\lim_{t\to\infty}\widetilde{\varepsilon}(t)t^{\delta} = 0$ for some $\delta > 0$, and $v' := v + \frac{h}{t}$.

(b) There exist constants $C, c \in (0,\infty)$ such that $\mathbb{P}$-a.s. for all t large enough, uniformly for $0 \leq h \leq t^{1/3}$ and $v, v + h/t \in V$

$$ce^{-Ch}\mathtt{E}^{\xi}_{u_0}\big[N^{\geq}(t, vt)\big] \leq \mathtt{E}^{\xi}_{u_0}\big[N^{\geq}(t, vt+h)\big] \leq Ce^{-ch}\mathtt{E}^{\xi}_{u_0}\big[N^{\geq}(t, vt)\big].$$

Proof. (a) Without loss of generality we can assume t to be large enough so that the events $\mathcal{H}_{vt}$ and $\mathcal{H}_{v't}$ occur and thus $\overline{\eta}^{\zeta}_{vt}(v)$ and $\overline{\eta}^{\zeta}_{v't}(v')$ exist and satisfy corresponding versions of (4.11). By Lemmas 4.12 and 4.13, the fraction in (5.1) can be approximated (up to a multiplicative constant that is irrelevant in the limit) by

$$\frac{Y_{v'}(v't)}{Y_v(vt)} = \frac{E_0\Big[\exp\Big\{\int_0^{H_{v't}} \zeta(X_s)\,\mathrm{d}s\Big\}; H_{v't} \le t\Big]}{E_0\Big[\exp\Big\{\int_0^{H_{vt}} \zeta(X_s)\,\mathrm{d}s\Big\}; H_{vt} \le t\Big]}.$$

Using the notation from (4.27), this can be rewritten in the same vein as in (4.39) as

$$\frac{E^{\zeta,\overline{\eta}^{\zeta}_{v't}(v')}\Big[\exp\big\{-\overline{\eta}^{\zeta}_{v't}(v')\sum_{i=1}^{v't}\widetilde{\tau}_i\big\}; \sum_{i=1}^{v't}\widetilde{\tau}_i \in (-\infty,0]\Big]\cdot\exp\big\{-S^{\zeta,v'}_{v't}(\overline{\eta}^{\zeta}_{v't}(v'))\big\}}{E^{\zeta,\overline{\eta}^{\zeta}_{vt}(v)}\Big[\exp\big\{-\overline{\eta}^{\zeta}_{vt}(v)\sum_{i=1}^{vt}\widehat{\tau}_i\big\}; \sum_{i=1}^{vt}\widehat{\tau}_i \in (-\infty,0]\Big]\cdot\exp\big\{-S^{\zeta,v}_{vt}(\overline{\eta}^{\zeta}_{vt}(v))\big\}}, \tag{5.2}$$

where, similarly as before $\widehat{\tau}_i := \tau_i - E^{\zeta,\overline{\eta}^{\zeta}_{vt}(v)}[\tau_i]$ and $\widetilde{\tau}_i := \tau_i - E^{\zeta,\overline{\eta}^{\zeta}_{v't}(v')}[\tau_i]$. By the same methods as in (4.40)–(4.43), the expectations in the numerator and denominator of (5.2) are both of order $t^{-1/2}$. Their ratio is thus bounded from above and below by positive finite constants and can be neglected in the limit taken in (5.1).

The remaining terms in (5.2) contribute to the minuend of (5.1) as

$$\begin{aligned}&\frac{1}{h}\big(S^{\zeta,v}_{vt}(\overline{\eta}^{\zeta}_{vt}(v)) - S^{\zeta,v'}_{v't}(\overline{\eta}^{\zeta}_{v't}(v'))\big)\\ &\quad= \frac{1}{h}\big(S^{\zeta,v}_{vt}(\overline{\eta}^{\zeta}_{vt}(v)) - S^{\zeta,v}_{vt}(\overline{\eta}^{\zeta}_{v't}(v'))\big) + \frac{1}{h}\big(S^{\zeta,v}_{vt}(\overline{\eta}^{\zeta}_{v't}(v')) - S^{\zeta,v'}_{v't}(\overline{\eta}^{\zeta}_{v't}(v'))\big).\end{aligned} \tag{5.3}$$

In order to show that the first summand on the right-hand side of (5.3) is negligible uniformly as $t \to \infty$, we write

$$\begin{aligned}&S^{\zeta,v}_{vt}(\overline{\eta}^{\zeta}_{v't}(v'))\\ &\quad= S^{\zeta,v}_{vt}(\overline{\eta}^{\zeta}_{vt}(v)) + (S^{\zeta,v}_{vt})'(\overline{\eta}^{\zeta}_{vt}(v))\,\big(\overline{\eta}^{\zeta}_{v't}(v') - \overline{\eta}^{\zeta}_{vt}(v)\big) + (S^{\zeta,v}_{vt})''(\widetilde{\eta})\,\big(\overline{\eta}^{\zeta}_{v't}(v') - \overline{\eta}^{\zeta}_{vt}(v)\big)^2,\end{aligned} \tag{5.4}$$

for some $\widetilde{\eta} \in \Delta$ with $\big|\widetilde{\eta} - \overline{\eta}^{\zeta}_{vt}(v)\big| \le \big|\overline{\eta}^{\zeta}_{v't}(v') - \overline{\eta}^{\zeta}_{vt}(v)\big|$. As observed below (4.31), one has $(S^{\zeta,v}_{vt})'(\overline{\eta}^{\zeta}_{vt}(v)) = 0$, so the second term vanishes. For the third one, note that by Lemma 4.4, $\mathbb{P}$-a.s. for all t large enough,

$$|\overline{\eta}^{\zeta}_{v't}(v') - \overline{\eta}^{\zeta}_{vt}(v')| \le \frac{Ch}{t}. \tag{5.5}$$

Moreover, by the characterizing property (4.16) of $\overline{\eta}^{\zeta}_{vt}(v)$, Lemma A.1 and the implicit function theorem, we see that $\overline{\eta}^{\zeta}_{vt}(v)$ is differentiable on V, with uniformly bounded derivative, on $\mathcal{H}_{vt}$. Therefore, on $\mathcal{H}_{vt}$, uniformly for $v \in V$ and h as in (5.1),

$$|\overline{\eta}^{\zeta}_{vt}(v') - \overline{\eta}^{\zeta}_{vt}(v)| \le \frac{Ch}{t}. \tag{5.6}$$

Recalling (4.32), we see that $(S^{\zeta,v}_{vt})''(\widetilde{\eta}) \le Ct$ uniformly in $v \in V$ and t large, $\mathbb{P}$-a.s. Combined with (5.4) to (5.6) we thus deduce that $\mathbb{P}$-a.s., the first term on the right-hand side of (5.3) satisfies

$$\Big|\frac{1}{h}\big(S^{\zeta,v}_{vt}(\overline{\eta}^{\zeta}_{vt}(v)) - S^{\zeta,v}_{vt}(\overline{\eta}^{\zeta}_{v't}(v'))\big)\Big| \le \frac{Ch}{t} \tag{5.7}$$

which is negligible in the limit considered in (5.1).

Plugging in the definitions, the second summand on the right-hand side of (5.3) satisfies

$$\frac{1}{h}\big(S^{\zeta,v}_{vt}(\overline{\eta}^{\zeta}_{v't}(v')) - S^{\zeta,v'}_{v't}(\overline{\eta}^{\zeta}_{v't}(v'))\big) = \frac{1}{h}\sum_{i=vt+1}^{v't} L^{\zeta}_i(\overline{\eta}^{\zeta}_{v't}(v')) \tag{5.8}$$

(where the sum should be interpreted as $-\sum_{i=v't+1}^{vt}$ if $v' < v$). The right-hand side of (5.8) can be approximated with the help of the following claim.

Claim 5.2. *For each $\varepsilon > 0$ and each $q \in \mathbb{N}$ there exists a constant $C = C(q, \varepsilon) < \infty$ such that for all t large enough,*

$$\mathbb{P}\Big(\sup_{\substack{v \in V \\ C(q,\varepsilon) \ln t \le |h| \le t\widetilde{\varepsilon}(t)}} \Big| \frac{1}{h} \sum_{i=vt+1}^{v't} L_i^\zeta(\overline{\eta}_{v't}^\zeta(v')) - L(\overline{\eta}(v)) \Big| > \varepsilon, \mathcal{H}_{v't}, \mathcal{H}_{vt} \Big) \le Ct^{-q}$$

with $\widetilde{\varepsilon}(t)$ and v' as in Lemma 5.1.

We postpone the proof of this claim after the proof of Lemma 5.1. (5.7) and Claim 5.2 together imply that the left-hand side of (5.3) $\mathbb{P}$-a.s. satisfies

$$\limsup_{t \to \infty} \sup \Big\{ \Big| \frac{1}{h} \big(S_{vt}^{\zeta, v}(\overline{\eta}_{vt}^\zeta(v)) - S_{v't}^{\zeta, v'}(\overline{\eta}_{v't}^\zeta(v')) \big) - L(\overline{\eta}(v)) \Big| : C(2, \varepsilon) \ln t \le |h| \le t\widetilde{\varepsilon}(t) \Big\} \le \varepsilon,$$

which is exactly what is necessary to prove Lemma 5.1(a).

(b) Using the same arguments as in the proof of (a), it is sufficient show that the exponential factors in (5.2) are bounded from above and below by exponential functions, that is the right-hand side of (5.3) is bounded away from 0 and ∞. However, for the second summand on the right-hand side this easily follows from (5.8), because $c^{-1} < L_i^\zeta(\overline{\eta}_{v't}^\zeta(v')) < c < 0$ uniformly in $i \ge 0$, $v \in V$, and ξ satisfying (POT). The first summand can be neglected for t sufficiently large due to (5.7). □

Proof of Claim 5.2. We rewrite

$$\begin{aligned} &\frac{1}{h} \sum_{i=vt+1}^{v't} L_i^\zeta(\overline{\eta}_{v't}^\zeta(v')) - L(\overline{\eta}(v)) \\ &= \frac{1}{h} \sum_{i=vt+1}^{v't} \big(L_i^\zeta(\overline{\eta}_{v't}^\zeta(v')) - L_i^\zeta(\overline{\eta}(v)) \big) + \frac{1}{h} \sum_{i=vt+1}^{v't} \big(L_i^\zeta(\overline{\eta}(v)) - L(\overline{\eta}(v)) \big). \end{aligned} \tag{5.9}$$

Observing that the family of functions $\big(\eta \mapsto L_i^\zeta(\eta)\big)_{i \in \mathbb{Z}, -\mathsf{es} \le \zeta(j) \le 0 \forall j \in \mathbb{Z}}$ is equicontinuous on Δ, Proposition 4.2, (5.5) and (5.6) yield that

$$\mathbb{P}\Big(\sup_{\substack{v \in V \\ \ln t \le |h| \le t\widetilde{\varepsilon}(t)}} \Big| \frac{1}{h} \sum_{i=vt+1}^{v't} \big(L_i^\zeta(\overline{\eta}_{v't}^\zeta(v')) - L_i^\zeta(\overline{\eta}(v)) \big) \Big| \ge \frac{\varepsilon}{2}, \mathcal{H}_{v't}, \mathcal{H}_{vt} \Big) \le Ct^{-q}. \tag{5.10}$$

Regarding the second summand on the right-hand side of (5.9), it suffices to observe that for $C(q, \varepsilon)$ large enough,

$$\mathbb{P}\Big(\sup_{\substack{x \in \Delta \\ C(q,\varepsilon) \ln t \le |h| \le t\widetilde{\varepsilon}(t)}} \Big| \frac{1}{h} \sum_{i=vt+1}^{v't} L_i^\zeta(x) - L(x) \Big| \ge \varepsilon/2 \Big) \le Ct^{-q}, \tag{5.11}$$

which follows from the Hoeffding type bound (Lemma A.4) using the same steps as in the proof of Claim 4.3. Combining (5.9)–(5.11) with (4.21) finishes the proof of the claim. □

We now deal with time perturbations.

Lemma 5.3. *(a) Let $\varepsilon(t)$ be an arbitrary function such that $\lim_{t \to \infty} \varepsilon(t) = 0$. Then $\mathbb{P}$-a.s.,*

$$\lim_{t \to \infty} \sup \Big\{ \Big| \frac{1}{h} \ln \frac{\mathtt{E}_{u_0}^\xi [N^{\ge}(t+h, vt)]}{\mathtt{E}_{u_0}^\xi [N^{\ge}(t, vt)]} - \big(\mathsf{es} - \overline{\eta}(v)\big) \Big| : \varepsilon(t)^{-1} \le |h| \le t\varepsilon(t), v, v' \in V \Big\} = 0. \tag{5.12}$$

where $v' := vt/(t+h)$.

(b) There exist constants $C, c \in (0, \infty)$ such that $\mathbb{P}$-a.s. for all t large enough, uniformly in $0 \le h \le t^{1/3}$ and $v, vt/(t+h) \in V$,

$$ce^{ch} \mathtt{E}_{u_0}^\xi \big[N^{\ge}(t, vt)\big] \le \mathtt{E}_{u_0}^\xi \big[N^{\ge}(t+h, vt)\big] \le Ce^{Ch} \mathtt{E}_{u_0}^\xi \big[N^{\ge}(t, vt)\big].$$

Proof. (a) Using Proposition 4.9 and the same arguments as in the proof of Lemma 5.1, the fraction in (5.12) can be approximated (up to a multiplicative constant that is irrelevant in the limit) by

$$e^{h\mathsf{es}} \cdot \frac{Y_{v'}^{\approx}(vt)}{Y_v^{\approx}(vt)}.$$

We may assume t to be large enough such that $\mathcal{H}_{vt}$ and $\mathcal{H}_{v't}$ hold true and then the previous can again be rewritten using (4.39) to obtain

$$e^{h\mathsf{es}} \cdot \frac{E^{\zeta,\overline{\eta}_{vt}^{\zeta}(v')}\Big[\exp\big\{-\overline{\eta}_{vt}^{\zeta}(v')\sum_{i=1}^{vt}\widetilde{\tau}_i\big\};\sum_{i=1}^{vt}\widetilde{\tau}_i\in[-K,0]\Big]\cdot\exp\big\{-S_{vt}^{\zeta,v'}(\overline{\eta}_{vt}^{\zeta}(v'))\big\}}{E^{\zeta,\overline{\eta}_{vt}^{\zeta}(v)}\Big[\exp\big\{-\overline{\eta}_{vt}^{\zeta}(v)\sum_{i=1}^{vt}\widehat{\tau}_i\big\};\sum_{i=1}^{vt}\widehat{\tau}_i\in[-K,0]\Big]\cdot\exp\big\{-S_{vt}^{\zeta,v}(\overline{\eta}_{vt}^{\zeta}(v))\big\}}, \tag{5.13}$$

where again $\widehat{\tau}_i := \tau_i - E^{\zeta,\overline{\eta}_{vt}^{\zeta}(v)}[\tau_i]$ and $\widetilde{\tau}_i := \tau_i - E^{\zeta,\overline{\eta}_{vt}^{\zeta}(v')}[\tau_i]$. As in the proof of Lemma 5.1, the ratio of the expectations in the numerator and denominator is asymptotically bounded from above and below and thus can be neglected in the limit taken in (5.12). The remaining terms in (5.13) contribute to the left-hand side of (5.12) as

$$\begin{aligned}\mathsf{es} &+ \frac{1}{h}\big(S_{vt}^{\zeta,v}(\overline{\eta}_{vt}^{\zeta}(v)) - S_{vt}^{\zeta,v'}(\overline{\eta}_{vt}^{\zeta}(v'))\big)\\ &= \mathsf{es} + \frac{1}{h}\big(S_{vt}^{\zeta,v}(\overline{\eta}_{vt}^{\zeta}(v)) - S_{vt}^{\zeta,v}(\overline{\eta}_{vt}^{\zeta}(v'))\big) + \frac{1}{h}\big(S_{vt}^{\zeta,v}(\overline{\eta}_{vt}^{\zeta}(v')) - S_{vt}^{\zeta,v'}(\overline{\eta}_{vt}^{\zeta}(v'))\big).\end{aligned} \tag{5.14}$$

Plugging in the definitions (4.27) and (4.23), the third summand on the right-hand side satisfies

$$\frac{1}{h}\big(S_{vt}^{\zeta,v}(\overline{\eta}_{vt}^{\zeta}(v')) - S_{vt}^{\zeta,v'}(\overline{\eta}_{vt}^{\zeta}(v'))\big) = \frac{1}{h}vt\overline{\eta}_{vt}^{\zeta}(v')\Big(\frac{1}{v}-\frac{1}{v'}\Big) = -\overline{\eta}_{vt}^{\zeta}(v') \xrightarrow{t\to\infty} -\overline{\eta}(v),$$

where the convergence follows from Proposition 4.2 and the continuity of $\overline{\eta}(\cdot)$. The second summand on the right-hand side of (5.14) can be shown to be negligible similarly to the proof of Lemma 5.1, completing the proof.

(b) Following the same steps as in (a), it is sufficient to observe that (5.14) is bounded from below and above by positive finite constants. □

5.2 Functional central limit theorem for the breakpoint

We now have all the ingredients to show our second main result, the invariance principle for the breakpoint, Theorem 2.8.

Proof of Theorem 2.3. Recall that we have to prove that $t \mapsto \frac{1}{\sigma_v\sqrt{n}}\big(\overline{m}_v(nt) - vnt\big)$ converges to standard Brownian motion, where

$$\overline{m}_v(t) = \sup\Big\{n\in\mathbb{N} : \mathtt{E}_{u_0}^{\xi}\big[N^{\geq}(t,n)\big] \geq \frac{1}{2}e^{t\lambda(v)}\Big\}$$

was defined in (2.7).

We assume that $u_0 = \mathbf{1}_{\{0\}}$ first. Let $u^{\geq}(t,x) := \mathtt{E}_0^{\xi}[N^{\geq}(t,x)]$, $t \geq 0$, $x \in \mathbb{Z}$, and extend it to $x \in \mathbb{R}$ by linear interpolation. Furthermore, set

$$U_v(t) := t\lambda(v) - \ln u^{\geq}(t,vt) - \ln 2. \tag{5.15}$$

Recalling the definition of σ_v^2 from (4.24), by Remark 4.14 the sequence

$$t \mapsto \frac{1}{\sqrt{\sigma_v^2 vn}}U_v(nt), \qquad n\in\mathbb{N}, \qquad \text{converges as } n\to\infty \text{ to Brownian motion.} \tag{5.16}$$

Obviously, $u^{\geq}(t,x)$ is decreasing in x, $\lim_{x\to\infty} u^{\geq}(t,x) = 0$, and $\lim_{t\to\infty}\frac{1}{t}\ln u^{\geq}(t,0) = \lambda(0) > \lambda(v)$, see Lemma A.3 also. Let $r = r(t)$ be the largest solution of the equation

$$u^{\geq}(t, vt+r) = \frac{1}{2}e^{t\lambda(v)}, \tag{5.17}$$

which exists $\mathbb{P}$-a.s. for t large enough by the previous considerations. Moreover, by the definition of $\overline{m}_v(t)$,

$$r(t) - 1 < \overline{m}_v(t) - vt \le r(t). \tag{5.18}$$

Combining equations (5.15) and (5.17), we see that $r(t)$ is the largest solution to

$$\ln \frac{u^{\ge}(t, vt + r(t))}{u^{\ge}(t, vt)} = U_v(t).$$

Let $\widetilde{\varepsilon}(t)$ be an arbitrary positive function with $\widetilde{\varepsilon}(t)t^{\frac{1}{4}} \to 0$ and $\widetilde{\varepsilon}(t)t^{\frac{1}{2}} \to \infty$ as $t \to \infty$. By the space perturbation Lemma 5.1, using also the monotonicity of $u^{\ge}(t,\cdot)$ and the fact that $L(\overline{\eta}(v)) < 0$, we obtain that for every

$$\delta \in (0, |L(\overline{\eta}(v))|), \tag{5.19}$$

$\mathbb{P}$-a.s. for all t large enough,

$$\overline{\varphi}_t(r(t))L(\overline{\eta}(v)) - \delta|r(t)| \le \ln \frac{u^{\ge}(t, vt + r(t))}{u^{\ge}(t, vt)} \le \underline{\varphi}_t(r(t))L(\overline{\eta}(v)) + \delta|r(t)|;$$

here, for $C_0 = C_0(\delta)$, the functions $\overline{\varphi}_t$ and $\underline{\varphi}_t$ are given by

$$\underline{\varphi}_t(r) = \sup\big\{s : s \le r \text{ and } C_0 \ln t \le |s| \le t\widetilde{\varepsilon}(t)\big\},$$
$$\overline{\varphi}_t(r) = \inf\big\{s : s \ge r \text{ and } C_0 \ln t \le |s| \le t\widetilde{\varepsilon}(t)\big\},$$

and satisfy $\underline{\varphi}_t(r) = \overline{\varphi}_t(r) = r$ for $C_0 \ln t \le |r| \le t\varepsilon(t)$ and $\underline{\varphi}_t \le \overline{\varphi}_t$. This implies that whenever

$$|U_v(t)| \in \big[C_0(|L(\overline{\eta}(v))| + \delta) \ln t, t\widetilde{\varepsilon}(t)(|L(\overline{\eta}(v))| - \delta)\big], \tag{5.20}$$

then, due to (5.19),

$$r(t) \in \Big[\frac{U_v(t)}{L(\overline{\eta}(v)) \pm \delta}, \frac{U_v(t)}{L(\overline{\eta}(v)) \mp \delta}\Big],$$

where the upper signs correspond to $U_v(t) > 0$ and the lower signs to $U_v(t) < 0$. In particular, since U_v satisfies the invariance principle (5.16), property (5.20) is satisfied with probability tending to 1 as $t \to \infty$. Since δ is arbitrary, it thus follows that in $\mathbb{P}$-distribution

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} r(n\cdot) = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \cdot \frac{U_v(n\cdot)}{L(\overline{\eta}(v))}$$

as processes defined on $[0, \infty)$, which together with (5.18) and (5.16) implies the claim of the theorem for

$$\overline{\sigma}_v = \frac{\sqrt{\sigma_v^2 v}}{|L(\overline{\eta}(v))|}. \tag{5.21}$$

The case of general u_0 satisfying (INI) then follows from Lemmas 4.13 and 5.1. This completes the proof. □

5.3 Invariance principle for the breakpoint inverse

We will later on need the following invariance principle for a generalized inverse of the breakpoint. We work only with $v = v_0$.

Theorem 5.4. *Let $T_0 = 0$ and*

$$T_n := \inf\Big\{t \ge 0 : \mathtt{E}^{\xi}_{u_0}[N^{\ge}(t, n)] \ge \frac{1}{2}\Big\}, \qquad n \ge 1, \tag{5.22}$$

be the breakpoint inverse. *Then there exists a $\mathbb{P}$-a.s. finite random variable $\mathcal{C} = \mathcal{C}(\xi)$ and a constant $C_1 < \infty$ such that $\mathbb{P}$-a.s. for all $n \ge 1$,*

$$\Big|T_n - \Big(\frac{n}{v_0} + \frac{1}{v_0 L(\overline{\eta}(v_0))} \sum_{i=1}^{n} \big(L_i^{\zeta}(\overline{\eta}(v_0)) - L(\overline{\eta}(v_0))\big)\Big)\Big| \le \mathcal{C} + C_1 \ln n. \tag{5.23}$$

In particular, the sequence

$$t \mapsto \frac{v_0 L(\overline{\eta}(v_0))}{\sqrt{\sigma_{v_0} n}}\Big(T_{nt} - \frac{nt}{v_0}\Big), \qquad n \geq 0,$$

of processes converges as $n \to \infty$ in $\mathbb{P}$-distribution to standard Brownian motion.

Proof. The proof is completely analogous to the proof of Theorem 2.3, using the time perturbation Lemma 5.3 instead of the space perturbation Lemma 5.1, and the relation $\mathtt{es} - \overline{\eta}(v_0) + v_0 L(\overline{\eta}(v_0)) = 0$ which follows from (A.5). For $n \geq \mathcal{N}_3$, where $\mathcal{N}_3$ has been defined in Remark 4.15, the formula (5.23) follows by writing $\ln u^{\geq}(n/v_0, n)$ explicitly as in Remark 4.15. The remaining terms with $n < \mathcal{N}_3$ are then bounded by the random variable $\mathcal{C}$. The invariance principle is then again a consequence of this formula and Remark 4.14. □

6 The breakpoint approximates the maximum

In this section we prove the main results about the position of the rightmost particle $M(t)$ and its median $m(t)$. We will see that those are well approximated by the breakpoint, and thus satisfy the same invariance principles.

It is elementary to obtain upper tail estimates for $M(t)$ and an upper bound on $m(t)$: the definition of $\overline{m}(t)$ and the Markov inequality imply directly that

$$\overline{m}(t) \geq m(t). \tag{6.1}$$

In addition, by Lemma 5.1(b), $\mathbb{P}$-a.s. for t large enough,

$$\mathtt{P}^{\xi}_{u_0}(M(t) \geq \overline{m}(t) + h) \leq \mathtt{E}^{\xi}_{u_0}[N(t, \overline{m}(t) + h)] \leq Ce^{-ch}, \qquad h \in (0, t^{1/3}). \tag{6.2}$$

Note that these estimates are rather imprecise. One expects (6.2) to hold with $m(t)$ instead of $\overline{m}(t)$ and $\overline{m}(t) - m(t) \asymp \ln t$. These bounds are however more than sufficient to show the stated functional limit theorems.

As usual in the branching random walk literature, the lower bounds are more difficult, and are obtained via second moment estimates on the so-called leading particles. Since $m(t)$ and $M(t)$ are stochastically increasing in the initial condition, we will assume, without loss of generality, that $u_0 = \mathbf{1}_{\{0\}}$ throughout this section.

6.1 Leading particles

We will consider a special class of particles $Y \in N(t)$ with trajectories satisfying

$$Y_t \geq \overline{m}(t) \qquad \text{and} \qquad H^Y_k \geq T_k - \alpha\psi^{\xi}(k) \text{ for all } 1 \leq k < \overline{m}(t), \tag{6.3}$$

where $H^Y_k = \inf\{s \geq 0 : Y_s = k\}$, $\alpha > 2$ is a fixed constant, T_k is the breakpoint inverse introduced in (5.22) of Theorem 5.4, and ψ^{ξ} is defined by

$$\psi^{\xi}(k) = \mathcal{C}(\xi) + C_1(1 \vee \ln k), \tag{6.4}$$

where $\mathcal{C}(\xi)$ and C_1 are as in (5.23). Analogously to the literature on homogeneous branching random walk, we call such particles *leading at time t*. We further set

$$N^{\mathcal{L}}_t = \big|\{Y \in N(t) : Y \text{ is leading at time } t\}\big|.$$

The probability of finding a leading particle at time t is bounded from below in the following proposition.

Proposition 6.1. *There exists a constant $\gamma > 0$ such that $\mathbb{P}$-a.s. for all t large enough*

$$\mathtt{P}^{\xi}_0(N^{\mathcal{L}}_t \geq 1) \geq t^{-\gamma}.$$

The proof of this proposition will be based on the classical Paley-Zygmund inequality

$$\mathtt{P}_0^\xi(N_t^{\mathcal{L}} \ge 1) \ge \frac{\mathtt{E}_0^\xi[N_t^{\mathcal{L}}]^2}{\mathtt{E}_0^\xi[(N_t^{\mathcal{L}})^2]}. \tag{6.5}$$

Estimates for the expectations on the right-hand side are provided in the following two subsections. Since we do not strive to find the optimal constant γ in this paper, we use γ to denote a generic large constant whose value can change during the computations.

6.1.1 First moment for the leading particles

Lemma 6.2. *There exists a constant $\gamma > 0$ such that $\mathbb{P}$-a.s. for all t large enough*

$$\mathtt{E}_0^\xi[N_t^{\mathcal{L}}] \ge t^{-\gamma}.$$

Proof. By the definition of $\overline{m}(t)$ we have $\mathtt{E}_0^\xi[N^{\ge}(t,\overline{m}(t))] \ge 1/2$, and thus by Proposition 3.1,

$$\begin{aligned}\mathtt{E}_0^\xi[N_t^{\mathcal{L}}] &\ge \frac{\mathtt{E}_0^\xi[N_t^{\mathcal{L}}]}{2\mathtt{E}_0^\xi[N^{\ge}(t,\overline{m}(t))]}\\ &\ge \frac{E_0\Big[\exp\big\{\int_0^t \zeta(X_s)\,\mathrm{d}s\big\}; X_t \ge \overline{m}(t), H_k \ge T_k - \alpha\psi^\xi(k)\,\forall k < \overline{m}(t)\Big]}{2E_0\Big[\exp\big\{\int_0^t \zeta(X_s)\,\mathrm{d}s\big\}; X_t \ge \overline{m}(t)\Big]}.\end{aligned} \tag{6.6}$$

Following the same steps as in the proof of the lower bound in Lemma 4.12, the numerator satisfies

$$\begin{aligned}&E_0\Big[e^{\int_0^t \zeta(X_s)\,\mathrm{d}s}; X_t \ge \overline{m}(t), H_k \ge T_k - \alpha\psi^\xi(k)\,\forall k < \overline{m}(t)\Big]\\ &\ge E_0\Big[e^{\int_0^{H_{\overline{m}(t)}} \zeta(X_s)\,\mathrm{d}s} E_{\overline{m}(t)}\big[e^{\int_0^r \zeta(X_s)\,\mathrm{d}s}; X_r \ge \overline{m}(t)\big]_{|r=t-H_{\overline{m}(t)}};\\ &\qquad H_{\overline{m}(t)} \in [t-K,t], H_k \ge T_k - \alpha\psi^\xi(k)\,\forall k < \overline{m}(t)\Big]\\ &\ge cE_0\Big[e^{\int_0^{H_{\overline{m}(t)}} \zeta(X_s)\,\mathrm{d}s}; H_{\overline{m}(t)} \in [t-K,t], H_k \ge T_k - \alpha\psi^\xi(k)\,\forall k < \overline{m}(t)\Big],\end{aligned}$$

where in the last inequality we used the fact that $\operatorname{ess\,inf}_{\zeta, r\le K, x\in\mathbb{Z}} E_x[e^{\int_0^r \zeta(X_s)\,\mathrm{d}s}; X_r \ge x] \ge c > 0$, due to (POT). On the other hand, by Lemma 4.12, the denominator of (6.6) is bounded from above by $CE_0\big[e^{\int_0^{H_{\overline{m}(t)}} \zeta(X_s)\,\mathrm{d}s}; H_{\overline{m}(t)} \in [t-K,t]\big]$. Finally, using that T_n is the (generalized) inverse of $\overline{m}(t)$ and that $\mathbb{P}$-a.s. we have $T_n \sim n/v_0$, we see from the previous reasoning that in order to show the lemma, it is sufficient to prove that $\mathbb{P}$-a.s., for all n large enough

$$\frac{E_0\Big[\exp\Big\{\int_0^{H_n} \zeta(X_s)\,\mathrm{d}s\Big\}; H_n \in [T_n-K,T_n], H_k \ge T_k - \alpha\psi^\xi(k)\,\forall k < n\Big]}{E_0\Big[\exp\Big\{\int_0^{H_n} \zeta(X_s)\,\mathrm{d}s\Big\}; H_n \in [T_n-K,T_n]\Big]} \ge n^{-\gamma}. \tag{6.7}$$

To prove (6.7), we set $\eta = \overline{\eta}(v_0)$ below and rewrite its left-hand side as

$$\begin{aligned}&\frac{E^{\zeta,\eta}\big[e^{-\eta H_n}; H_n \in [T_n-K,T_n], H_k \ge T_k - \alpha\psi^\xi(k)\,\forall k < n\big]}{E^{\zeta,\eta}\big[e^{-\eta H_n}; H_n \in [T_n-K,T_n]\big]}\\ &\qquad\ge c\cdot\frac{P^{\zeta,\eta}\big(H_n \in [T_n-K,T_n], H_k \ge T_k - \alpha\psi^\xi(k)\,\forall k < n\big)}{P^{\zeta,\eta}\big(H_n \in [T_n-K,T_n]\big)}\\ &\qquad\ge c\cdot P^{\zeta,\eta}\big(H_n \in [T_n-K,T_n], H_k \ge T_k - \alpha\psi^\xi(k)\,\forall k < n\big).\end{aligned}$$

Setting $\widehat{H}_n := H_n - E^{\zeta,\eta}H_n$ and $R_n := T_n - E^{\zeta,\eta}H_n$, the last probability can be written as

$$P^{\zeta,\eta}\Big(\widehat{H}_n \in [R_n-K,R_n], \widehat{H}_k \ge R_k - \alpha\psi^\xi(k)\,\forall k < n\Big). \tag{6.8}$$

The next two claims will show that, after rescaling, the processes R_n and $\widehat{H}_n$ behave like Brownian motions. To approximate R_n, whose increments are not stationary, we introduce an auxiliary process with stationary increments

$$R'_n := \sum_{i=1}^{n} \rho_i, \qquad n \geq 1,$$

where

$$\rho_i := \frac{1}{v_0 L(\eta)} \big(L_i^{\zeta}(\eta) - L(\eta) \big) - \big(E^{\zeta,\eta}[\tau_i] - \mathbb{E}[E^{\zeta,\eta}[\tau_i]] \big), \qquad i \geq 1. \tag{6.9}$$

Lemma 6.3. *The random variables R_n and R'_n are adapted to the filtration $\mathcal{F}_n = \sigma(\xi(i) : i \leq n)$, and R'_n approximates R_n in the sense that $\mathbb{P}$-a.s.,*

$$|R_n - R'_n| \leq \psi^{\xi}(n), \qquad \textit{for all } n \geq 0, \tag{6.10}$$

with ψ^{ξ} as in (6.4). Moreover, the sequence of increments (ρ_n) is bounded, stationary and there exist some constants $c, C \in (0, \infty)$ such that

$$\big| \mathbb{E}[\rho_{n+m} \,|\, \mathcal{F}_n] \big| \leq C e^{-cm}. \tag{6.11}$$

Finally, there is $\sigma_1^2 \in (0, \infty)$ such that both processes, $[0, \infty) \ni t \mapsto n^{-1/2} R_{nt}$ and $[0, \infty) \ni t \mapsto n^{-1/2} R'_{nt}$, converge as $n \to \infty$ in $\mathbb{P}$-distribution to a Brownian motion with variance σ_1^2.

Proof. The adaptedness of (R_n) and (R'_n) to $(\mathcal{F}_n)$, as well as the stationarity and the boundedness of (ρ_n) follow directly from their definitions, recalling the assumption (POT). The estimate (6.10) follows from (5.23) of Theorem 5.4 after a straightforward computation. Furthermore, Lemma A.2 yields

$$\big| \mathbb{E}[L_{n+m}^{\zeta}(\eta) - L(\eta) \,|\, \mathcal{F}_n] \big| \leq C e^{-cm},$$

and analogically, bearing in mind that $(L_i^{\zeta})'(\eta) = E^{\zeta,\eta}[\tau_i]$,

$$\big| \mathbb{E}\big[E^{\zeta,\eta}[\tau_{n+m}] - \mathbb{E}[E^{\zeta,\eta}[\tau_{n+m}]] \,\big|\, \mathcal{F}_n \big] \big| \leq C e^{-cm},$$

proving (6.11).

Finally, observing that the increments of R'_n are centered, the functional central limit theorem for $n^{-1/2} R_{n\cdot}$ follows directly from a functional central limit theorem for stationary mixing sequences, see e.g. Theorem 11 and Corollary 12 of [MPU06], the assumptions of which can be checked easily from (6.11). The functional central limit theorem for $n^{-1/2} R'_{n\cdot}$ then follows from (6.10). □

Claim 6.4. *There is $\sigma_2^2 \in (0, \infty)$ such that $\mathbb{P}$-a.s., under $P^{\zeta,\eta}$, $n^{-1/2} \widehat{H}_{n\cdot}$ converges to a Brownian motion with variance σ_2^2.*

Proof. Since the τ_i's are independent under $P^{\zeta,\eta}$, $\widehat{H}_n$ is a sum of independent and centered random variables, which have uniformly exponential tails. Moreover, the sequence of the variances of the increments is stationary under $\mathbb{P}$. The claim then follows easily by a functional version of the Lindeberg-Feller central limit theorem (see e.g. [GS69, Theorem 9.3.1]). □

Remark 6.5. In view of the last claim and Lemma 6.3, the probability in (6.8) can approximatively be viewed as the probability that one Brownian motion stays above another, quenched, Brownian motion. This problem was recently studied in [MM15a] for the case of two independent Brownian motions, where it was shown that this probability behaves like $n^{-\gamma}$ with γ depending on the variances of the Brownian motions. More importantly, it was proved there that $\gamma > 1/2$ whenever the variance of the quenched Brownian motion is positive. That implies that the price for a particle to be leading should be larger than in the homogeneous case, resulting thus in a larger backlog of $m(t)$ behind $\overline{m}(t)$.

In this paper, the situation is more intricate due to the dependencies of the random variables involved. Hence, we do not strive for the optimal γ. Nevertheless, our proof partially builds on certain ideas appearing in [MM15a].

We proceed by showing that $\mathbb{P}$-a.s. the probability (6.8) can be bounded from below by $n^{-\gamma}$, for n large. In view of (6.10),

$$P^{\zeta,\eta}\Big(\widehat{H}_n \in [R_n - K, R_n], \widehat{H}_k \geq R_k - \alpha\psi^{\xi}(k)\,\forall 1 \leq k < n\Big)$$
$$\geq P^{\zeta,\eta}\Big(\widehat{H}_n \in [R_n - K, R_n], \widehat{H}_k \geq R'_k - (\alpha-1)\psi^{\xi}(k)\,\forall 1 \leq k < n\Big).$$

Defining the process (β_k) to have distribution

$$\beta_k \overset{d}{=} \widehat{H}_k - R'_k, \quad k \geq 1, \tag{6.12}$$

and $\beta_0 = 0$, we can rewrite the latter probability as

$$P^{\zeta,\eta}\big(\beta_k \geq -(\alpha-1)\psi^{\xi}(k)\,\forall 1 \leq k < n, \beta_n \in I_n\big), \tag{6.13}$$

where $I_n = [R_n - R'_n - K, R_n - R'_n]$. Since $\alpha > 2$ we also have that $R_n - R'_n - K \geq -(\alpha-1)\psi^{\xi}(n)$ for n large enough.

To bound (6.13) we construct β as follows. Let β^1, β^2 be two copies of β that are conditionally independent given $\sigma(\xi(i), i \in \mathbb{Z})$, and let Σ_n be a random variable independent of β^1, β^2, uniformly distributed on $\{1, \dots, n-1\}$. Defining β via

$$\beta_k = \begin{cases} \beta^1_k, & \text{for } 1 \leq k \leq \Sigma_n, \\ \beta^1_{\Sigma_n} + (\beta^2_k - \beta^2_{\Sigma_n}), & \text{for } \Sigma_n < k \leq n, \end{cases} \tag{6.14}$$

it is not difficult to see that this has the correct distribution. We also write $\overline{\beta}_k = \beta^2_{n-k} - \beta^2_n$ for 'β^2 running backwards from n'. Observe that

$$\beta_n = \beta^1_{\Sigma_n} - \overline{\beta}_{n-\Sigma_n}. \tag{6.15}$$

The following lemma, the proof of which is postponed to the end of this subsection, provides a control on the process β.

Lemma 6.6. *(a) There is $\gamma' > 0$ such that $\mathbb{P}$-a.s. for all n large enough*

$$P^{\zeta,\eta}\big(\beta^1_k \geq 0\,\forall 1 \leq k \leq n, \beta^1_n \geq n^{1/4}\big) \geq n^{-\gamma'},$$
$$P^{\zeta,\eta}\big(\overline{\beta}_k \geq 0\,\forall 1 \leq k \leq n, \overline{\beta}_n \geq n^{1/4}\big) \geq n^{-\gamma'}.$$

(b) There is $C_2 > 0$ such that $\mathbb{P}$-a.s. for n large enough,

$$P^{\zeta,\eta}\Big(\max_{1\leq k\leq n} |\beta^1_k - \beta^1_{k-1}| \vee |\overline{\beta}_k - \overline{\beta}_{k-1}| \leq C_2 \ln n\Big) \geq 1 - n^{-3\gamma'}.$$

(c) Let $\delta \in (0,1)$. There is $c > 0$ such that $\mathbb{P}$-a.s., for all $x > 0$,

$$P^{\zeta,\eta}\big(\beta_1 \in [x, x+\delta]\big) \geq c\delta e^{-x/c}.$$

We now complete the lower bound of (6.13). Observe that with (6.14) and (6.15) we get

$$\begin{aligned} &\{\beta_k \geq -(\alpha-1)\psi^{\xi}(k)\,\forall 1 \leq k < n, \beta_n \in I_n\} \\ &\qquad \supset \{\beta^1_k - \beta^1_1 \geq 0\,\forall 1 \leq k \leq n,\, \beta^1_n \geq n^{\frac14}\} \cap \{\overline{\beta}_k \geq 0\,\forall 0 \leq k \leq n,\, \overline{\beta}_n \geq n^{\frac14}\} \\ &\qquad\quad \cap \{\beta^1_1 \in I_n - (\beta^1_{\Sigma_n} - \beta^1_1) + \overline{\beta}_{n-\Sigma_n}\}. \end{aligned} \tag{6.16}$$

The first two events on the right-hand side are independent under $P^{\zeta,\eta}$, and their probabilities are larger than $n^{-\gamma'}$ by Lemma 6.6(a). In addition, on the event from Lemma 6.6(b), if the first two events on the right-hand side of (6.16) occur, there is $J \in \{1, \dots, n\}$ such that $I_n - (\beta^1_J - \beta^1_1) + \overline{\beta}_{n-J} \subset [0, 2C_2 \ln n]$. Moreover, $\mathbb{P}(\Sigma_n = J) = 1/(n-1)$. Hence, using Lemma 6.6(c), conditionally on the occurrence of the first two events, we can bound the probability of the third event on the right-hand side from below by $c'n^{-1}e^{-C_2 \ln n/c} \geq n^{-\gamma''}$. Combining these estimates proves that $\mathbb{P}$-a.s. for n large, (6.13), and thus also (6.8), is larger than $n^{-\gamma}$ with $\gamma > 2\gamma' + \gamma''$. This completes the proof of (6.7) and thus of Lemma 6.2. □

We proceed by proving Lemma 6.6 that we used in the last proof.

Proof of Lemma 6.6. To show (b), recall (6.12) and note that the random variables $T_i - T_{i-1}$, $i \geq 1$, are uniformly bounded from above, due to (POT), and that $\tau_i = H_i - H_{i-1}$ have uniform exponential tails under $P^{\zeta,\eta}$. Furthermore, the ρ_i defined in (6.9) are bounded due to Lemma 6.3. Using a union bound and (6.12), this readily establishes (b).

To prove (c), noting that R'_1 is bounded since the (ρ_n) are due to Lemma 6.3, it is sufficient to show that there exists $c \in \mathbb{R}$ such that $\mathbb{P}$-a.s. we have $P^{\zeta,\eta}(\tau_1 \in I) \geq c\delta e^{-x/c}$, where $I = T_1 + x + [0, \delta]$. To see this, recall that under $P^{\zeta,\eta}$, X is a Markov chain whose jump rate from 0 is bounded uniformly in ζ, again by (POT). If the waiting time of X at 0 is in I and then X jumps to the right, then the required event is realized, giving the lower bound (c).

Claim (a) is the most difficult. We will only prove it for β^1, the case of $\overline{\beta}$ being handled analogously. For the sake of notational convenience, we write β instead of β^1 in what follows. Also, we often split the random environment ξ into two parts $\underline{\xi}(j) = (\xi(k))_{k \leq j}$ and $\overline{\xi}(j) = (\xi(k))_{k>j}$. To simplify the notation we write β_k for β^1_k. Set $t_0 = t_{-1} = 0$ and $t_i = 2^i$, for $i \geq 1$. Fix $a \in (0, \infty)$ and define random variables Z_i by

$$\begin{aligned} Z_i &:= \operatorname*{ess\,inf}_{\underline{\xi}(t_{i-2})} \inf_{x \geq a t_{i-1}^{1/2}} P^{\zeta,\eta}\big(\beta_{t_i} \geq a t_i^{1/2}, \beta_k \geq 0 \,\forall k \in \{t_{i-1}, \dots, t_i\} \,\big|\, \beta_{t_{i-1}} = x\big) \\ &= \operatorname*{ess\,inf}_{\underline{\xi}(t_{i-2})} P^{\zeta,\eta}\big(\beta_{t_i} \geq a t_i^{1/2}, \beta_k \geq 0 \,\forall k \in \{t_{i-1}, \dots, t_i\} \,\big|\, \beta_{t_{i-1}} = a t_{i-1}^{1/2}\big), \quad i \geq 1. \end{aligned}$$

Here, $\operatorname{ess\,inf}_{\underline{\xi}(t_{i-2})}$ means taking the essential supremum with respect to $\underline{\xi}(t_{i-2})$ and leaving the remaining ξ random. The second equality then follows from the obvious monotonicity of the considered event in the starting position. Observe that the random variable Z_i is $\sigma(\xi(k), t_{i-2} < k \leq t_i)$ measurable, that is the sequence (Z_i) is 1-dependent.

Setting $i(n) = \lceil \log_2 n \rceil$, using the Markov property, for n large enough,

$$P^{\zeta,\eta}(\beta_k \geq 0 \,\forall k \leq n, \beta_n \geq n^{1/4}) \geq \prod_{i=1}^{i(n)} Z_i = \exp\Big\{ \sum_{i=1}^{i(n)} \ln Z_i \Big\}.$$

If we show that $\mathbb{P}$-a.s.,

$$\limsup_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} (-\ln Z_i) \leq c < \infty, \tag{6.17}$$

then the first half of claim (a) will follow with $\gamma' > c/\ln 2$.

We claim first that $-\ln Z_i < \infty$, $\mathbb{P}$-a.s. Indeed, this follows from (6.12) in combination with the fact that the increments τ_i of $\widehat{H}_k$ can be arbitrarily large, while the increments of R'_k are uniformly bounded due to Lemma 6.3. We now prove that the $(-\ln Z_i)$ have uniformly small exponential moments. Recalling that Z_n is a 1-dependent sequence, (6.17) then follows by standard arguments.

We thus claim that there is a sufficiently small $\theta > 0$ such that for all i large enough

$$\mathbb{E}[\exp\{-\theta \ln Z_i\}] \leq c < \infty, \tag{6.18}$$

Throughout the proof of this inequality, i is considered fixed and we often omit it from the notation. To gain more independence, again, we introduce $\bar{\rho}_k^{(j)}$, $j < k$, by

$$\bar{\rho}_k^{(j)} := \operatorname*{ess\,sup}_{\underline{\xi}(j)} \rho_k.$$

Note that ρ_k is a $\sigma(\xi(n), n \leq k)$-measurable random variable and thus $\bar{\rho}_k^{(j)}$ is $\sigma(\xi(n), j < n \leq k)$-measurable. We further write $\overline{R}_n^{(j)} = \sum_{k=1}^{n} \bar{\rho}_k^{(j)}$ and note that the increments of $\overline{R}^{(j)}$ are larger than the increments of R'.

Let M_R be the essential supremum of the absolute value of the increments ρ_k of R'. Set $L := at_i^{1/2}$, $r_0 := t_{i-1}$, and define

$$s_0 := \inf\Big\{k \geq r_0 : R'_k - R'_{r_j} \geq \frac{L}{8}\Big\} \wedge t_i.$$

Further, recursively for $j \geq 1$, we define

$$\begin{aligned} r_{j+1} &:= s_j + \Big\lceil \frac{L}{8M_R} \Big\rceil, \\ s_{j+1} &:= \inf\Big\{k \geq r_{j+1} : \overline{R}^{(s_j)}_k - \overline{R}^{(s_j)}_{r_{j+1}} \geq \frac{L}{8}\Big\} \wedge \big(r_{j+1} + (t_i - t_{i-1})\big). \end{aligned} \tag{6.19}$$

Heuristically, s_j is the first time when $\overline{R}$ (and thus possibly also R') 'increases considerably after time r_j'; due to the definition of β_k in (6.12), such a behavior of R' is potentially dangerous for the event in Z_i, in that it might lead to Z_i being very small. By definition, s_{j+1} depends only on $\xi(l)$ with $l > s_j$, so the increments $s_j - r_j$ are independent under $\mathbb{P}$ and bounded by $t_i - t_{i-1}$.

For $j \geq 0$ consider the events

$$\begin{aligned} \mathcal{G}_j &= \Big\{\beta_{s_j} \geq 2L, \inf_{r_j \leq l \leq s_j} \widehat{H}_l - \widehat{H}_{r_j} \geq -\frac{L}{8}\Big\}, \\ \mathcal{G}'_j &= \Big\{\inf_{s_j \leq l \leq r_{j+1}} \widehat{H}_l - \widehat{H}_{s_j} \geq -\frac{L}{8}\Big\}, \end{aligned}$$

and define

$$J = \inf\{j : s_j - r_j \geq t_i - t_{i-1}\}.$$

Finally, set $\mathcal{G} = \bigcap_{j=0}^{J} \mathcal{G}_j \cap \bigcap_{j=0}^{J-1} \mathcal{G}'_j$.

We claim that this construction ensures that

$$Z_i \geq P^{\zeta,\eta}\big(\mathcal{G} \,|\, \beta_{t_{i-1}} = at_{i-1}^{1/2}\big). \tag{6.20}$$

To see this, observe that in each of the time intervals $[r_j, s_j]$ and $[s_j, r_{j+1}]$, the process $\overline{R}$ (and thus also R') moves upwards by at most $L/8$ by definition of these intervals. On the other hand, on $\mathcal{G}$ the process $\widehat{H}$ moves downwards by at most $L/8$ in any of these intervals. Since in the probability defining Z_i we condition on $\beta_{r_0} = L/\sqrt{2} > L/2$ and, on $\mathcal{G}$, $\beta_{s_j} \geq 2L$, this ensures that $\beta_k \geq 0$ for $k \in [r_0, s_0]$ and $\beta_k \geq L$ for $k \in [s_0, s_J]$. Moreover, on $\mathcal{G}$, $s_J \geq t_i$, proving (6.20).

Using the independence of the increments of $\widehat{H}$ under the measure $P^{\zeta,\eta}$, the monotonicity of $x \mapsto P^{\zeta,\eta}(\mathcal{G}_j \,|\, \beta_{r_j} = x)$, and the fact that J is $\sigma(\xi(x) : x \in \mathbb{Z})$-measurable, we get

$$P^{\zeta,\eta}\big(\mathcal{G} \,|\, \beta_{t_{i-1}} = at_{i-1}^{1/2}\big) \geq P^{\zeta,\eta}\big(\mathcal{G}_0 \,|\, \beta_{r_0} = L/\sqrt{2}\big) \prod_{j=1}^{J} P^{\zeta,\eta}(\mathcal{G}_j \,|\, \beta_{r_j} = 2L) \prod_{j=0}^{J-1} P^{\zeta,\eta}(\mathcal{G}'_j). \tag{6.21}$$

It is not difficult to show, using the independence and the uniform exponential tail of the increments of $\widehat{H}$ as well as the fact that they are centered, that if i is large enough, $P^{\zeta,\eta}(\mathcal{G}'_j) \geq \frac{1}{2}$ for all j. On the other hand, for $j \geq 1$,

$$P^{\zeta,\eta}(\mathcal{G}_j \,|\, \beta_{r_j} = 2L) \geq P^{\zeta,\eta}\Big(\widehat{H}_{s_j} - \widehat{H}_{r_j} \geq \frac{5L}{2}, \inf_{r_j \leq l \leq s_j} \widehat{H}_l - \widehat{H}_{r_j} \geq -\frac{L}{8}\Big) \geq c \exp\Big\{-\frac{L^2}{c(s_j - r_j)}\Big\},$$

where to obtain the last inequality we used a moderate deviation argument with a sufficiently small constant $c > 0$. By changing the constant c, the same lower bound holds for the first term on the right-hand side of (6.21) as well.

Coming back to (6.18), using (6.20) and (6.21), recalling that the increments of R' are exponentially mixing (cf. (6.11)) and that the intervals $[r_j, s_j]$ are separated by spaces of length $\frac{L}{8M_R}$, we obtain

$$
\begin{aligned}
\mathbb{E}[\exp\{-\theta \ln Z_i\}] &\le C\,\mathbb{E}\Big[\exp\Big\{-\theta J \ln\frac{c}{2} + \theta\sum_{j=0}^{J}\frac{L^2}{c(s_j-r_j)}\Big\}\Big] \\
&\le \sum_{k=1}^{\infty}\Big(\frac{2}{c}\Big)^{\theta k}\mathbb{E}\Big[\exp\Big\{\frac{\theta L^2}{c(s_k-r_k)}\Big\}\mathbf{1}_{s_k-r_k=t_i-t_{i-1}}\prod_{j=0}^{k-1}\exp\Big\{\frac{\theta L^2}{c(s_j-r_j)}\Big\}\mathbf{1}_{s_j-r_j<t_i-t_{i-1}}\Big] \qquad (6.22)\\
&= \sum_{k=1}^{\infty}\Big(\frac{2}{c}\Big)^{\theta k}\exp\Big\{\frac{2\theta L^2}{ct_i}\Big\}\prod_{j=0}^{k-1}\mathbb{E}\Big[\exp\Big\{\frac{\theta L^2}{c(s_j-r_j)}\Big\}\mathbf{1}_{s_j-r_j<t_i/2}\Big],
\end{aligned}
$$

where in the last equality we used the independence of the $s_j - r_j$'s under $\mathbb{P}$. To estimate the last expectation, we write it as

$$
\begin{aligned}
&\int_0^{\infty}\mathbb{P}\Big(\exp\Big\{\frac{\theta L^2}{c(s_j-r_j)}\Big\}\mathbf{1}_{s_j-r_j<t_i/2} > a\Big)\,\mathrm{d}a \\
&\quad\le \exp\Big\{\frac{2\theta L^2}{ct_i}\Big\}\mathbb{P}(s_j-r_j<t_i/2) + \int_{\exp\{2\theta L^2/ct_i\}}^{\infty}\mathbb{P}\Big(\exp\Big\{\frac{\theta L^2}{c(s_j-r_j)}\Big\} > a\Big)\,\mathrm{d}a.
\end{aligned}
\qquad (6.23)
$$

Substituting $a = \exp\{\theta\frac{L^2}{cy}\}$, the second summand can be written as

$$
\int_0^{t_i/2}\mathbb{P}(s_j-r_j<y)\frac{\theta L^2}{cy^2}e^{\theta L^2/cy}\,\mathrm{d}y. \qquad (6.24)
$$

Recalling the definition (6.19) of s_j, for i sufficiently large using (6.11) and definitions of L and $\bar\rho_k^{(j)}$, the probability in the argument of the integral satisfies

$$
\mathbb{P}(s_j-r_j<y) = \mathbb{P}\Big(\max_{0\le m\le y}\sum_{k=1}^{m}\bar\rho_{r_j+k}^{(s_{j-1})} \ge \frac{L}{8}\Big) \le \mathbb{P}\Big(\max_{1\le m\le y}\sum_{k=1}^{m}\rho_{r_j+k} \ge \frac{L}{9}\Big).
$$

Inequality (6.11) can be also used to verify the assumption of Azuma's inequality for mixing sequences of Lemma A.4 for the sequence ρ_k, and thus

$$
\mathbb{P}\Big(\sum_{k=1}^{m}\rho_{r_j+k} \ge \frac{L}{9}\Big) \le Ce^{-cL^2/m},
$$

for some constants C and c and all admissible m. This inequality extends to a maximal inequality, as follows from [KM11, Theorem 1],

$$
\mathbb{P}\Big(\max_{0\le m\le y}\sum_{k=1}^{m}\rho_{r_j+k} \ge \frac{L}{9}\Big) \le Ce^{-cL^2/y}.
$$

Inserting these back into (6.24) implies that the second summand in (6.23) is smaller than

$$
\int_0^{t_i/2}\frac{C\theta L^2}{y^2}e^{\frac{(\theta-c)L^2}{c'y}}\,\mathrm{d}y = \int_0^{\frac{1}{2a^2}}\frac{C\theta}{z^2}e^{\frac{(\theta-c)}{c'z}}\,\mathrm{d}z,
$$

which can be made arbitrarily small by choosing θ small. In addition, the first summand in (6.23) is strictly smaller than 1 by the functional central limit theorem from Lemma 6.3, hence the right-hand side of (6.23) is strictly smaller than one for all $\theta > 0$ sufficiently small. Therefore, for θ small enough the sum in (6.22) converges, which implies (6.18) and completes the proof of Lemma 6.6. □

6.1.2 Second moment for the leading particles

We now estimate the second moment of the number of leading particles needed for the application of (6.5). The proof is relatively short because we do not try to get the optimal power γ below.

Lemma 6.7. *There exists a constant $\gamma < \infty$ such that $\mathbb{P}$-a.s. for all t large enough,*

$$\mathtt{E}_0^\xi[(N_t^{\mathcal{L}})^2] \le t^\gamma.$$

Proof. Recall the definition (6.3) of leading particles and define a random function $\varphi^\xi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\varphi^\xi(s) := k \qquad \text{for all } s \in [T_k - \alpha\psi^\xi(k), T_{k+1} - \alpha\psi^\xi(k+1)), \quad k \ge 0,$$

where ψ^ξ as in (6.4) and $T_0 - \alpha\psi^\xi(0) := -\infty$, by convention. By (3.2) of Proposition 3.1 we have

$$\begin{aligned}\mathtt{E}_0^\xi[(N_t^{\mathcal{L}})^2] = \mathtt{E}_0^\xi[N_t^{\mathcal{L}}] + 2\int_0^t E_0\Big[\exp\Big\{\int_0^s \xi(X_r)\,\mathrm{d}r\Big\}\xi(X_s)\mathbf{1}_{X_r \le \varphi^\xi(r)\forall r\in[0,s]} \\ \times \Big(E_{X_s}\Big[\exp\Big\{\int_0^{t-s}\xi(X_r)\,\mathrm{d}r\Big\}; X_r \le \varphi^\xi(s+r)\,\forall r \in [0,t-s], X_{t-s} \ge \overline{m}(t)\Big]\Big)^2\Big]\,\mathrm{d}s.\end{aligned} \tag{6.25}$$

The first summand on the right-hand side satisfies, by Lemma 5.1(b) and the definition of $\overline{m}(t)$,

$$\mathtt{E}_0^\xi[N_t^{\mathcal{L}}] \le \mathtt{E}_0^\xi[N^{\ge}(t,\overline{m}(t))] \le C\,\mathtt{E}_0^\xi[N^{\ge}(t,\overline{m}(t)+1)] \le C/2.$$

Since $\xi(X_s) \le \mathtt{es}$, the second summand on the right-hand side of (6.25) is bounded from above by

$$\begin{aligned}&2\mathtt{es}\int_0^t \sum_{k=-\infty}^{\varphi^\xi(s)} E_0\Big[\exp\Big\{\int_0^s \xi(X_r)\,\mathrm{d}r\Big\}\mathbf{1}_{X_s=k}\Big(E_k\Big[\exp\Big\{\int_0^{t-s}\xi(X_r)\,\mathrm{d}r\Big\}; X_{t-s} \ge \overline{m}(t)\Big]\Big)^2\Big]\,\mathrm{d}s \\ &= 2\mathtt{es}\int_0^t \sum_{k=-\infty}^{\varphi^\xi(s)} \mathtt{E}_0^\xi[N(s,k)]\mathtt{E}_k^\xi[N^{\ge}(t-s,\overline{m}(t))]^2\,\mathrm{d}s.\end{aligned} \tag{6.26}$$

By the first moment formula (3.1) of Proposition 3.1, the Markov property, Lemma 5.1(b), and the definition of $\overline{m}(t)$,

$$\begin{aligned}\mathtt{E}_0^\xi[N(s,k)]\mathtt{E}_k^\xi[N^{\ge}(t-s,\overline{m}(t))] &= \mathtt{E}_0^\xi\big[|\{Y \in N(t), Y_t \ge \overline{m}(t), Y_s = k\}|\big] \\ &\le \mathtt{E}_0^\xi\big[N^{\ge}(t,\overline{m}(t))\big] \le C\,\mathtt{E}_0^\xi\big[N^{\ge}(t,\overline{m}(t)+1)\big] \le C,\end{aligned}$$

and thus $\mathtt{E}_k^\xi[N^{\ge}(t-s,\overline{m}(t))] \le C/\mathtt{E}_0^\xi[N(s,k)]$. To exploit this inequality we treat separately four ranges of parameters $s \in [0,t]$ and $k \le \varphi^\xi(s)$.

(1.) By Lemma 4.13, $\mathtt{E}_0^\xi[N(s,k)] \ge c\mathtt{E}_0^\xi[N^{\ge}(s,k)]$ whenever $k \ge \mathcal{N}_1$ (cf. (4.21)) and $k/s \in V$. Therefore, since $k \le \varphi^\xi(s)$,

$$\mathtt{E}_0^\xi[N(s,k)] \ge c\,\mathtt{E}_0^\xi[N^{\ge}(s,k)] \ge c\,\mathtt{E}_0^\xi[N^{\ge}(s,\varphi^\xi(s))] \ge c\mathtt{E}_0^\xi[N^{\ge}(T_l - \alpha\psi^\xi(l), l)]$$

for $l = l(s)$ such that $s \in [T_l - \alpha\psi^\xi(l), T_{l+1} - \alpha\psi^\xi(l+1))$. Moreover, since $T_l/l \to 1/v_0$, for some $c > 0$ and $s \ge \mathcal{S}(\xi)$ we have $l(s) \ge cs$. Therefore, for $s \ge \mathcal{S}(\xi)$, $k \ge \mathcal{N}_1(\xi)$, and $k/s \in V$, by Lemma 5.1(b) with $h = -\alpha\psi^\xi(l)$ and in combination with $\mathtt{E}_0^\xi[N^{\ge}(T_l, l)] = O(1)$,

$$\mathtt{E}_0^\xi[N(s,k)] \ge \mathcal{C}'(\xi)t^{-\gamma}$$

for some $\gamma \in (0,\infty)$. We thus have for $\mathcal{N}_1 \le k \le \varphi^\xi(s)$, $s \ge \mathcal{S}(\xi)$ and $k/s \in V$

$$\mathtt{E}_0^\xi[N(s,k)]\mathtt{E}_k^\xi[N^{\ge}(t-s,\overline{m}(t))]^2 \le c\,\mathtt{E}_0^\xi[N(s,k)]^{-1} \le \mathcal{C}''(\xi)t^\gamma. \tag{6.27}$$

(2.) Let $\bar{v} > 0$ be the asymptotic speed of the maximal particle in the homogeneous branching random walk with branching rate $\mathtt{ei}$ (cf. (POT)) and assume that V is fixed so that it contains $\bar{v}/2$ in its interior. Since $\xi(x) \geq \mathtt{ei}$, by a straightforward comparison argument and properties of the homogeneous branching random walk, we have $\mathtt{E}_0^\xi[N(s,k)] \geq C$ for all $s \in [0,t]$ and $|k/s| \leq \bar{v}/2$. Therefore, on this domain,

$$\mathtt{E}_0^\xi[N(s,k)]\mathtt{E}_k^\xi[N^{\geq}(t-s,\overline{m}(t))]^2 \leq c\,\mathtt{E}_0^\xi[N(s,k)]^{-1} \leq C. \tag{6.28}$$

(3.) For $k \leq 0$ and $s \in [0,t]$, by the Feynman-Kac formula

$$\begin{aligned}2\mathtt{E}_k^\xi[N^{\geq}(t,\overline{m}(t))] &\geq 2E_k\Big[\exp\Big\{\int_0^{t-s}\xi(X_r)\,\mathrm{d}r\Big\}\mathbf{1}_{X_{t-s}\geq\overline{m}(t)}\mathbf{1}_{X_t\geq\overline{m}(t)}\Big]\\ &\geq 2\mathtt{E}_k^\xi[N^{\geq}(t-s,\overline{m}(t))]P_0(X_s\geq 0)\\ &\geq \mathtt{E}_k^\xi[N^{\geq}(t-s,\overline{m}(t))].\end{aligned}$$

Therefore,

$$\begin{aligned}\mathtt{E}_0^\xi[N(s,k)]\mathtt{E}_k^\xi[N^{\geq}(t-s,\overline{m}(t))]^2 &\leq 2\mathtt{E}_0^\xi[N(s,k)]\mathtt{E}_k^\xi[N^{\geq}(t-s,\overline{m}(t))]\mathtt{E}_{\mathbf{1}_{-\mathbb{N}_0}}^\xi[N^{\geq}(t,\overline{m}(t))]\\ &\leq C\mathtt{E}_0^\xi[N(s,k)]\mathtt{E}_k^\xi[N^{\geq}(t-s,\overline{m}(t))],\end{aligned}$$

where we took advantage of Lemma 4.13 to infer the last inequality. In particular, using the Markov property we infer that for every $s \in [0,t]$,

$$\sum_{k=-\infty}^{0}\mathtt{E}_0^\xi[N(s,k)]\mathtt{E}_k^\xi[N^{\geq}(t-s,\overline{m}(t))]^2 \leq C\mathtt{E}_0^\xi[N^{\geq}(t,\overline{m}(t))] \leq C. \tag{6.29}$$

(4.) The remaining domain in (6.26) not controlled by (1.)–(3.) is a subset of

$$\mathcal{B}^\xi = \{(s,k)\in\mathbb{R}_+\times\mathbb{N} : \bar{v}s/2 \leq k \leq \varphi^\xi(s), s+k \leq \tilde{\mathcal{C}}(\xi)\}$$

for some $\tilde{\mathcal{C}}(\xi)$ depending on $\mathcal{N}_1$ and $\mathcal{S}$. A slight adaptation of the previous arguments yields

$$\begin{aligned}C &\geq \mathtt{E}_0^\xi[N^{\geq}(t+1,\overline{m}(t+1))]\\ &\geq \mathtt{E}_0^\xi[N(1+s,k)]\mathtt{E}_k^\xi\big[N^{\geq}\big(t-s,\overline{m}(t)+\big(\overline{m}(t+1)-\overline{m}(t)\big)\big)\big]\\ &\geq C\mathtt{E}_0^\xi[N(1+s,k)]\mathtt{E}_k^\xi\big[N^{\geq}\big(t-s,\overline{m}(t)\big)\big],\end{aligned}$$

where in the last inequality we used Lemma 5.1(b) and the fact that $\overline{m}(t+1)-\overline{m}(t)$ is bounded (using perturbation Lemmas 5.1(b) and 5.3(b) again). It is easy to see that there is $c(\xi)\in(0,\infty)$ such that for all $(s,k)\in\mathcal{B}^\xi$ we have $\mathtt{E}_0^\xi[N(s+1,k)] \geq c^{-1}(\xi) > 0$ and $\mathtt{E}_0^\xi[N(s,k)] \leq c(\xi)$. Hence, for all $(s,k)\in\mathcal{B}^\xi$

$$\mathtt{E}_0^\xi[N(s,k)]\mathtt{E}_k^\xi[N^{\geq}(t-s,\overline{m}(t))]^2 \leq Cc^3(\xi). \tag{6.30}$$

Using inequalities (6.27)–(6.30) in their respective domains in the summation and integration in (6.26) (recalling that V contains $\bar{v}/2$ in its interior) it is easy to find the required upper bound for the second summand on the right-hand side of (6.25) and to complete the proof of the lemma. □

Combining Lemmas 6.2 and 6.7 with (6.5) completes the proof of Proposition 6.1.

6.2 Proof of Theorem 2.1

By inserting the estimates from Lemmas 6.2 and 6.7 into the Paley-Zygmund inequality (6.5) we obtain $\mathtt{P}_0^\xi(N_t^{\mathcal{L}} \geq 1) \geq t^{-\gamma}$ for all large t, $\mathbb{P}$-a.s. To complete the proof of the lower bound in Theorem 2.1 we need to amplify this estimate, using a technique adapted from the homogeneous branching random walk literature (see e.g. [McD95]). The first step is the following lemma guaranteeing that with very high probability the number of particles in the origin grows exponentially in time.

Lemma 6.8. *There exists $C_3 > 1$ and $t_0 < \infty$ such that such that for all $t \geq t_0$, and $\mathbb{P}$-a.e. ξ,*

$$\mathsf{P}_0^\xi(N(t,0) \leq C_3^t) \leq C_3^{-t}.$$

Proof. Recall from (POT) that the essential infimum ei of the ξ is strictly positive. By the monotonicity of $N(t,0)$ in ξ which can be ensured by a straightforward coupling, it suffices to show the claim for the homogeneous branching random walk with branching rate ei. We write $\mathsf{P}_0^{\mathsf{ei}}$ for the law of this process starting in 0.

For $t \geq 0$ and $\varepsilon > 0$, let $D_\varepsilon(t/3)$ be the set of *direct* offsprings of the initial particle until time $t/3$ which are at sites $[-\varepsilon t, \varepsilon t]$ at time $t/3$. Then, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathsf{P}_0^{\mathsf{ei}}\big(|D_\varepsilon(t/3)| \leq \delta t/3\big) \leq e^{-\delta t/3}. \tag{6.31}$$

Indeed, the probability that the initial particle leaves $[-\varepsilon t/2, \varepsilon t/2]$ before $t/3$ is smaller than $e^{-c(\varepsilon)t}$. If it stays in this interval, it produces more than $t\,\mathsf{ei}/4$ direct offsprings with probability larger than $1 - e^{-ct}$, by large deviations for the Poisson distribution with parameter $t\,\mathsf{ei}/3$, and every of these offsprings stays in $[-\varepsilon t, \varepsilon t]$ with probability at least $1 - e^{-c(\varepsilon)t}$ again.

For a particle $Y \in D_\varepsilon(t/3) \subset N(t/3)$, we denote by $A_Y(2t/3)$ the set of *all* offsprings it produced between times $t/3$ and $2t/3$ and which are at the site $Y_{t/3}$ at time $2t/3$. We claim that there exists $c > 1$ such that

$$\mathsf{P}_0^{\mathsf{ei}}\Big(\big|A_Y(2t/3)\big| \geq c^t\Big) > 0 \tag{6.32}$$

Indeed, under P^{ei}, it is well-known (e.g., it follows from the Feynman-Kac formula) that the expected number of particles in 0 grows exponentially. Hence, we can fix $r > 0$ such that $\mathsf{E}^{\mathsf{ei}}[N(r,0)] =: \mu > 1$, and consider an auxiliary process evolving as follows

- start with one particle at an arbitrary site $x \in \mathbb{Z}$ at time 0,
- particles evolve independently as a continuous time simple random walk and split into two at rate ei,
- and at each time rn, $n \in \mathbb{N}$, all the particles not at x are killed.

Let Z_n be the number of particles at x at time rn in this auxiliary process. It is easy to see that Z_n is a supercritical Galton-Watson process and thus it survives with a positive probability, $\mathsf{P}_0^{\mathsf{ei}}(Z_n > 0\,\forall n \geq 0) \geq p > 0$, and on the event of survival it grows exponentially, $\mathsf{P}_0^{\mathsf{ei}}(Z_n \geq c^n \,|\, Z_n > 0\,\forall n \geq 0) \geq 1/2$ for some $c > 1$. Hence, for every $Y \in D_\varepsilon(t/3)$, $\mathsf{P}_0^{\mathsf{ei}}(|A_Y(2t/3)| \geq c^t) \geq p/2$ at all times such that $2t/3 = rn$ for some $n \in \mathbb{N}$. A straightforward extension to all times then yields (6.32).

Combining (6.31) and (6.32) implies that

$$\mathsf{P}_0^{\mathsf{ei}}\big(N(2t/3, [-\varepsilon t, \varepsilon t]) \geq c^t\big) \geq 1 - e^{-c' t}.$$

Moreover, the constant c does not depend on ε. As a consequence, choosing $\varepsilon > 0$ small enough such that

$$P_{\varepsilon t}(X_{t/3} = 0) \geq \Big(\frac{1+c}{2}\Big)^{-t/3},$$

and using an easy large deviation argument we obtain that

$$\mathsf{P}_0^{\mathsf{ei}}\Big(N(t,0) < \frac{c^t}{2}\Big(\frac{1+c}{2}\Big)^{-t/3} \;\Big|\; N(2t/3, [-\varepsilon t, \varepsilon t]) \geq c^t\Big) \leq e^{-c'' t}.$$

This completes the proof of the lemma. □

We now obtain a lower bound on $M(t)$.

Proposition 6.9. *For any $q \in \mathbb{N}$ there exists a constant $C^{(q)} < \infty$ such that for $\mathbb{P}$-a.a. ξ, for all t large enough*

$$\mathsf{P}_0^\xi(M(t) \geq \overline{m}(t) - C^{(q)} \ln t) \geq 1 - 2t^{-q}.$$

Proof. Without loss of generality we assume that $q > \gamma$ for γ as in Proposition 6.1. We fix $r = c_1 \ln t$ where c_1 is chosen so that for C_3 of Lemma 6.8 we have $C_3^{-r} = t^{-q}$, and further choose $C^{(q)}$ large enough so that $\overline{m}(t-r) \geq \overline{m}(t) - C^{(q)} \ln t$. To see that this is possible, observe that by the perturbation Lemmas 5.1 and 5.3 we have for some $c, c' \in (0, \infty)$

$$\begin{aligned}
\mathtt{E}_0^\xi[N^{\geq}(t-r, \overline{m}(t) - C^{(q)} \ln t)] &\geq e^{-cr} \mathtt{E}_0^\xi[N^{\geq}(t, \overline{m}(t) - C^{(q)} \ln t)] \\
&\geq e^{-cr} e^{c' C^{(q)} \ln t} \mathtt{E}_0^\xi[N^{\geq}(t, \overline{m}(t))] \\
&\geq \frac{1}{2} e^{-cr} e^{c' C^{(q)} \ln t},
\end{aligned}$$

and fix $C^{(q)}$ so that the right-hand side is smaller than $1/2$.

Set $x := \overline{m}(t) - C^{(q)} \ln t$ and observe that by considering separately the events $\{N(r,0) < C_3^r\}$, $\{N(r,0) \geq C_3^r\}$ and using the Markov property and the independence of the particles in the second case

$$\begin{aligned}
\mathtt{P}_0^\xi(M(t) \geq x) &= \mathtt{P}_0^\xi(N^{\geq}(t,x) \geq 1) \\
&\geq 1 - \mathtt{P}_0^\xi(N(r,0) \leq C_3^r) - \big(\mathtt{P}_0^\xi(N^{\geq}(t-r,x) < 1)\big)^{C_3^r}. \\
&\geq 1 - t^{-q} - \big(\mathtt{P}_0^\xi(N_{t-r}^{\mathcal{L}} < 1)\big)^{C_3^r}.
\end{aligned}$$

Here, for the last inequality we used Lemma 6.8 as well as $x \leq \overline{m}(t-r)$ and so $N^{\geq}(t-r,x) \geq N_{t-r}^{\mathcal{L}}$. Proposition 6.1 then implies

$$\big(\mathtt{P}_0^\xi(N_{t-r}^{\mathcal{L}} < 1)\big)^{C_3^r} \leq (1 - t^{-\gamma})^{t^q} \leq t^{-q}$$

for t large enough. This completes the proof. □

Proof of Theorem 2.1 and Proposition 2.6. By Proposition 6.9 and the Borel-Cantelli lemma (controlling non-integer t by standard estimates), we have $M(t) \geq \overline{m}(t) - C^{(2)} \ln t$, $\mathbb{P} \times \mathtt{P}_0^\xi$-a.s. for all t large enough, and thus $\mathbb{P}$-a.s., $m(t) \geq \overline{m}(t) - C^{(2)} \ln t$, for such t as well. By the monotonicity and the independence of the particles, these lower bounds hold for an arbitrary initial condition satisfying (INI). These facts combined with (6.2) and $m(t) \leq \overline{m}(t)$ (cf. (6.1)), complete the proof of Theorem 2.1 and Proposition 2.6. □

7 BRWRE and the randomized Fisher-KPP

In this section we prove the central limit theorem for the front of the randomized Fisher-KPP equation. We begin by establishing the connection between the BRWRE and this Fisher-KPP equation. Its proof is a straightforward adaptation of [McK75], who proved the corresponding result in the case of the homogeneous BBM (see also [INW68a, INW68b, INW69]).

Proposition 7.1. *If $(\xi(x))_{x \in \mathbb{Z}}$ is a bounded potential and $f : \mathbb{Z} \to [0,1]$, then*

$$w(t,x) := 1 - \mathtt{E}_x^\xi\Big[\prod_{Y \in N(t)} f(Y_t)\Big]$$

solves

$$\frac{\partial w}{\partial t} = \Delta_d w + \xi(x) w(1-w)$$

with initial condition $w(0,\cdot) = 1 - f$. In particular, $w(t,x) = \mathtt{P}_x^\xi(M(t) \geq 0)$ solves this equation with $f = \mathbf{1}_{-\mathbb{N}}$, i.e. $w(0,\cdot) = \mathbf{1}_{\mathbb{N}_0}$.

Proof. Actually, we show that $v := 1 - w$ solves

$$\frac{\partial v}{\partial t} = \Delta_d v - \xi(x) v(1-v)$$

with initial condition $v(0,\cdot) = f$, which will establish the claim.

According to whether or not the original particle has split into two before time t, the Feynman-Kac formula in combination with the Markov property at time s supplies us with

$$v(t,x) = E_x\Big[e^{-\int_0^t \xi(X_r)\,\mathrm{d}r} f(X_t)\Big] + \int_0^t E_x\Big[\xi(X_s)e^{-\int_0^s \xi(X_r)\,\mathrm{d}r} v^2(t-s, X_s)\Big]\,\mathrm{d}s$$

Using the reversibility of the random walk, and substituting s by $t-s$, this can be written as

$$v(t,x) = \sum_{y\in\mathbb{Z}} f(y) E_y\Big[e^{-\int_0^t \xi(X_r)\,\mathrm{d}r}\mathbf{1}_x(X_t)\Big] + \int_0^t \sum_{y\in\mathbb{Z}} v^2(s,y) E_y\Big[\xi(y) e^{-\int_0^{t-s}\xi(X_r)\,\mathrm{d}r}\mathbf{1}_x(X_{t-s})\Big]\,\mathrm{d}s. \tag{7.1}$$

Differentiation then yields

$$\begin{aligned}
\frac{\partial v}{\partial t}(t,x) &= \sum_{y\in\mathbb{Z}} f(y) E_y\Big[-\xi(x) e^{-\int_0^t \xi(X_r)\,\mathrm{d}r}\mathbf{1}_x(X_t)\Big] \\
&\quad + \sum_{y\in\mathbb{Z}} f(y) E_y\Big[e^{-\int_0^t \xi(X_r)\,\mathrm{d}r}(\Delta_d \mathbf{1}_x)(X_t)\Big] \\
&\quad + \xi(x) v^2(t,x) \\
&\quad + \int_0^t \sum_{y\in\mathbb{Z}} v^2(s,y) E_y\Big[-\xi(y)\xi(x) e^{-\int_0^{t-s}\xi(X_r)\,\mathrm{d}r}\mathbf{1}_x(X_{t-s})\Big]\,\mathrm{d}s \\
&\quad + \int_0^t \sum_{y\in\mathbb{Z}} v^2(s,y) E_y\Big[\xi(y) e^{-\int_0^{t-s}\xi(X_r)\,\mathrm{d}r}(\Delta_d\mathbf{1}_x)(X_{t-s})\Big]\,\mathrm{d}s.
\end{aligned}$$

Comparing this expression with the representation (7.1), the second and fifth summands together yield $\Delta_d v$, the third is $\xi(x)v^2$, and the first and fourth together supply us with $-\xi(x)v$, which finishes the proof of the first claim. The second claim is a straightforward consequence of the first one. □

Proof of Theorem 2.9. Observe first that the initial conditions in Theorem 2.9 and the second claim of Proposition 7.1 are related by the reflection $x \mapsto -x$. Hence, setting $\tilde{\xi}(x) = \xi(-x)$, it is easy to see from the last proposition that the front $\widehat{m}(t)$ of the Fisher-KPP equation defined in (2.15) can be represented as

$$\widehat{m}(t) = \sup\Big\{x\in\mathbb{Z} : \mathtt{P}^{\tilde{\xi}}_{-x}(M(t)\geq 0) \geq \frac{1}{2}\Big\}.$$

Comparing this to the definition (2.4) of $m(t)$, we see that the role of x and the origin is reversed, and the environment is reflected. This complication is easy to resolve. By the translation and reflection invariance of the environment ξ, for every $x\in\mathbb{Z}$,

$$\mathbb{P}\Big(\mathtt{P}^{\tilde{\xi}}_{-x}(M(t)\geq 0)\geq \frac{1}{2}\Big) = \mathbb{P}\Big(\mathtt{P}^{\xi}_{0}(M(t)\geq x)\geq \frac{1}{2}\Big).$$

The central limit theorem for $\widehat{m}(t)$ then follows directly from the central limit theorem for $m(t)$. □

8 Open questions

We collect here some open questions which naturally arise from the investigations of this article.

1. Can we say that $m(t)$ lags at least $\Omega(\ln t)$ behind $\overline{m}(t)$?
2. For $x\in\mathbb{Z}$ fixed, is the function $[0,\infty)\ni t\mapsto u(t,x)$ increasing? It is not hard to see that generally this is not the case on $[0,\infty)$; however, is it true for t large enough?
3. Is the family $M(t)-m(t)$, $t\geq 0$, tight? In the case of homogeneous BBM, it already follows from the convergence to a traveling wave solution (see [KPP37]) that this is the case. In the case of spatially random branching rates this remains an open question.

We expect our results to transfer to the continuum setting where the space $\mathbb{Z}$ is replaced by $\mathbb{R}$ under suitable regularity and mixing assumptions on ξ.

A Appendix

We prove here several auxiliary results that are used through the text. Most of them use rather standard techniques, but we did not find any suitable reference for them.

A.1 Properties of logarithmic moment generating functions

Lemma A.1. *The functions L, L_i^ζ, and $\overline{L}_n^\zeta$ defined in (4.5)–(4.7) are infinitely differentiable on $(-\infty,0)$ and satisfy for $\eta \in (-\infty,0)$,*

$$L'(\eta) = \mathbb{E}\Big[\frac{E^\zeta[H_1 \exp\{\eta H_1\}]}{E^\zeta[\exp\{\eta H_1\}]}\Big] = \mathbb{E}\big[E^{\zeta,\eta}[H_1]\big], \tag{A.1}$$

$$(L_i^\zeta)'(\eta) = \frac{E^\zeta[\tau_i \exp\{\eta\tau_i\}]}{E^\zeta[\exp\{\eta\tau_i\}]} = E^{\zeta,\eta}[\tau_i], \tag{A.2}$$

and thus also $(\overline{L}_n^\zeta)'(\eta) = \frac{1}{n}E^{\zeta,\eta}[H_n]$. Further

$$\begin{aligned} L''(\eta) &= \mathbb{E}\Big[\frac{E^\zeta[H_1^2 \exp\{\eta H_1\}]E^\zeta[\exp\{\eta H_1\}] - E^\zeta[H_1\exp\{\eta H_1\}]^2}{E^\zeta[\exp\{\eta H_1\}]^2}\Big] \\ &= \mathbb{E}\big[E^{\zeta,\eta}[H_1^2] - E^{\zeta,\eta}[H_1]^2\big] > 0, \end{aligned} \tag{A.3}$$

$$\begin{aligned} (L_i^\zeta)''(\eta) &= \frac{E^\zeta[\tau_i^2 \exp\{\eta\tau_i\}]E^\zeta[\exp\{\eta\tau_i\}] - E^\zeta[\tau_i\exp\{\eta\tau_i\}]^2}{E^\zeta[\exp\{\eta\tau_i\}]^2} \\ &= \big(E^{\zeta,\eta}[\tau_i^2] - E^{\zeta,\eta}[\tau_i]^2\big) > 0, \end{aligned} \tag{A.4}$$

and thus also $(\overline{L}_n^\zeta)''(\eta) = \frac{1}{n}\big(E^{\zeta,\eta}[H_n^2] - E^{\zeta,\eta}[H_n]^2\big)$

Moreover, $L_i^\zeta(\eta)$ are uniformly bounded random variables, that is for every $\Delta \subset (-\infty,0)$ compact, there is $c = c(\Delta) \in (0,\infty)$ such that

$$\sup_{\eta\in\Delta} \operatorname{ess\,sup} |L_i^\zeta(\eta)| \le c,$$

and analogous statements hold for $(L_i^\zeta)'$ and $(L_i^\zeta)''$ as well.

Proof. The fact that L and $\overline{L}_n^\zeta$ are infinitely differentiable follows easily from the dominated convergence theorem which allows to interchange the differentiation with the expectations. The first equalities in (A.1)–(A.4) can be then obtained by a direct computation from the definitions of the corresponding functions. The second equalities follow from the definition (4.4) of $P^{\zeta,\eta}$. The strict inequalities in (A.3) and (A.4) follow from the fact that, as H_1, τ_i are non-degenerate random variables, Jensen's inequality provides us with a strict inequality.

To prove the last claim, it is sufficient to observe that $\zeta(x) \mapsto L_i^\zeta$, $\zeta(x) \mapsto (L_i^\zeta)'$ and $\zeta(x) \mapsto (L_i^\zeta)''$ are increasing in $\zeta(x)$ and $\zeta(x) \in [\mathsf{ei} - \mathsf{es}, 0]$ $\mathbb{P}$-a.s., therefore e.g.

$$-\infty < \ln E_{i-1}[e^{H_i(\mathsf{ei}-\mathsf{es}-\min\Delta)}] \le \inf_{\eta\in\Delta} \operatorname{ess\,inf} L_i^\zeta(\eta) \le \sup_{\eta\in\Delta} \operatorname{ess\,sup} L_i^\zeta(\eta) \le \ln E_{i-1}[e^{H_i \max\Delta}] < \infty,$$

where the finiteness of the expectations on the left- and right-hand side follows from standard random walk properties. The proofs for $(L_i^\zeta)'$ and $(L_i^\zeta)''$ are analogous. □

Lemma A.2. *Let $\mathcal{F}_k = \sigma(\xi(i) : i \le k)$ and Δ be a compact interval in $(-\infty,0)$. Then there exists a constant $C_\Delta < \infty$ such that for all $0 \le i < j$ and $\eta \in \Delta$, $\mathbb{P}$-a.s.,*

$$\big|\mathbb{E}[L_j^\zeta(\eta) \,|\, \mathcal{F}_i] - L(\eta)\big| \le C_\Delta e^{-(j-i)/C_\Delta},$$

and similarly

$$\big|\mathbb{E}[(L_j^\zeta)'(\eta) \,|\, \mathcal{F}_i] - L'(\eta)\big| \le C_\Delta e^{-(j-i)/C_\Delta}.$$

Proof. We only prove the first inequality, the second one being derived in a similar manner.

By translation invariance we may assume without loss of generality $0 = i < j$. Write $L_j^\zeta(\eta) = \ln(A + B)$ where

$$A = E_{j-1}\Big[\exp\Big\{\int_0^{H_j} (\zeta(X_s) + \eta)\,\mathrm{d}s\Big\}, \inf_{0\le s\le H_j} X_s > 0\Big]$$

and

$$B = E_{j-1}\Big[\exp\Big\{\int_0^{H_j} (\zeta(X_s) + \eta)\,\mathrm{d}s\Big\}, \inf_{0\le s\le H_j} X_s \le 0\Big].$$

Let K_t denote the number of jumps that the random walk (X_n) has made up to time $t > 0$, which has Poisson distribution with parameter t. Then, since $\operatorname{ess\,sup}\zeta = 0$, for $\delta > 0$ sufficiently small, uniformly over $\eta \in \Delta$,

$$B \le E_{j-1}[e^{\eta H_j}; H_j \ge \delta j] + P_{j-1}[K_{\delta j} \ge j] \le ce^{-j/c},$$

where the last inequality follows from standard large deviations for the Poisson random variable. On the other hand, due to (4.3), there is $c' \in (0, 1)$ such that $A \in (c', 1)$, $\mathbb{P}$-a.s. Therefore, since $\ln(1 + x) \le x$, we have that $\mathbb{P}$-a.s.,

$$\ln A \le L_j^\zeta(\eta) \le \ln A + \ln(1 + \tfrac{B}{A}) \le \ln A + ce^{-j/c}.$$

Using this, since $\ln(A)$ is independent of $\mathcal{F}_0$ by definition, we infer that

$$\big|\mathbb{E}[L_j^\zeta(\eta) - L(\eta) \,|\, \mathcal{F}_0]\big| \le \big|\mathbb{E}[\ln(A) - \mathbb{E}[\ln(A)] \,|\, \mathcal{F}_0]\big| + 2ce^{-j/c} = 2ce^{-j/c},$$

which finishes the proof of the lemma. □

A.2 Basic properties of the Lyapunov exponent

We prove here various properties of the Lyapunov exponent λ defined in (2.2) that are used throughout the paper.

Lemma A.3. *The function $\lambda : \mathbb{R} \to \mathbb{R}$ is well-defined, non-random, even, and concave. It satisfies $\lambda(0) = \mathsf{es}$, $\lambda(v) < \mathsf{es}$ for every $v \ne 0$, and $\lim_{v\to\infty} \lambda(v)/v = -\infty$. In particular, there exists a unique $v_0 \in (0, \infty)$ such that $\lambda(v_0) = 0$. Moreover, for $v_c = \frac{1}{L'(0)}$, with $L'(0)$ denoting the derivative from the left, we have that λ is linear on $[0, v_c]$, and strictly concave on (v_c, ∞). On the latter interval it is given by the formula*

$$\lambda(v) = \mathsf{es} - vL^*(1/v). \tag{A.5}$$

Proof. For $\alpha \in (0, 1)$ and $v_1, v_2 \in \mathbb{R}$ the Markov property yields

$$\begin{aligned} &\ln E_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}; X_t = \lfloor(\alpha v_1 + (1-\alpha)v_2)t\rfloor\Big] \\ &\quad\ge \ln E_0\Big[\exp\Big\{\int_0^{(1-\alpha)t} \xi(X_s)\,\mathrm{d}s\Big\}; X_{(1-\alpha)t} = \lfloor(1-\alpha)v_2 t\rfloor\Big] \\ &\quad+ \ln E_{\lfloor(1-\alpha)v_2 t\rfloor}\Big[\exp\Big\{\int_0^{\alpha t} \xi(X_s)\,\mathrm{d}s\Big\}; X_{\alpha t} = \lfloor(\alpha v_1 + (1-\alpha)v_2)t\rfloor\Big]. \end{aligned} \tag{A.6}$$

Hence, choosing $v_1 := v_2 := v$ and using Kingman's subadditive ergodic theorem [Lig85] as well as the Feynman-Kac formula (Proposition 3.1), we obtain that for each $v \in \mathbb{R}$, the limit $\lambda(v)$ exists and is non-random. In addition, λ is an even function since X is symmetric simple random walk and the $(\xi(x))_{x\in\mathbb{Z}}$ are i.i.d. by assumption.

Dividing both sides of inequality (A.6) by t and taking the limit $t \to \infty$, the left-hand side converges $\mathbb{P}$-a.s. to $\lambda(\alpha v_1 + (1-\alpha)v_2)$, and the first summand on the right-hand side converges to $(1-\alpha)\lambda(v_2)$. Further, the second summand converges to $\alpha\lambda(v_1)$ in distribution, since it has the same distribution (up to possibly a small error introduced by the use of the floor function, and which is irrelevant in the limit) as

$$\frac{1}{t}\ln E_0\Big[\exp\Big\{\int_0^{\alpha t}\xi(X_s)\,\mathrm{d}s\Big\}\mathbf{1}_{X_{\alpha t}=\lfloor \alpha v_1 t\rfloor}\Big],$$

which converges $\mathbb{P}$-a.s. to $\alpha\lambda(v_1)$. The concavity of λ then follows.

The proof of $\lambda(0) = \mathsf{es}$ is standard but we include it for the sake of completeness. By the Feynman-Kac formula and (4.2),

$$\lambda(0) = \mathsf{es} + \lim_{t\to\infty}\frac{1}{t}\ln E_0\Big[\exp\Big\{\int_0^t \zeta(X_s)\,\mathrm{d}s\Big\}; X_t = 0\Big]. \tag{A.7}$$

Since $\zeta(x) \le 0$, the upper bound $\lambda(0) \le \mathsf{es}$ follows trivially. To show the lower bound, fix $\varepsilon > 0$ arbitrarily and note that by standard i.i.d. properties of ζ's, there is $c(\varepsilon) > 0$ such that $\mathbb{P}$-a.s. for t large enough, there is an interval $I_t \subset [-t^{1/4}, t^{1/4}]$ of length at least $c(\varepsilon)\ln t$ such $\zeta(j) \ge -\varepsilon$ for all $j \in I_t \cap \mathbb{Z}$. Consider now the event $\mathcal{A}_t = \{X_0 = X_t = 0, X_s \in I_t\,\forall x \in [t^{1/2}, t - t^{1/2}]\}$. By a local central limit theorem, $P_0(X_{t^{1/2}} \in I_t) \ge ct^{-1/4}$. By standard spectral estimates for the simple random walk, for any $m \in I_t$,

$$P_m(X_s \in I_t\,\forall s \le t - 2t^{1/2}) \ge e^{-ct/\ln t},$$

and, by a local central limit theorem again, $P_m(X_{t^{1/2}} = 0) \ge ct^{-1/4}$. The Markov property thus yields $P_0(\mathcal{A}_t) \ge e^{-ct/\ln t}$. Going back to (A.7), restricting the expectation to $\mathcal{A}_t$,

$$\lambda(0) \ge \mathsf{es} + \frac{1}{t}\limsup_{t\to\infty} P_0(\mathcal{A}_t)e^{-2\mathsf{es}t^{1/2}}e^{-\varepsilon(t-2t^{1/2})} \ge \mathsf{es} - \varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrarily, $\lambda(0) = \mathsf{es}$ follows.

The fact $\lim_{v\to\infty}\lambda(v)/|v| = -\infty$ follows from (POT) and large deviation properties of the continuous time simple random walk X.

Claim (A.5) is shown in the proof of Theorem 2.8 in Section 4.6. The strict concavity of $\lambda(v)$ for $v \in (v_c, \infty)$ is a consequence of the strict convexity of $L^*(1/v)$ on this interval, which in turn follows from definition (4.12) of v_c, the strict convexity of L on $(-\infty, 0)$ and standard properties of the Legendre transform.

Finally, to show the linearity of λ on $[0, v_c]$, observe that

$$\begin{aligned} u(t, vt) &= e^{t\mathsf{es}}E_0\Big[\exp\Big\{\int_0^t \zeta(X_s)\,\mathrm{d}s\Big\}; X_t = vt\Big] \\ &\le e^{t\mathsf{es}}E_0\Big[\prod_{i=1}^{vt-1}\exp\Big\{\int_{H_{i-1}}^{H_i}\zeta(X_s)\,\mathrm{d}s\Big\}\Big] = e^{t\mathsf{es}}e^{\sum_{i=1}^{vt}L_i^\zeta(0)}. \end{aligned}$$

Taking logarithms and letting $t \to \infty$, it follows that $\lambda(v) \le \mathsf{es} + vL(0) = \mathsf{es} - vL^*(1/v_c)$, where for the inequality we used (4.10), and for the equality we took advantage of (4.12) again. Concavity, $\lambda(0) = \mathsf{es}$ and (A.5) then imply the matching lower bound, proving the claimed linearity.

□

A.3 Hoeffding type inequality for mixing sequences

We repeatedly make use of the following concentration inequality for mixing sequences. We state it here for Readers convenience.

Lemma A.4 ([Rio13, Theorem 2.4]). *Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of real valued bounded random variables on some $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{F}_i = \sigma(X_j, j \le i)$. Suppose that there are real numbers $m_i > 0$, $i \in \{1, \dots, n\}$ such that*

$$\sup_{j\in\{i+1,\dots,n\}}\Big(\|X_i^2\|_\infty + 2\Big\|X_i\sum_{k=i+1}^{j}\mathbb{E}[X_k|\mathcal{F}_i]\Big\|_\infty\Big) \le m_i, \qquad \textit{for all } i \le n.$$

Then for every $a > 0$,

$$\mathbb{P}\big(|X_1 + \cdots + X_n| \geq a\big) \leq \sqrt{e} \exp\Big\{ - \frac{a^2}{2\sum_{i=1}^n m_i} \Big\}.$$

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