RANDOMLY TRAPPED RANDOM WALKS

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We introduce a general model of trapping for random walks on graphs. We give the possible scaling limits of these *Randomly Trapped Random Walks* on $\mathbb{Z}$. These scaling limits include the well known Fractional Kinetics process, the Fontes-Isopi-Newman singular diffusion as well as a new broad class we call *Spatially Subordinated Brownian Motions*. We give sufficient conditions for convergence and illustrate these on two important examples.

1. Introduction. We present here a general class of trapping mechanisms for random walks. This class includes the usual 'effective' models of trapping, from the Continuous Time Random Walks (CTRW) (see [24]), to the Bouchaud Trap Models (BTM) (see [9, 10, 11, 12] and [4]). It is in fact much wider. This higher level of generality is needed for the study of random walks on classical random structures, where the trapping is not introduced *ab initio* as in the CTRW or the BTM, but is created by the complexity of the underlying geometry. We introduce the class of models for general graphs, but restrict the study, in this paper, to the case of the line $\mathbb{Z}$. We obtain a rather complete understanding of the asymptotic behavior of these trapped walks on $\mathbb{Z}$. We give first a description of all possible scaling limits, and then proceed to give wide sufficient conditions for convergence to each of the possible scaling limits. We illustrate this by two simple examples, one effective and the other geometric, where we exhibit a rich transition picture between those different asymptotic regimes.

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The behavior of these models in higher dimension or other graphs is open. It seems clear that, when the underlying graph is transient, the asymptotic behavior should be much simpler. One might even risk the conjecture that, when the underlying graph is transient, the Brownian and the Fractional Kinetics scaling limits obtained both for the CTRW or the BTM, should be prevalent in general.

Consider a graph $G = (V, E)$, where V denotes the set of vertices, and E the set of edges. A general 'trapping landscape' on the graph G will be given by a collection $\boldsymbol{\pi} = (\pi_x)_{x\in V}$ of probability measures on $(0, \infty)$. Consider now the continuous-time random process $X := (X_t)_{t\geq 0}$ defined on V as follows: X_t stays at a vertex say $x \in V$, for a random duration sampled from the distribution π_x and then moves on to one of the neighbors of x, chosen uniformly at random. If the process X visits x again at a later time, the random duration of this next visit at x is sampled again and independently, from the distribution π_x. We will call the process X the trapped random walk (TRW) defined by the trapping landscape $\boldsymbol{\pi} = (\pi_x)_{x\in V}$.

This structure contains the important and very well-studied class of continuous time random walks (CTRW) as the simple particular case where the trapping landscape is constant, i.e. π_x is independent of $x \in V$. So, in particular the possible scaling limits, on the graph $\mathbb{Z}^d$, include the Brownian Motion (BM) and the Fractional Kinetics (FK) models (see [23]).

We will study in fact a much richer class of models, by considering the case of random trapping landscapes, i.e. the situation where the landscape $(\pi_x)_{x\in V}$ is given as an i.i.d. sample of a distribution on the space of probability measures on $(0, \infty)$. The random collection $(\pi_x)_{x\in V}$ is now a random environment. We have thus one extra layer of randomness and call the random process X defined as above, for every fixed (or quenched) realization of environment, a Randomly Trapped Random Walk (RTRW).

This richer class contains the Bouchaud Trap Model. This is the case where the probability measures π_x are chosen as exponential distributions with mean $\tau(x)$, and the $\tau(x)$'s are chosen as i.i.d. random variables in $(0, \infty)$. The scaling limits of this model in dimension 2 and above include the Brownian Motion and the Fractional Kinetics models (see [7, 5, 25, 6]), and in dimension one, the Fontes-Isopi-Newman (or FIN) singular diffusion (see [14, 17] and also [13]).

The new class of RTRW's also contains completely new examples which have motivated this general study. These examples are of random walks in random media, where the trapping mechanism is not imposed a priori, but is a consequence of the geometric characteristics of the medium. For instance one of our main motivations is given by the random walk on an incipient

critical Galton Watson tree (introduced by Kesten in [22], see also [2]). This incipient critical tree can be seen as made of a one-dimensional backbone, and of very long dead-ends (in fact finite critical trees) attached to this backbone. The trapping landscape is here of geometric origin: the projection of the random walk along the backbone is trapped by the very long sojourns in the dead-ends. We are also interested in the similar problem of the random walk on the invasion percolation cluster on a regular tree (see [1]). These two examples will not be treated here, but in a forthcoming work.

In this paper, we build the foundation by studying the general question of understanding the scaling limits of our general class of RTRW's in dimension one. We call this general class of limit processes Randomly Trapped Brownian Motions (RTBM's). These processes are all obtained through random time-changes of Brownian Motion. The needed class of time changes is rich and complex. The class of RTBM's contains naturally the scaling limits of the examples mentioned above, i.e. the Brownian Motion, the FK dynamics, and the FIN-diffusion. But it also contains very interesting new processes, which we call Spatially Subordinated Brownian Motions (SSBM). The class of geometric models mentioned above (the random walk on the incipient critical tree and the invasion percolation cluster) have scaling limits that belong to these new classes of models, hence the necessity of the general study done here.

In order to begin the discussion about the asymptotic behavior of the process X that we have defined above, we remark that its structure is a priori quite simple. It is given by a random time-change of the standard discrete-time random walk, say $Y = (Y_n)_{n\geq 0}$, on the graph G. Indeed, we first define $S(n)$, the 'clock process', i.e. the sum of the random trapping durations along the first n steps of the random walk $(Y_n)_{n\geq 0}$. More precisely, consider an random array of independent positive numbers $(s_x^k)_{k\geq 1, x\in V}$ where for every fixed vertex $x \in V$ the numbers $(s_x^k)_{k\geq 1}$ are an i.i.d. sample with common distribution π_x. Also define $L(x,n)$ to be the local time of the random walk Y, i.e. the number of visits of the site x before (and including) time n.

$$L(x,n) = \sum_{k=0}^{n} \mathbb{1}_{\{Y_k = x\}}. \tag{1.1}$$

The clock process is simply defined as

$$S(n) = \sum_{k=0}^{n-1} s_{Y_k}^{L(Y_k,k)} = \sum_{x\in G} \sum_{k=1}^{L(x,n-1)} s_x^k. \tag{1.2}$$

Then, clearly the process X is the time change of the simple random walk

Y by this random additive functional, i.e.

$$X_t = Y_n \quad \text{if } S(n) \le t < S(n+1). \tag{1.3}$$

It is thus perfectly natural, at least when $G = \mathbb{Z}$, to expect that the possible scaling limits will be random time-changes of Brownian Motion. But it might not be obvious that the asymptotic behavior of the time-change can be as rich as we find it to be. In the case of the FK processes, it is clear that this time change is a stable subordinator, and is independent of the underlying Brownian Motion. In the case of the FIN diffusion, this time-change is not independent of the underlying Brownian Motion and is very singular since it retains the randomness of the spatial information contained in the traps.

In the general situation, the time-change will be even more complex. We show in our first result (Theorem 2.7), that the asymptotic behavior is in general a mixture of an FK type situation, and of the new class of processes, the Spatially Subordinated Brownian Motions (SSBM). These processes are again defined by a time change of Brownian Motion, where the time change retains some of the randomness of the spatial information about deep traps, in a much more intricate fashion than in the FIN case.

In order to illustrate this new class of processes, we also show in this article that very simple models give rise to them, much simpler indeed that the two geometric models mentioned above. We start with the simplest of such models, which we call the model with 'transparent traps': Consider the Bouchaud trap model with the following twist: at site $x \in \mathbb{Z}$ the process X can, with positive probability, ignore the trap. This model exhibits different regimes where the scaling limits can be very different. They include the Brownian Motion, the FK dynamics, the FIN diffusion and in a critical regime a new example of our wide class of SSBM's. This model is interesting since, although very simple, it contains this rich array of limiting behaviors and this new transition. In fact it contains, in a very simple way, the main mechanism: the possibility to ignore somewhat the deep traps.

As a next step, and building on this intuition, we give finally a complete study of a simple geometric example, much closer to the cases of the random walk on the incipient critical tree and invasion percolation cluster. We study the random walk on comb models. This model is also rich. If one add a drift towards the teeth of the comb, then various regimes mentioned above are also present in this model.

2. Statement of results. In this section we provide precise statements of our results. We begin by describing the processes that will later appear

as possible scaling limits of RTRW's on $\mathbb{Z}$. We will define first the Fractional Kinetics processes, then introduce our new class of Spatially Subordinated Brownian Motions, and then specialize this definition to introduce the Fontes-Isopi-Newman (or FIN) diffusion.

DEFINITION 2.1 (Fractional Kinetics). Let $(B_t)_{t\geq 0}$ be a standard one-dimensional Brownian Motion and let $(V_t^\alpha)_{t\geq 0}$ be an α-stable subordinator (for some $\alpha \in (0,1)$) independent of B. Let $\psi_t^\alpha := \inf\{s \geq 0 : V_s^\alpha > t\}$. The Fractional Kinetics process of index α, Z^α, is defined as

$$Z_t^\alpha := B_{\psi_t^\alpha}.$$

Next we define *Spatially Subordinated Brownian Motions* (SSBM's). Let $\mathfrak{F}^*$ be the set of Laplace exponents of subordinators (i.e. of non-decreasing Lévy processes), that is the set of continuous functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ that can be expressed as

$$f(\lambda) = f_{\mathtt{d},\Pi}(\lambda) := \mathtt{d}\lambda + \int_{\mathbb{R}_+} (1 - e^{-\lambda t})\Pi(dt) \tag{2.1}$$

for a $\mathtt{d} \geq 0$ and a measure Π satisfying $\int_{(0,\infty)}(1 \wedge t)\Pi(dt) < \infty$. We endow $\mathfrak{F}^*$ with topology of pointwise convergence and the corresponding Borel σ-algebra.

Let $\mathbb{F}$ be a σ-finite measure on $\mathfrak{F}^*$ and let $(x_i, f_i)_{i\in\mathbb{N}}$ be a Poisson point process on $\mathbb{R} \times \mathfrak{F}^*$ with intensity $dx \otimes \mathbb{F}$. Let $(S_t^i)_{t\geq 0}$, $i \in \mathbb{N}$, be a family of processes, such that, conditioned on a realization of $(x_i, f_i)_{i\in\mathbb{N}}$, $(S^i)_{i\in\mathbb{N}}$ is distributed as an independent sequence of subordinators, where the Laplace exponent of S^i is given by f_i. We will assume that the measure $\mathbb{F}$ satisfies the following assumption:

$$\sum_{i:x_i\in[0,1]} S_1^i < \infty \qquad \text{almost surely.} \tag{2.2}$$

Let B be a one-dimensional standard Brownian Motion started at the origin, independent of the $(S^i)_{i\in\mathbb{N}}$, and $\ell(x,t)$ be its local time. Define

$$\phi_t := \sum_{i\in\mathbb{N}} S^i_{\ell(x_i,t)} \tag{2.3}$$

and $\psi_t := \inf\{s \geq 0 : \phi_s > t\}$.

DEFINITION 2.2 (Spatially Subordinated Brownian Motion). The process $B^{\mathbb{F}}$ defined as

$$B_t^{\mathbb{F}} := B_{\psi_t}$$

is called an $\mathbb{F}$-Spatially Subordinated Brownian Motion.

REMARK 2.3. The assumption (2.2) ensures that ϕ_t is finite for all $t \geq 0$ and hence the $\mathbb{F}$-SSBM is well defined.

The FIN diffusion is a particular case of a SSBM. It is in fact a Markovian SSBM, which has been introduced as the scaling limit of the BTM on $\mathbb{Z}$ in [14], see also [3]. For every $v > 0$, consider the atomic measure δ_{f_v} concentrated on the linear function $f_v(\lambda) = v\lambda$. For $\gamma \in (0,1)$, consider the measure $\mathbb{F}$ on $\mathfrak{F}^*$ defined by

$$\mathbb{F}^\gamma = \int_0^\infty \gamma v^{-1-\gamma} \delta_{f_v} dv. \tag{2.4}$$

DEFINITION 2.4 (Fontes-Isopi-Newman diffusion). For $\gamma \in (0,1)$, the $\mathbb{F}^\gamma$-SSBM is the FIN-diffusion of index γ (FIN$_\gamma$) .

To see that this definition agrees with the usual one, it is sufficient to observe that the Lévy process S_t corresponding to the Laplace exponent f_v satisfies $S_t = tv$ and thus ϕ_t can be written as $\sum_i v_i \ell(x_i, t)$ for a Poisson process (x_i, v_i) on $\mathbb{R} \times (0, \infty)$ with intensity $dx\, \gamma v^{-1-\gamma} dv$.

Finally, we will define processes which are constructed as mixtures of the SSBM's and the FK-processes. Let $\mathbb{F}$ be a σ-finite measure on $\mathfrak{F}^*$ satisfying (2.2) and $(x_i, f_i)_{i\geq 0}$, $(S^i)_{i\in\mathbb{N}}$ be as in Definition 2.2. Let $(V^\gamma)_{t\geq 0}$ be an γ-stable subordinator (for some $\gamma \in (0,1)$) independent of the processes $(S^i)_{i\in\mathbb{N}}$, and B be a Brownian Motion independent of the $(S^i)_{i\in\mathbb{N}}$ and V^γ. Let $\ell(x,t)$ be the local time of B. Define

$$\phi_t := \sum_{i\in\mathbb{N}} S^i_{\ell(x_i,t)} + V^\gamma_t \tag{2.5}$$

and $\psi_t := \inf\{s \geq 0 : \phi_s > t\}$.

DEFINITION 2.5 (FK-SSBM mixture). The process $(B_{\psi_t})_{t\geq 0}$ is called an FK-SSBM mixture.

REMARK 2.6. Note that the SSBM and the FK-processes are both particular cases of FK-SSBM mixtures. The SSBM is obtained by taking $V^\gamma \equiv 0$ (i.e. the 'trivial' γ-stable subordinator), and the FK process is recovered by taking $\mathbb{F}$ to be a zero measure.

2.1. *Classification Theorem.* The first result we present is a classification theorem which characterizes the set of limiting processes of RTRW's with an i.i.d. trapping landscape.

Consider $P \in M_1(M_1((0,\infty)))$ (that is P is a probability measure on the space of probability measures on $(0,\infty)$). Let $\boldsymbol{\pi}$ be the corresponding i.i.d. trapping landscape, that is an i.i.d. sequence $\boldsymbol{\pi} = (\pi_z)_{z\in\mathbb{Z}}$, $\pi_z \in M_1((0,\infty))$ with marginal P defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Given a realisation of $\boldsymbol{\pi}$, let $(s_x^i)_{x\in\mathbb{Z},i\geq 1}$ be an independent collection of random variables such that s_x^i has distribution π_x, and let X be the RTRW whose random trapping landscape is $\boldsymbol{\pi}$, defined as in (1.1)–(1.3). We write $P^{\boldsymbol{\pi}}$ for the law of X given $\boldsymbol{\pi}$. The distribution of X is then the semi-direct product $\mathbb{P} \times P^{\boldsymbol{\pi}}$.

THEOREM 2.7. *Assume that there is a non-decreasing function ρ such that the processes*

$$X_t^\varepsilon = \varepsilon X_{\rho(\varepsilon)^{-1}t}, \qquad t \geq 0, \tag{2.6}$$

converge as $\varepsilon \to 0$ in $(\mathbb{P} \times P^{\boldsymbol{\pi}})$-distribution on the space $D(\mathbb{R}_+)$ of cadlag functions endowed with Skorokhod topology to a process U satisfying the non-triviality assumption

$$\limsup_{t\to\infty} |U_t| = \infty, \quad \text{almost surely.} \tag{2.7}$$

Then one of the two following possibilities occurs:

(i) $\rho(\varepsilon) = \varepsilon^2 L(\varepsilon)$ for a function L slowly varying at 0. Then there exists $c > 0$ such that $U_t = (B_{c^{-1}t})_{t\geq 0}$ where B is a standard Brownian Motion.

(ii) $\rho(\varepsilon) = \varepsilon^\alpha L(\varepsilon)$ for $\alpha > 2$ and a function L slowly varying at 0. Then U is a FK-SSBM mixture $(B_{\psi_t})_{t\geq 0}$. Moreover, index γ of the γ-stable subordinator associated to B_{ψ_t} equals $2/\alpha$ and the intensity measure $\mathbb{F}$ satisfies the scaling relation

$$a\mathbb{F}(A) = \mathbb{F}(\sigma_a^\alpha A), \qquad \text{for every } A \in \mathcal{B}(\mathfrak{F}^*), a > 0, \tag{2.8}$$

where $\sigma_a^\alpha : \mathfrak{F}^* \to \mathfrak{F}^*$ is defined by

$$\sigma_a^\alpha(f)(\lambda) = af(a^{-\alpha}\lambda). \tag{2.9}$$

REMARK 2.8. The map σ_a^α maps the Laplace exponent of a Lévy process V to the Laplace exponent of the Lévy process $a^{-\alpha}V(a\cdot)$.

2.2. *Convergence Theorems.* We now present sufficient conditions for the convergence to the processes described above. Let X be, as above, a RTRW with i.i.d. trapping landscape whose marginal is $P \in M_1(M_1((0,\infty)))$.

2.2.1. *Convergence to Brownian Motion.* We start by presenting general criteria for the convergence to the Brownian Motion. For any probability measure $\nu \in M_1((0,\infty))$ we define $m(\nu)$ to be its mean,

$$m(\nu) = \int_{\mathbb{R}_+} x\nu(dx). \tag{2.10}$$

THEOREM 2.9. *Assume that*

$$M := \int m(\pi)P(d\pi) \in (0,\infty). \tag{2.11}$$

Then, $\mathbb{P}$-a.s., as $\varepsilon \to 0$, the rescaled RTRW $(\varepsilon X_{M^{-1}\varepsilon^{-2}t})_{t\geq 0}$ converges to a standard Brownian Motion, in $P^{\boldsymbol{\pi}}$-distribution on the space $D(\mathbb{R}_+)$.

REMARK 2.10. Observe that Theorem 2.9 is a quenched result: the convergence holds for $\mathbb{P}$-a.e. realization of the trapping landscape $\boldsymbol{\pi}$.

2.2.2. *Convergence to the Fractional Kinetics process.* We now deal with the convergence to the FK process. Let, as usual, X be a RTRW with i.i.d. trapping landscape $\boldsymbol{\pi}$ whose marginal is P. We write

$$\hat{\pi}(\lambda) := \int_0^\infty e^{-\lambda t}\pi(dt) \tag{2.12}$$

for the Laplace transform of a probability measure over $(0,\infty)$, and set

$$\Gamma(\varepsilon) := \mathbb{E}[1 - \hat{\pi}_0(\varepsilon)]. \tag{2.13}$$

It is easy to see that Γ is strictly increasing on $\mathbb{R}_+$, taking values in $[0, \Gamma_{\max})$ for some $0 < \Gamma_{\max} < 1$. Therefore, the inverse Γ^{-1} is well defined on this interval. For ε small enough, we can thus introduce the inverse time scale q_{FK} by

$$q_{\mathrm{FK}}(\varepsilon) = \Gamma^{-1}(\varepsilon^2). \tag{2.14}$$

THEOREM 2.11. *Assume that*

$$q_{\mathrm{FK}}(\varepsilon) = \varepsilon^\alpha L(\varepsilon) \tag{2.15}$$

for some $\alpha > 2$ and a slowly varying function L. In addition assume that

$$\lim_{\varepsilon\to 0} \varepsilon^{-3}\mathbb{E}\big[\big(1-\hat{\pi}_0(q_{\mathrm{FK}}(\varepsilon))\big)^2\big] = 0. \tag{2.16}$$

Then, as $\varepsilon \to 0$, the rescaled RTRW $(\varepsilon X_{q_{\mathrm{FK}}(\varepsilon)^{-1}t})_{t\geq 0}$ converges in $P^{\boldsymbol{\pi}}$-distribution on $D(\mathbb{R}_+)$ to the FK process with parameter $\gamma = 2/\alpha$, in $\mathbb{P}$-probability.

In addition, if ε^{-3} in (2.16) is replaced by $\varepsilon^{-4-\delta}$, $\delta > 0$, then the convergence in $P^{\boldsymbol{\pi}}$-distribution holds $\mathbb{P}$-a.s.

Remark 2.12. Due to (2.14), (2.15) is equivalent to

$$\Gamma(\varepsilon) = \varepsilon^{2/\alpha}\tilde{L}(\varepsilon) = \varepsilon^{\gamma}\tilde{L}(\varepsilon), \tag{2.17}$$

for some slowly varying function $\tilde{L}$.

2.2.3. *Convergence to Spatially Subordinated Brownian Motions.* Here we present sufficient conditions for the convergence to the SSBM processes introduced in Definition 2.2. We assume that X is a RTRW with an i.i.d. random trapping landscape $\boldsymbol{\pi} = (\pi_z)_{z\in\mathbb{Z}}$ with marginal $P \in M_1(M_1((0,\infty)))$.

We recall that $m(\nu)$ denotes the mean of the probability distribution ν, see (2.10). Our first assumption is that the distribution of $m(\pi_0)$ has heavy tails.

Assumption (**HT**). There exists $\gamma \in (0,1)$ and a non-vanishing slowly varying function at infinity $L : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$P[\pi \in M_1((0,\infty)) : m(\pi) > u] = u^{-\gamma}L(u). \tag{2.18}$$

Remark 2.13. We define $V \in D(\mathbb{R})$ by

$$V_x = \begin{cases} \sum_{i=1}^{\lfloor x\rfloor} m(\pi_i), & x \geq 1, \\ 0, & x \in [0,1), \\ \sum_{i=\lfloor x\rfloor+1}^{0} m(\pi_i), & x < 0. \end{cases} \tag{2.19}$$

Under Assumption (HT), there exists a function $d(\varepsilon)$ satisfying $d(\varepsilon) = \varepsilon^{-1/\gamma}\tilde{L}(\varepsilon)$ for a function $\tilde{L}$ slowly varying at 0, such that $(d(\varepsilon)^{-1}V_{\varepsilon^{-1}x})_{x\in\mathbb{R}}$ converges in distribution on $D(\mathbb{R})$ to a (two-sided) γ-stable subordinator with Lévy measure $\gamma v^{-1-\gamma}\,dv$. In addition we may assume that d is strictly decreasing and continuous.

Next, we prepare the statement of the second assumption. For each $a \in \mathbb{R}_+$, let π^a be a random measure having the law of π_0 conditioned on $m(\pi_0) = a$. Let

$$q(\varepsilon) := \varepsilon d(\varepsilon)^{-1}, \tag{2.20}$$

where d is as in Remark 2.13.

For $\varepsilon > 0$, define $\Psi_\varepsilon : M_1((0,\infty)) \to C(\mathbb{R}_+)$ by

$$\Psi_\varepsilon(\nu)(\lambda) := \varepsilon^{-1}(1 - \hat{\nu}(q(\varepsilon)\lambda)), \qquad \nu \in M_1((0,\infty)), \lambda \geq 0. \tag{2.21}$$

Observe that $\Psi_\varepsilon(\nu)$ is the Laplace exponent of a pure jump Lévy process whose jumps have intensity ε^{-1} and the size of jumps divided by $q(\varepsilon)$ has distribution ν. In particular, $\Psi_\varepsilon(\nu) \in \mathfrak{F}^*$ for every $\nu \in M_1((0,\infty))$. Our second assumption is

Assumption (**L**). There exists $\mathbb{F}_1 \in M_1(\mathfrak{F}^*)$ such that

$$\text{law of } \Psi_\varepsilon(\pi^{d(\varepsilon)}) \xrightarrow{\varepsilon \to 0} \mathbb{F}_1. \tag{2.22}$$

In addition, $\mathbb{F}_1$ is non-trivial, that is

$$\mathbb{F}_1 \neq \delta_{\mathbf{0}} \tag{2.23}$$

where $\mathbf{0}$ is the identically zero function.

Remark 2.14. Observe that $\Psi_\varepsilon(\pi^{d(\varepsilon)})$ is a Laplace exponent of a subordinator S such that $\mathbb{E}[S_1] = \varepsilon^{-1} d(\varepsilon) q(\varepsilon) = 1$. The measure $\mathbb{F}_1$ thus gives the full mass to the set $\mathfrak{F} \subset \mathfrak{F}^*$ of functions $f : \mathbb{R}_+ \to \mathbb{R}$ that can be written as $f(\lambda) = \mathtt{d}\lambda + c\int(1 - e^{-\lambda t})\Pi(dt)$ for $\mathtt{d} + c \leq 1$ and Π satisfying $\int_{\mathbb{R}_+} t\Pi(dt) = 1$. In particular, $f(\lambda) \leq \lambda$.

Theorem 2.15. *Assume that (HT) and (L) hold. Then, as $\varepsilon \to 0$, $(\varepsilon X_{q(\varepsilon)^{-1}t})_{t \geq 0}$ converges on $D(\mathbb{R}_+)$ in $\mathbb{P} \times P^{\boldsymbol{\pi}}$-distribution to a SSBM process $(B^{\mathbb{F}}_t)_{t \geq 0}$ introduced in Definition 2.2. The intensity measure $\mathbb{F}$ which determines the law of the limiting process is given by*

$$\mathbb{F}(df) := \int_0^\infty \gamma v^{-\gamma - 1} \mathbb{F}_v(df)\, dv, \tag{2.24}$$

where (recall (2.9) for the notation)

$$\mathbb{F}_v := \mathbb{F}_1 \circ \sigma_{v^\gamma}^{1+1/\gamma}. \tag{2.25}$$

REMARK 2.16. Observe that the scaling relation (2.8) is satisfied for $\mathbb{F}$ in (2.24) and $\alpha = 1 + \frac{1}{\gamma}$. Indeed, since $\sigma_a^{1+1/\gamma}\sigma_b^{1+1/\gamma} = \sigma_{ab}^{1+1/\gamma}$, for any $A \in \mathcal{B}(\mathfrak{F}^*)$,

$$\begin{aligned}\mathbb{F}(\sigma_a^{1+1/\gamma}A) &= \int \gamma v^{-1-\gamma}\mathbb{F}_v(\sigma_a^{1+1/\gamma}A)dv \\ &= \int \gamma v^{-1-\gamma}\mathbb{F}_1(\sigma_{av^\gamma}^{1+1/\gamma}A)dv \\ &= a\int \gamma u^{-1-\gamma}\mathbb{F}_1(\sigma_u^{1+1/\gamma}A)du = a\mathbb{F}(A).\end{aligned} \tag{2.26}$$

2.2.4. *Convergence to the FIN diffusion.* Next we present a theorem which gives sufficient conditions for convergence to the FIN diffusion. Recall that $m(\nu)$ denotes the expectation $\nu \in M_1((0,\infty))$, and define $m_2(\nu) = \int_{\mathbb{R}_+} t^2\,\nu(dt)$ to be its second moment. As before we let π^a stand for a random measure having the distribution of π_0 given $m(\pi_0) = a$. Define random variable $m_2(a) := m_2(\pi^a)$.

THEOREM 2.17. *Assume that (HT) holds and let $d(\varepsilon)$ be as in Remark 2.13. In addition, let $\varepsilon d(\varepsilon)^{-2}m_2(d(\varepsilon)) \xrightarrow{\varepsilon\to 0} 0$ in distribution. Then, $(\varepsilon^{-1}X_{q(\varepsilon)t})_{t\geq 0}$ converges to the FIN_γ diffusion in the sense of Theorem 2.15.*

We conclude the introduction with a description of the organization of the paper. In Section 3 we will define two examples of RTRW's for which we will prove convergence results. First we will define the transparent traps model and we will state the theorem which describes its phase diagram (see Theorem 3.2). Then we will define the comb model and we will present Theorem 3.5 which deals with its possible scaling limits.

Sections 4 and 5 contain the main definitions which will be used through the paper. In Section 4 we give the precise definitions of Trapped Random Walks and Trapped Brownian motions. In Section 5 we give the definitions and examples of Randomly Trapped Random Walks and Randomly Trapped Brownian Motions. In Section 6 we prove a general result from which one can deduce convergence of trapped processes from the convergence of their respective *trap measures*.

The bulk of the paper is Section 7 where we deal with limits of RTRW's. In Subsection 7.1 we prove the classification of the all possible limits of RTRW's with i.i.d. trapping landscape stated in Theorem 2.7. In Subsection 7.2 we prove Theorem 2.9 which deals with the convergence to the Brownian Motion. The convergence to the FK process stated in Theorem 2.11 will be proved in Subsection 7.3. In Subsection 7.4 we will prove the convergence to

the SSBM stated in Theorem 2.15. In Subsection 7.5 we prove Theorem 2.17 which states the convergence to the FIN diffusion.

Finally, Section 8 deals with the proof of the theorems for the transparent traps model and the comb model. In Subsection 8.1 we will prove Theorem 3.2 and in Subsection 8.2 we will prove Theorem 3.5.

The appendix collects, for reader's convenience, several known results from the random measure theory that are used through the paper.

3. Examples. In this section we define two examples of RTRW's. We also present the theorems which describe their phase-diagrams.

3.1. *Transparent traps model.* The simplest model which we will treat is the *Trap model with transparent traps*. Let $\alpha, \beta > 0$, and let $(\tau_x)_{x\in\mathbb{Z}}$ be a i.i.d. sequence of positive random variables which satisfy

$$\lim_{u\to\infty} u^{\alpha}\mathbb{P}(\tau_0 > u) = c \in (0,\infty), \tag{3.1}$$

and $\mathbb{P}(\tau_x > 1) = 1$. For each $x \in \mathbb{Z}$, consider the random probability distribution $\pi_x := (1 - \tau_x^{-\beta})\delta_1 + \tau_x^{-\beta}\delta_{\tau_x}$.

Definition 3.1 (Trap model with transparent traps). Let X be the RTRW with random trapping landscape $(\pi_x)_{x\in\mathbb{Z}}$. Then X is the called the trap model with transparent traps.

The reason for his name is the following. When X reaches $x \in \mathbb{Z}$ it is trapped there for time τ_x with probability $\tau_x^{-\beta}$, otherwise it does not 'see' the trap and just stays at x for a unit of time. The phase-diagram of the transparent traps model is given by the following theorem.

Theorem 3.2. *The trap model with transparent traps has the following scaling behavior:*

- *(i) If $\alpha + \beta > 1$, then for $m := \mathbb{E}(m(\pi_0)) < \infty$, the process $\varepsilon X_{m\varepsilon^{-2}t}$ converges to a standard Brownian Motion in the sense of Theorem 2.9.*
- *(ii) If $\alpha + \beta < 1$ and $\alpha > \beta$, then for $\gamma = \alpha/(1-\beta)$ and $q(\varepsilon) = \varepsilon^{1+1/\gamma}$, the process $\varepsilon X_{q(\varepsilon)^{-1}t}$ converges to FIN_γ in the sense of Theorem 2.15.*
- *(iii) If $\alpha+\beta < 1$ and $\alpha < \beta$, then for $\kappa = \alpha+\beta$ and $q(\varepsilon) = \varepsilon^{2/\kappa}$, the process $\varepsilon X_{q(\varepsilon)^{-1}t}$ converges to a Fractional Kinetics process with parameter κ. in the sense of Theorem 2.11.*
- *(iv) If $\alpha + \beta < 1$ and $\alpha = \beta$, then for $q(\varepsilon) = \varepsilon^{1/\alpha}$ the process $\varepsilon X_{q(\varepsilon)^{-1}t}$ converges, in the sense of Theorem 2.15, to a SSBM process, which will be referred as a "Poissonian" SSBM.*

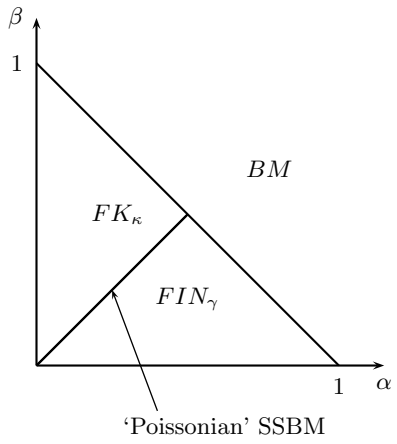

FIG 1. *Phase diagram for the transparent traps model*

REMARK 3.3. In the case $\alpha+\beta = 1$ which is not covered by the theorem, the scaling limit is a Brownian Motion, but a logarithmic correction should be added to the scaling. We do not consider this case here, for the sake of brevity.

3.2. *Comb model.* The comb model is a 'geometric' RTRW on a graph that looks like a comb with randomly long teeth. More precisely, consider an i.i.d. family N_z, $z \in \mathbb{Z}$, satisfying

$$\mathbb{P}(N_0 = n) = \mathcal{Z}^{-1} n^{-1-\alpha}, \qquad n \geq 1 \tag{3.2}$$

for some $\alpha > 0$ and a normalizing constant $\mathcal{Z} = \mathcal{Z}(\alpha)$. Let G_z be the graph with vertices $\{(z,0),(z,1),\ldots,(z,N_z)\}$ and with nearest-neighbor edges, and let G_{comb} be the tree-like graph composed by a backbone $\mathbb{Z}$ with leaves $(G_z)_{z\in\mathbb{Z}}$; $(z,0) \in G_z$ is identified with $z \in \mathbb{Z}$ on the backbone. By projecting the simple random walk on G_{comb} to the backbone we obtain a RTRW denoted X^{comb}.

We will see later that the behavior of X^{comb} is not very rich. When $\alpha > 1$ the teeth are 'short' and the mean time spent on them has a finite expectation, thus X^{comb} is diffusive and Brownian Motion is its scaling limit. On the other hand, when $\alpha < 1$, then the teeth may be 'long', and the expectation of the mean trapping time is infinite. However, as it is rather unlikely for the random walk on G_{comb} to reach the tip of long teeth, it takes many visits to a tooth to discover that it is long. This indicates that in this case the FK process is the limit.

To obtain a richer behavior, we need to increase the chance that the random walk on G_{comb} hits the tips of the teeth. Therefore we add a small drift pointing to the tips, as follows. Let Y^{comb} be a random walk on G_{comb} which on the backbone behaves like the simple random walk on G_{comb},

$$\mathbb{P}[Y^{\text{comb}}_{k+1} = z \pm 1 | Y^{\text{comb}}_k = z] = \mathbb{P}[Y^{\text{comb}}_{k+1} = (z,1) | Y^{\text{comb}}_k = z] = \tfrac{1}{3}, \tag{3.3}$$

and, when on the tooth G_z, it performs a random walk with a drift $g(N_z) \geq 0$ pointing away from the backbone, reflecting at the tip: for any z and $0 < n < N_z$,

$$\mathbb{P}[Y^{\text{comb}}(k+1) = (z, n+1) | Y^{\text{comb}}(k) = (z,n)] = (1 + g(N_z))/2, \tag{3.4}$$

$$\mathbb{P}[Y^{\text{comb}}(k+1) = (z, n-1) | Y^{\text{comb}}(k) = (z,n)] = (1 - g(N_z))/2, \tag{3.5}$$

$$\mathbb{P}[Y^{\text{comb}}(k+1) = (z, N_z - 1) | Y^{\text{comb}}(k) = (z,N_z)] = 1. \tag{3.6}$$

We will choose g as

$$g(N) = \min(\beta N^{-1} \log(N), 1) \tag{3.7}$$

for some $\beta \geq 0$. The case $\beta = 0$ corresponds to the comb model without drift.

Definition 3.4 (Comb model). We define X^{comb} as then the projection of Y^{comb} to the backbone. More precisely $X^{\text{comb}}_t = z$ iff $Y^{\text{comb}}_{\lfloor t \rfloor} \in G_z$.

The next theorem describes the asymptotic behavior of X^{comb}.

Theorem 3.5. *The comb model has the following scaling behavior:*

(i) If $\alpha > 1$ and $1+2\beta < \alpha$, then for some $m \in (0, \infty)$, the process $\varepsilon X^{\text{comb}}_{m\varepsilon^{-2}t}$ converges to a standard Brownian Motion in the sense of Theorem 2.9.

(ii) If $\alpha > 1$ and $1 + 2\beta > \alpha$, then for $\gamma = \alpha/(1+2\beta)$ there exists a regularly varying function $q(\varepsilon)$ of index $1 + 1/\gamma$, such that the process $\varepsilon X^{\text{comb}}_{q(\varepsilon)^{-1}t}$ converges to FIN_γ in the sense of Theorem 2.15.

(iii) If $\alpha < 1$, then for $\kappa = \frac{1+\alpha}{2(1+\beta)}$, there exists a regularly varying function $q(\varepsilon)$ of index $2/\kappa$ such that the process $\varepsilon X^{\text{comb}}_{q(\varepsilon)^{-1}t}$ converges to a Fractional Kinetics process with parameter κ in the sense of Theorem 2.11.

Remark 3.6. We expect that on the line $\alpha = 1 + 2\beta$ the scaling limit is Brownian Motion. We have not studied the behavior on the line $\alpha = 1$.

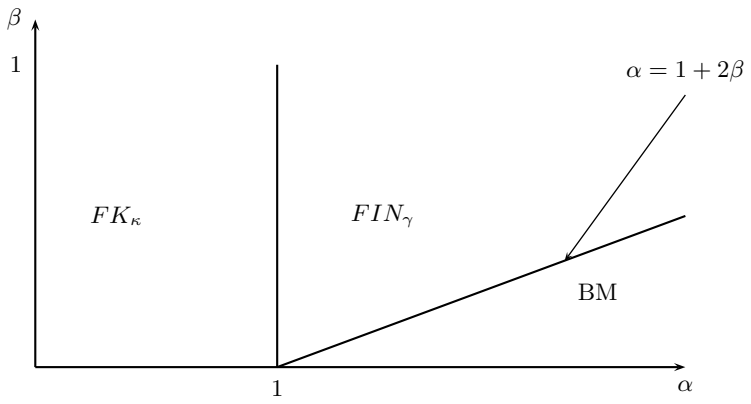

FIG 2. *Phase diagram for the comb model.*

4. Trapped Random walks and Trapped Brownian Motions.

4.1. *Trapped Random Walk.* In this section we give the definitions of several classes of processes which we will use through the paper.

4.1.1. *Time changed random walk.* We first consider 'deterministic' time change. Let $Z = (Z_k)_{k\geq 0}$ be a simple symmetric discrete-time random walk on $\mathbb{Z}$, $Z_0 = 0$, and let $(s_x^i)_{x\in\mathbb{Z},i\in\mathbb{N}}$ (with $\mathbb{N} = \{1, 2, \dots\}$) be a family of positive numbers. We define *time changed random walk* as the continuous-time $\mathbb{Z}$-valued process following the same trajectory as Z, characterized by stating that the duration of the i-th visit of X to $x \in \mathbb{Z}$ is s_x^i.

Alternatively, the time changed random walk can be defined using the following procedure, which will be more suitable for generalization into a continuous setting. Consider an atomic measure on $\mathbb{H} := \mathbb{R}\times\mathbb{R}_+ = \mathbb{R}\times[0,\infty)$ given by

$$\mu := \sum_{x\in\mathbb{Z},i\in\mathbb{N}} s_x^i \delta_{(x,i)}. \tag{4.1}$$

Let

$$L(x,t) := \sum_{i=1}^{\lfloor t\rfloor} \mathbb{1}_{\{Z_i=\lfloor x\rfloor\}}, \qquad t \geq 0, x \in \mathbb{R} \tag{4.2}$$

be the local time of Z. For a Borel-measurable function $f : \mathbb{R} \to \mathbb{R}_+$, define the set $U_f \subset \mathbb{H}$ of points under the graph of f by

$$U_f := \{(x, y) \in \mathbb{H} : y \leq f(x)\}. \tag{4.3}$$

Let $\phi[\mu, Z] : \mathbb{R}_+ \to \mathbb{R}_+$ be the function

$$\phi[\mu, Z]_t := \mu(U_{L(\cdot,t)}), \qquad t \geq 0, \tag{4.4}$$

and let $\psi[\mu, Z]$ be its right-continuous generalized inverse

$$\psi[\mu, Z]_t := \inf\{s > 0 : \phi[\mu, Z]_s > t\}, \qquad t \geq 0. \tag{4.5}$$

Definition 4.1. The **μ-time changed random walk** $(Z[\mu]_t)_{t\geq 0}$ is the process given by

$$Z[\mu]_t := Z_{\psi[\mu,Z]_t}, \qquad t \geq 0. \tag{4.6}$$

Remark 4.2. (a) If $\mu(\mathbb{H}) < \infty$, $Z[\mu]$ is not defined for times $t > \mu(\mathbb{H})$ and might not be defined for $t = \mu(\mathbb{H})$.

(b) It is easy to see that the functions $\phi[\mu, Z]$ and $\psi[\mu, Z]$ are non-decreasing and right-continuous. Hence, $Z[\mu]$ has right-continuous trajectories.

4.1.2. *Trapped random walk.* We want of course consider random time changes. One natural way how to introduce randomness is to require that the duration of every visit to $x \in \mathbb{Z}$ is distributed according to some probability distribution π_x, which may depend on x, assuming also that the durations of the visits are independent, and independent of the direction of the jumps of the random walk Z. We will call such random time change *trapped random walk* with (deterministic) *trapping landscape* $\pi = (\pi_x)_{x\in\mathbb{Z}}$.

More precisely, extending Definition 4.1, we may define the trapped random walk as follows:

Definition 4.3 (Trapped random walk). Let $\boldsymbol{\pi} = (\pi_x)_{x\in\mathbb{Z}}$ be a sequence of probability measures on $(0, \infty)$, $(s_x^i)_{i\in\mathbb{N},x\in\mathbb{Z}}$ an independent family of random variables such for every $x \in \mathbb{Z}$, $(s_x^i)_{i\in\mathbb{N}}$ is an i.i.d. sequence distributed according to π_x. Let μ be a random measure on $\mathbb{H}$ defined as in (4.1), and let Z be a simple symmetric random walk independent of $(s_x^i)_{x\in\mathbb{Z},i\in\mathbb{N}}$. The μ-time changed random walk $Z[\mu]$ is then called **trapped random walk** (TRW) with **trap measure** μ and **trapping landscape** $\boldsymbol{\pi}$.

We present three examples of TRWs.

EXAMPLE 4.4 (Montrol-Weiss continuous-time random walk). Let $\pi_x = \pi_0$ for all $x \in \mathbb{Z}$, and assume that π_0 satisfies the tail condition

$$\lim_{u\to\infty} u^{\gamma}\pi_0([u,\infty)) = c \tag{4.7}$$

for some $\gamma \in (0,1)$ and $c \in (0,\infty)$. In this case the durations of visits $(s_x^i)_{i\in\mathbb{N},x\in\mathbb{Z}}$ form an i.i.d. family with marginal π_0, and the trapped random walk $Z[\mu]$ is a one-dimensional continuous-time random walk à la Montroll-Weiss (see [24]).

EXAMPLE 4.5 (Geometric TRW). Let $(G_x)_{x\in\mathbb{Z}}$ be a family of rooted finite graphs, and let G be the graph obtained by attaching the graphs G_x to vertices of $\mathbb{Z}$. More precisely, denote by $V(G_x)$ the set of vertices of G_x, and assume that $(V(G_x))_{x\in\mathbb{Z}}$ are pairwise disjoint. Then G is the graph whose set of vertices is $V(G) := \cup_{x\in\mathbb{Z}} V(G_z)$, and its set of edges $E(G)$ is determined by: $(y,z) \in E(G)$ iff one of the following conditions hold

- There exists $x \in \mathbb{Z}$ such that $y, z \in V(G_x)$ and y and z are neighbors in G_x .
- There exists $x \in \mathbb{Z}$ such that y is the root of G_x and z is the root of G_{x+1}.
- There exists $x \in \mathbb{Z}$ such that y is the root of G_x and z is the root of G_{x-1}.

Hence, G is a graph consisting of a copy of $\mathbb{Z}$ (called the *backbone*) from which emerge *branches* $G_x, x \in \mathbb{Z}$. We will naturally identify the backbone with $\mathbb{Z}$.

Let $Y := (Y_k)_{k\geq 0}$ be a discrete time, symmetric random walk on G with $Y_0 = 0$. We can project Y to the backbone to obtain a continuous time $\mathbb{Z}$-valued process $W := (W_t)_{t\geq 0}$ given by $W_t = x \in \mathbb{Z}$ iff $Y_{\lfloor t\rfloor} \in G_x$. We call W *Geometric trapped random walk*. Its waiting times are of course related to the distribution of the return time to the root for the simple random walks on the finite graphs G_x.

EXAMPLE 4.6 (Markovian random walk on $\mathbb{Z}$). The trapped random walk is in general not Markovian. However, when for a family of positive numbers $(m_x)_{x\in\mathbb{Z}}$, π_x is the exponential distribution with mean m_x, then the trapped random walk $Z[\mu]$ with trapping landscape $(\pi_z)_{z\in\mathbb{Z}}$ is Markovian. The total jump rate at x is m_x^{-1}.

4.2. *Trapped Brownian Motion.* We now define continuous counterparts of the previously defined processes.

4.2.1. *Time changed Brownian Motion.*

DEFINITION 4.7 (μ-time changed Brownian Motion). Let μ be a deterministic measure on $\mathbb{H}$, and B be a standard one-dimensional Brownian Motion. Denote by $\ell(x,t)$ a bi-continuous version of the local time of B, and define

$$\begin{aligned} \phi[\mu,B]_t &:= \mu(U_{\ell(\cdot,t)}), \\ \psi[\mu,B]_t &:= \inf\{s>0 : \phi[\mu,B]_s > t\}. \end{aligned} \tag{4.8}$$

The **μ-time changed Brownian Motion** $(B[\mu]_t)_{t\geq 0}$ is the process given by

$$B[\mu]_t := B_{\psi[\mu,B]_t}, \qquad t \geq 0. \tag{4.9}$$

REMARK 4.8. It is easy to see that the functions $\phi[\mu,B]$, and $\psi[\mu,B]$ are non-decreasing and right-continuous. Hence, $B[\mu]$ has right-continuous trajectories.

4.2.2. *Trapped Brownian Motion.* Before defining the class of Trapped Brownian Motions, we recall the definition of random measure with independent increments (see §10 of [20]).

DEFINITION 4.9. A random measure μ on $\mathbb{H}$ is called a **measure with independent increments** iff for every two disjoint sets $A, B \in \mathcal{B}(\mathbb{H})$, the random variables $\mu(A)$ and $\mu(B)$ are independent.

For any random measure μ and $A \in \mathcal{B}(\mathbb{R})$ we define the μ-trapping process $(\mu\langle A\rangle_t)_{t\geq 0}$ by

$$\mu\langle A\rangle_t := \mu(A \times [0,t]). \tag{4.10}$$

Note that, if μ is a measure with independent increments and A, B are disjoint Borel subsets of $\mathbb{R}$, then $\mu\langle A\rangle$ and $\mu\langle B\rangle$ are independent processes.

DEFINITION 4.10 (Lévy trap measure). A random measure μ on $\mathbb{H}$ is called **Lévy trap measure** when $\mu\langle A\rangle$ is a Lévy process for every bounded $A \in \mathcal{B}(\mathbb{R})$.

DEFINITION 4.11 (Trapped Brownian Motion). Let μ be a random measure on $\mathbb{H}$ and B be a standard one-dimensional Brownian Motion. Suppose that (i) μ is independent from B, (ii) μ is a measure with independent increments, (iii) μ is a Lévy trap measure. Then $B[\mu]$ is called **Trapped Brownian Motion** (TBM) with **trap measure** μ.

The class of TBMs includes the following processes:

EXAMPLE 4.12 (Speed-measure changed Brownian Motion). Fix $\rho \in M(\mathbb{R})$ (cf. the appendix for the notation) and let Leb_+ be the Lebesgue measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. Define $\mu := \rho \otimes \mathrm{Leb}_+$. Then μ is a (deterministic) Lévy trap measure. Furthermore, as μ is deterministic, it is also a measure with independent increments.

The TBM $B[\mu]$ is simply a time change of Brownian Motion with speed measure ρ. Indeed, this time change B^ρ is usually defined as

$$(B^\rho_t)_{t\geq 0} := (B_{\psi_\rho(t)})_{t\geq 0}. \tag{4.11}$$

for $\phi_\rho(s) := \int_{\mathbb{R}} \ell(x,s)\rho(dx)$ and $\psi_\rho(t) := \inf\{s > 0 : \phi_\rho(s) > t\}$. By Fubini's theorem, it is easy to see that

$$\phi_\rho(s) = \int_{\mathbb{R}} \int_0^{\ell(x,s)} dy\, \rho(dx) = (\rho \otimes \mathrm{Leb}_+)(U_{\ell(\cdot,s)}) = \phi[\rho \otimes \mathrm{Leb}_+, B]_s. \tag{4.12}$$

This implies that B^ρ equals $B[\mu]$.

EXAMPLE 4.13 (Fractional Kinetics process). Let $\mathcal{P} = (x_i, y_i, z_i)_{i\in\mathbb{N}}$ be a Poisson point process on $\mathbb{H} \times (0,\infty)$ with intensity measure

$$\varrho = \gamma z^{-1-\gamma} dx\, dy\, dz, \qquad \gamma \in (0,1). \tag{4.13}$$

Define the random measure μ_{FK} on $\mathbb{H}$ as

$$\mu_{\mathrm{FK}} = \mu^\gamma_{\mathrm{FK}} := \sum_i z_i \delta_{(x_i,y_i)}. \tag{4.14}$$

It is easy to see that for every compact $K \subset \mathbb{H}$, $\mu_{\mathrm{FK}}(K)$ has a γ-stable distribution with the scaling parameter proportional to the Lebesgue measure of K. Further, as $\mathcal{P}$ is a Poisson point process, we have that $\mu_{\mathrm{FK}}(K_1)$ and $\mu_{\mathrm{FK}}(K_2)$ are independent when K_1, K_2 are disjoint. Thus μ_{FK} is a measure with independent increments, and $\mu_{\mathrm{FK}}\langle A\rangle$ is a stable Lévy process for each bounded $A \in \mathcal{B}(\mathbb{R})$, and thus μ_{FK} is a Lévy trap measure.

The TBM $B[\mu]$ corresponding to this measure is the FK process introduced in Definition 2.1. To see this, it is enough to show that the process $(\phi[\mu, B]_t)_{t\in\mathbb{R}_+}$ is a γ-stable subordinator that is independent of B.

This can be proved as follows. Fix a realization of the Brownian Motion B. Then its local time is also fixed. As $\mathrm{Leb}(U_{\ell(\cdot,t)}) = t$ and $U_{\ell(\cdot,s)}$, $(U_{\ell(\cdot,t)} \setminus U_{\ell(\cdot,s)})$ are disjoint sets for every $s < t$, we have that $\phi[\mu, B]_t$ has γ-stable distribution with the scaling parameter proportional to t, and $\phi[\mu, B]_t - \phi[\mu, B]_s$

is independent of $\phi[\mu, B]_s$. Hence, for every realization of B, $\phi[\mu, B]$ is a γ-stable subordinator, and thus $\phi[\mu, B]$ is a γ-stable subordinator independent of B.

The last important example goes in the direction of the SSBM.

EXAMPLE 4.14. Let $k \in \mathbb{N} \cup \{\infty\}$ and $((S_t^i)_{t\geq 0})_{i<k}$ be a family of independent subordinators. Let $(x_i)_{i<k}$ be real numbers. Denoting by dS^i the Lebesgue-Stieltjes measure corresponding to S^i, it is immediate that

$$\mu(dx \otimes dy) := \sum_{i<k} \delta_{x_i}(dx) \otimes dS^i(y) \tag{4.15}$$

is a Lévy trap measure with independent increments. The TBM $B[\mu]$ is a process which is always located at some x_i.

5. Randomly Trapped Random Walk and Randomly Trapped Brownian Motion. The classes of trapped random walks and trapped Brownian Motions are too small to include some processes that we want to consider, in particular Bouchaud's trap model, the FIN diffusion and the projections of the random walk on IIC, IPC. More precisely, quenched distributions of these models (given corresponding random environments) are trapped random walks. If we want to consider averaged distributions, we need to introduce larger classes, *Randomly Trapped Random Walks* and *Randomly Trapped Brownian Motion*. Their corresponding random measures will be constructed as mixtures of the respective trap measures.

The mixture of random measures is defined as follows. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let for every $\omega \in \Omega$, μ_ω be a random measure on $\mathbb{H}$ defined on some other probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. The random measure $\mu : \Omega \times \tilde{\Omega} \to M(\mathbb{H})$ given by

$$\mu(\omega, \tilde{\omega})(A) = \mu_\omega(\tilde{\omega})(A), \qquad A \in \mathcal{B}(\mathbb{H}). \tag{5.1}$$

is called mixture of μ_ω with respect to $\mathbb{P}$. For reader's convenience, Proposition A.1 ensuring the existence of the mixtures is included in the appendix.

5.1. *Randomly Trapped Random Walk.*

DEFINITION 5.1 (Randomly Trapped Random Walk). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mu_\omega)_{\omega\in\Omega}$ a family of *trap measures* on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ indexed by $\omega \in \Omega$. Let μ be the mixture of $(\mu_\omega)_{\omega\in\Omega}$ with respect to $\mathbb{P}$, and Z a simple random walk independent of μ. Then the μ-time changed random walk $Z[\mu]$ is called **Randomly Trapped Random Walk** (RTRW) with **trap measure** μ.

DEFINITION 5.2 (Random trapping landscape). Let $Z[\mu]$ be a RTRW where μ is the mixture of $(\mu_\omega)_{\omega\in\Omega}$ w.r.t. $\mathbb{P}$. Let $\boldsymbol{\pi} := (\pi_x)_{x\in\mathbb{Z}} : \Omega \to M_1((0,\infty))^{\mathbb{Z}}$ be defined by stating that, for each $\omega \in \Omega$, $\boldsymbol{\pi}(\omega)$ is the trapping landscape of $Z[\mu_\omega]$. $\boldsymbol{\pi}(\omega)$ is called the **random trapping landscape of** μ.

Let $\mathbf{P} = \mathbb{P} \circ \boldsymbol{\pi}^{-1}$ be the distribution of $\boldsymbol{\pi}$ on $M_1((0,\infty))^{\mathbb{Z}}$. If $\mathbf{P}$ is a product measure, that is $\mathbf{P} = \bigotimes_{x\in\mathbb{Z}} P^x$ for some $P_x \in M_1(M_1((0,\infty)))$, $x \in \mathbb{Z}$, then the coordinates of the random trapping landscape $(\pi_x)_{x\in\mathbb{Z}}$ are independent. In this case we say that the random trapping landscape $\boldsymbol{\pi}$ is **independent**. If $\mathbf{P} = \bigotimes_{x\in\mathbb{Z}} P$ for some $P \in M_1(M_1((0,\infty)))$, then the $(\pi_x)_{x\in\mathbb{Z}}$ are i.i.d., and we say the random trapping landscape $\boldsymbol{\pi}$ is **i.i.d.**

As usual we give some examples of RTRWs.

EXAMPLE 5.3 (Bouchaud Trap Model). The symmetric one-dimensional **Bouchaud trap model** (BTM) is a symmetric continuous time random walk X on $\mathbb{Z}$ with random jump rates. More precisely, to each vertex $x \in \mathbb{Z}$ we assign a positive number τ_x where $(\tau_x)_{x\in\mathbb{Z}}$ is an i.i.d. sequence of positive random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\lim_{u\to\infty} u^\gamma \mathbb{P}[\tau_z \geq u] = c, \qquad \gamma \in (0,1), c \in (0,\infty). \tag{5.2}$$

Each visit of X to $x \in \mathbb{Z}$ lasts an exponentially distributed time with mean τ_x.

It can be seen easily that the BTM is a RTRW. Its random trapping landscape is given by

$$\boldsymbol{\pi}(\omega) = (\nu_{\tau_x(\omega)})_{x\in\mathbb{Z}}, \tag{5.3}$$

where ν_a is the exponential distribution with mean a. As τ_x are i.i.d., the random trapping landscape $\boldsymbol{\pi}$ is i.i.d.

EXAMPLE 5.4 (Trap model with transparent traps). The trap model with transparent traps defined in Section 3.1 is a particular case of RTRW. In Section 8.1 we will study the scaling limits of this process.

The following three examples of RTRW are of geometric nature. The first (and the easiest) one is studied in this paper, the behavior of the next two examples will be considered a follow up paper.

EXAMPLE 5.5 (Comb model). The Comb model defined in Section 3.2 is a RTRW. Its scaling limits are given in Theorem 3.5 which we prove in Section 8.2.

EXAMPLE 5.6 (Incipient critical Galton-Watson tree). Let T be a rooted, regular tree of forward degree $g > 1$. Let us perform critical percolation on T and denote by $\mathcal{C}_n$ the percolation cluster of the root conditioned on reaching level n, that is conditioned on having a vertex whose graph-distance from the root is n. By letting $n \to \infty$ the trees $\mathcal{C}_n$ converge to the *Incipient infinite cluster* (IIC) (for details of this construction see [22]). The IIC is an infinite random tree and it can be shown that it has a single path to infinity, that is, there is a single unbounded nearest neighbor path started at the root. Such path is called the *backbone*. The backbone is obviously isomorphic (as a graph) to $\mathbb{N}$, hence the IIC can be seen as $\mathbb{N}$ adorned with dangling branches. We denote $\mathcal{L}_k$ the branch emerging from the k-th vertex of the backbone. Let $(Y^{\mathrm{IIC}}_k)_{k\in\mathbb{N}}$ be a simple random walk on the IIC starting from the root. Let W^{IIC} be the projection of Y^{IIC} to the backbone. More precisely, let $(W^{\mathrm{IIC}}_t)_{t\geq 0}$ be a continuous-time random walk taking values in $\mathbb{N}$ defined by stating that $W^{\mathrm{IIC}}_t = k$ if and only if $Y^{\mathrm{IIC}}_{\lfloor t\rfloor} \in \mathcal{L}_k$. Then W^{IIC} is a RTRW (disregarding for the moment the fact that it takes values on $\mathbb{N}$ instead of $\mathbb{Z}$). In this case the branches $(\mathcal{L}_k)_{k\in\mathbb{N}}$ play the role of traps.

EXAMPLE 5.7 (Invasion Percolation Cluster). One can also consider, instead of the Incipient infinite cluster, the *Invasion percolation cluster* (IPC) on a regular tree . The construction of the IPC is as follows: Recall that T denotes a rooted, regular tree of forward degree $g > 1$. Let $(w_x)_{x\in T}$ be an i.i.d. sequence of random variables uniformly distributed over $(0, 1)$. Set $\mathcal{I}^0 := \{\text{root of } T\}$ and

$$\mathcal{I}^{n+1} := \mathcal{I}^n \cup \big\{x : d(x, \mathcal{I}^n) = 1 \text{ and } w_x = \min\{w_z : d(\mathcal{I}^n, z) = 1\}\big\}, \tag{5.4}$$

where d is the graph distance in T. That is, $\mathcal{I}^{n+1}$ is obtained from $\mathcal{I}^n$ by adding the vertex on the outer boundary of $\mathcal{I}_n$ with the smallest 'weight'. The **Invasion percolation cluster** on T is defined as $\cup_{n\in\mathbb{N}}\mathcal{I}^n$. The IPC will be denoted as $\mathcal{I}^\infty$. It can be shown (see [1]) that, as the IPC, the IIC possesses a single path to infinity. We can define a RTRW W^{IPC} in the same way we have defined W^{IIC}.

5.2. *Randomly Trapped Brownian Motion.* Finally, we define the randomly trapped Brownian Motion analogously to RTRW.

DEFINITION 5.8 (Randomly trapped Brownian Motion). Let a random measure μ be the mixture of $(\mu_\omega)_{\omega\in\Omega}$ with respect to $\mathbb{P}$, where for each $\omega \in \Omega$, μ_ω is a trap measure of a TBM. Furthermore, let us suppose that μ is independent of the Brownian motion B. Then $B[\mu]$ is called **randomly trapped Brownian motion** (RTBM) with **trap measure** μ.

EXAMPLE 5.9 (FIN diffusion). Let $\mathcal{P} = (x_i, v_i)_{i\in\mathbb{N}}$ be a Poisson point process on $\mathbb{R}\times(0,\infty)$ with intensity measure $\gamma dx v^{-1-\gamma}dv$, $\gamma \in (0,1)$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For each $\omega \in \Omega$, let $\mu_\omega := \sum_{i\in\mathbb{N}} \delta_{x_i(\omega)} \otimes v_i(\omega)\,\mathrm{Leb}_+$. By Proposition A.1, the mixture of $(\mu_\omega)_{\omega\in\Omega}$ w.r.t. $\mathbb{P}$ exists and thus there exists the mixture μ_{FIN} of $(\mu_\omega)_{\omega\in\Omega}$ w.r.t. $\mathbb{P}$.

Recalling Example 4.12, it is easy to see that $B[\mu_\omega]$ is a time change of B with speed measure $\rho(dx) = \sum_i v_i(\omega)\delta_{x_i(\omega)}(dx)$. Comparing this with Definition 2.4, we see that the RTBM corresponding to μ_{FIN}, $B[\mu_{\mathrm{FIN}}]$, is a FIN diffusion.

EXAMPLE 5.10 (Spatially Subordinated Brownian Motion). Recall from (2.1) that $\mathfrak{F}^*$ is the set of Laplace exponents of subordinators. Let $\mathbb{F}$ be a σ-finite measure on $\mathfrak{F}^*$ satisfying the assumption appearing in (2.2) and let $(x_i, f_i)_{i\geq 0}$ be a Poisson point process on $\mathbb{R}\times\mathfrak{F}^*$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with intensity $dx\otimes\mathbb{F}$. Let $(S^i_t)_{t\geq 0}$, $i \geq 0$, be a family of independent subordinators, Laplace exponent of S^i being f_i, defined on a probability space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$.

For a given realization of $(x_i, f_i)_{i\geq 0}$, we set similarly as in Example 4.14,

$$\mu_{(x_i,f_i)}(dx\otimes dy) = \sum_{i\geq 0} \delta_{x_i}(dx)\otimes dS^i(y), \tag{5.5}$$

Hence, the measure $\mu_{(x_i,f_i)}$ is a Lévy trap measure with independent increments on $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$.

Using Proposition A.1, we can show that the mixture of $(\mu_{(x_i(\omega),f_i(\omega))})_{\omega\in\Omega}$ w.r.t. $\mathbb{P}$,

$$\mu^{\mathbb{F}}_{\mathrm{SSBM}}(\omega,\tilde\omega) := \mu_{(x_i(\omega),f_i(\omega))}(\tilde\omega) \tag{5.6}$$

is a random measure. The corresponding RTBM is the $\mathbb{F}$-Spatially Subordinated Brownian Motion introduced in Definition 2.2.

6. Convergence of processes.

We study now the convergence of various classes of processes introduced in the previous section.

6.1. *Convergence of time changed random walks.*

We start by presenting the basic convergence theorems for μ-time changed random walks and μ-time changed Brownian Motions. These theorems allow to deduce the convergence of processes (TRWs, TBMs, RTRWs, RTBMs) from the convergence of their associated random measures. This, in turn, makes possible to use the well developed theory of convergence of random measures, see e.g. [18].

First we need few additional definitions. We say that a random measure μ is **dispersed** if

$$\mu(\{(x,y) \in \mathbb{H} : y = f(x)\}) = 0 \quad \text{almost surely, for all } f \in C_0(\mathbb{R}, \mathbb{R}_+) \tag{6.1}$$

(here C_0 stands for the space of continuous functions with compact support). We say that a random measure μ is **infinite** if $\mu(\mathbb{H}) = \infty$, almost surely. We say that μ is **dense** if its support is $\mathbb{H}$, almost surely.

We write $D(\mathbb{R}_+)$, $D(\mathbb{R})$ for the sets of real-valued *cadlag* functions on $\mathbb{R}_+$, or $\mathbb{R}$, respectively. We endow these sets either with the standard Skorokhod J_1-topology, or with the so called M_1-topology, and write $D(\mathbb{R}_+, J_1)$, $D(\mathbb{R}_+, M_1)$ when we want to stress the topology used. Also $D(\mathbb{R}_+, U)$ will denote $D(\mathbb{R}_+)$ endowed with the uniform topology. For definitions and properties of these topologies see [27], Chapters 12 and 13.

Let μ be a random measure and $\varepsilon > 0$. We define the scaled random measure $\mathfrak{S}_\varepsilon(\mu)$ by

$$\mathfrak{S}_\varepsilon(\mu)(A) := \mu(\varepsilon^{-1}A), \qquad \text{for each } A \in \mathcal{B}(\mathbb{H}). \tag{6.2}$$

Our first theorem states that the convergence of μ-time changed random walks can be deduced from the convergence of associated measures. As it does not complicate the proof, we allow μ being random.

THEOREM 6.1 (convergence of time changed random walks). *Let μ^ε, $\varepsilon > 0$, be a family of infinite random measures supported on $\mathbb{Z} \times \mathbb{N}$, and let Z be a simple random walk independent of them. Assume that there exists a non-decreasing function $q : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\varepsilon \to 0} q(\varepsilon) = 0$, such that, as $\varepsilon \to 0$, $q(\varepsilon)\mathfrak{S}_\varepsilon(\mu^\varepsilon)$ converges vaguely in distribution to a dispersed, infinite, dense random measure μ. Then the corresponding time changed random walks $Z[\mu^\varepsilon]$ converge after rescaling to the time changed Brownian Motion $B[\mu]$,*

$$(\varepsilon Z[\mu^\varepsilon]_{q(\varepsilon)^{-1}t})_{t \geq 0} \xrightarrow{\varepsilon \to 0} (B[\mu]_t)_{t \geq 0}. \tag{6.3}$$

in distribution on $D(\mathbb{R}_+, J_1)$. Here B is a Brownian Motion independent of μ.

The next theorem, which we will not need later in the paper, gives a similar criteria for convergence of time changed Brownian Motions. We present it as it has intrinsic interest and because its proof is a simplified version of the proof of Theorem 6.1.

THEOREM 6.2. *Let μ^ε, $\varepsilon > 0$, be a family of infinite random measures on $\mathbb{H}$, and let B be a Brownian Motion independent of them. Assume that, as $\varepsilon \to 0$, μ^ε converges vaguely in distribution to a dispersed, infinite, dense random measure μ. Then the corresponding time changed Brownian Motions $B[\mu^\varepsilon]$ converge to $B[\mu]$,*

$$(B[\mu^\varepsilon]_t)_{t\geq 0} \xrightarrow{\varepsilon\to 0} (B[\mu]_t)_{t\geq 0}, \tag{6.4}$$

in distribution on $D(\mathbb{R}_+, J_1)$.

PROOF OF THEOREM 6.2. As μ^ε converges vaguely in distribution to μ, in virtue of the Skorokhod representation theorem, there exist random measures $(\bar\mu^\varepsilon)_{\varepsilon>0}$ and $\bar\mu$ on $\mathbb{H}$ defined on a common probability space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, such that $\bar\mu^\varepsilon$ is distributed as μ^ε, $\bar\mu$ is distributed as μ and $\bar\mu^\varepsilon$ converges vaguely to $\bar\mu$ as $\varepsilon \to 0$, $\tilde{\mathbb{P}}$-a.s. Without loss of generality, we can suppose that on the space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ there is defined a one-dimensional standard Brownian Motion $(B_t)_{t\geq 0}$ independent of $(\bar\mu^\varepsilon)_{\varepsilon>0}$ and $\bar\mu$.

First, we show that $\phi[\bar\mu^\varepsilon, B] \to \phi[\bar\mu, B]$ in $D(\mathbb{R}_+, M_1)$, $\tilde{\mathbb{P}}$-a.s. as $\varepsilon \to 0$: Using that $\bar\mu$ is a dispersed random measure,

$$\tilde{\mathbb{P}}[\bar\mu(\partial U_{\ell(\cdot,t)}) = 0, \forall 0 \leq t \in \mathbb{Q}] = 1, \tag{6.5}$$

where ∂A denotes the boundary of A in $\mathbb{H}$. Since $U_{\ell(\cdot,t)}$ is a bounded set, this implies that for all $0 \leq t \in \mathbb{Q}$

$$\phi[\bar\mu^\varepsilon, B]_t = \bar\mu^\varepsilon(U_{\ell(\cdot,t)}) \xrightarrow{\varepsilon\to 0} \bar\mu(U_{\ell(\cdot,t)}) = \phi[\bar\mu, B]_t, \qquad \tilde{\mathbb{P}}\text{-a.s.} \tag{6.6}$$

Since, by [27, Theorem 12.5.1 and 13.6.3], on the set of monotonous functions the convergence on $D(\mathbb{R}_+, M_1)$ is equivalent to pointwise convergence on a dense subset including 0 and since $\phi[\bar\mu^\varepsilon, B]$ and $\phi[\bar\mu, B]$ are non-decreasing in t, we know that

$$\phi[\bar\mu^\varepsilon, B] \to \phi[\bar\mu, B] \tag{6.7}$$

in $D(\mathbb{R}_+, M_1)$, $\tilde{\mathbb{P}}$-a.s., as claimed.

Since the random measures $\bar\mu^\varepsilon$ and $\bar\mu$ are infinite, the functions $\phi[\bar\mu^\varepsilon, B]$ and $\phi[\bar\mu, B]$ are unbounded. As, by hypothesis, $\bar\mu$ is dense, then the function $\phi[\bar\mu, B]$ will be strictly increasing. Hence, [27, Corollary 13.6.4] allows us to deduce uniform convergence of $\psi[\bar\mu^\varepsilon, B]$ to $\psi[\bar\mu, B]$ from (6.7).

Using the continuity of the Brownian paths and [27, Theorem 13.2.2], we get that $B[\bar\mu^\varepsilon]_t \to B[\bar\mu]_t$ in the J_1-topology. $\bar\mu^\varepsilon$ and $\bar\mu$ are distributed as the μ^ε and μ respectively, the convergence in distribution of $B[\mu^\varepsilon]$ to $B[\mu]$ follows. □

PROOF OF THEOREM 6.1. As $q(\varepsilon)\mathfrak{S}_\varepsilon(\mu^\varepsilon)$ converges vaguely in distribution to μ, we can, in virtue of the Skorokhod representation theorem, construct random measures $(\bar\mu^\varepsilon)_{\varepsilon>0}$ and $\bar\mu$ defined on a common probability space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, such that $\bar\mu^\varepsilon$ is distributed as $q(\varepsilon)\mathfrak{S}_\varepsilon(\mu^\varepsilon)$, $\bar\mu$ is distributed as μ, and $\bar\mu^\varepsilon$ converges vaguely to $\bar\mu$ as $\varepsilon \to 0$, $\tilde{\mathbb{P}}$-a.s. Without loss of generality, we can suppose that on the space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ there is defined a one-dimensional standard Brownian Motion $(B_t)_{t\geq 0}$ independent of $(\bar\mu^\varepsilon)_{\varepsilon>0}$ and $\bar\mu$.

Set $B^\varepsilon_t := \varepsilon^{-1}B_{\varepsilon^2 t}$. For each $\varepsilon > 0$, we define a sequence of stopping times $(\sigma^\varepsilon_k)_{k=0}^\infty$ by $\sigma^\varepsilon_0 := 0$,

$$\sigma^\varepsilon_k := \inf\{t > \sigma^\varepsilon_{k-1} : B^\varepsilon_t \in \mathbb{Z} \setminus \{B^\varepsilon_{\sigma^\varepsilon_{k-1}}\}\}. \tag{6.8}$$

Then, the process $(Z^\varepsilon_k)_{k\in\mathbb{N}}$ defined by $Z^\varepsilon_k := B^\varepsilon_{\sigma^\varepsilon_k}$ is a simple symmetric random walk on $\mathbb{Z}$. We define the local time of Z^ε as $L^\varepsilon(x,s) := \sum_{i=0}^{\lfloor s\rfloor} 1_{\{Z^\varepsilon_i = \lfloor x\rfloor\}}$. Define

$$\bar\phi^\varepsilon_s = q(\varepsilon)^{-1}\mathfrak{S}_{\varepsilon^{-1}}(\bar\mu^\varepsilon)(U_{L^\varepsilon(\cdot,s)}), \qquad s \geq 0, \varepsilon > 0. \tag{6.9}$$

Note that $q(\varepsilon)^{-1}\mathfrak{S}_{\varepsilon^{-1}}(\bar\mu^\varepsilon)$ is distributed as μ^ε. Hence, $(\bar\phi^\varepsilon_t)_{t\geq 0}$ is distributed as $(\mu^\varepsilon(U_{L^1(\cdot,t)}))_{t\geq 0} = (\phi[\mu^\varepsilon]_t)_{t\geq 0}$. Hence, denoting $\bar\psi^\varepsilon_t := \inf\{s > 0 : \bar\phi^\varepsilon_s > t\}$, we see that for each $\varepsilon > 0$, the process $(Z^\varepsilon_{\bar\psi^\varepsilon_t})_{t\geq 0}$ is distributed as $(Z[\mu^\varepsilon]_t)_{t\geq 0}$.

The proof of Theorem 6.1 relies on the following two lemmas.

LEMMA 6.3. *For each $t \geq 0$, there exists a random compact set K_t such that $\bigcup_{\varepsilon>0} \operatorname{supp} L^\varepsilon(\varepsilon^{-1}\cdot, \varepsilon^{-2}t)$ is contained in K_t.*

PROOF. By the strong Markov property for the Brownian Motion B, for each $\varepsilon > 0$, $(\sigma^\varepsilon_k - \sigma^\varepsilon_{k-1})_{k>0}$ is an i.i.d. sequence with $\tilde{\mathbb{E}}[\sigma^\varepsilon_i - \sigma^\varepsilon_{i-1}] = 1$. Thus, by the strong law of large numbers for triangular arrays, $\tilde{\mathbb{P}}$-almost surely, there exists a (random) constant C such that $\varepsilon^2\sigma^\varepsilon_{\lfloor\varepsilon^{-2}t\rfloor} \leq C$ for all $\varepsilon > 0$. Thus, for each $\varepsilon > 0$, the support of $L^\varepsilon(\varepsilon^{-1}\cdot, \varepsilon^{-2}t)$ is contained in the support of $\ell(\cdot, C)$. Therefore, it is sufficient to choose $K_t = \operatorname{supp}(\ell(\cdot, C))$. □

LEMMA 6.4. $(q(\varepsilon)\bar\phi^\varepsilon_{\varepsilon^{-2}t})_{t\geq 0} \xrightarrow{\varepsilon\to 0} (\phi[\bar\mu, B]_t)_{t\geq 0}$ $\tilde{\mathbb{P}}$-*a.s. on* $(D(\mathbb{R}_+), M_1)$.

PROOF. It is easy to see that

$$q(\varepsilon)\bar\phi^\varepsilon_{t\varepsilon^{-2}} = \mathfrak{S}_{\varepsilon^{-1}}(\bar\mu^\varepsilon)(U_{L^\varepsilon(\cdot,\varepsilon^{-2}t)}) = \bar\mu^\varepsilon(U_{\varepsilon L^\varepsilon(\varepsilon^{-1}\cdot,\varepsilon^{-2}t)}) \tag{6.10}$$

By [8, Chapter IV; Theorem 2.1], for each $t \geq 0$, $\tilde{\mathbb{P}}$-a.s., $\varepsilon L^\varepsilon(\varepsilon^{-1}x, \varepsilon^{-2}t) \xrightarrow{\varepsilon\to 0} \ell(x,t)$ uniformly in x. Thus for any $\eta > 0$ there exists ε_η such that, if $\varepsilon < \varepsilon_\eta$

we will have that $\varepsilon L^\varepsilon(\varepsilon^{-1}\cdot, \varepsilon^{-2}t) \le \ell(\cdot,t)+\eta$. Note that $\ell(\cdot,t)+\eta$ is not compactly supported. Let $h_\eta : \mathbb{H} \to \mathbb{R}_+$ be a continuous function which for every $t \ge 0$ coincides with $\ell(\cdot,t)+\eta$ on K_t, $h_\eta(\cdot,t) \le \eta$ outside K_t, and $h_\eta(\cdot,t)$ is supported on $[\inf K_t - \eta, \sup K_t + \eta]$. Using Lemma 6.3 we find that $\varepsilon L^\varepsilon(\varepsilon^{-1}\cdot, \varepsilon^{-2}t) \le h_\eta(\cdot,t)$. Thus

$$\bar\mu^\varepsilon(U_{\varepsilon L^\varepsilon(\varepsilon^{-1}\cdot,\varepsilon^{-2}t)}) \le \bar\mu^\varepsilon(U_{h_\eta(\cdot,t)}). \tag{6.11}$$

As $\bar\mu$ is a dispersed random measure, for fixed t, $\bar\mu(\partial U_{h_\eta(\cdot,t)}) = \bar\mu(\partial U_{\ell(\cdot,t)}) = 0$, $\tilde{\mathbb{P}}$-a.s. For any $\delta > 0$ and all ε small enough (depending on δ), as $\bar\mu^\varepsilon$ converges vaguely to $\bar\mu$,

$$\bar\mu^\varepsilon(U_{h_\eta(\cdot,t)}) \le \bar\mu(U_{h_\eta(\cdot,t)}) + \delta/2. \tag{6.12}$$

For each $\delta > 0$ there exists $\eta > 0$ such that $\bar\mu(U_{h_\eta(\cdot,t)}) \le \bar\mu(U_{\ell(\cdot,t)}) + \delta/2$. Combining this with (6.10)–(6.12), we find that

$$\limsup_{\varepsilon\to 0} q(\varepsilon)\bar\phi^\varepsilon_{t\varepsilon^{-2}} = \limsup_{\varepsilon\to 0} \bar\mu^\varepsilon(U_{\varepsilon L^\varepsilon(\varepsilon^{-1}\cdot,\varepsilon^{-2}t)}) \le \phi[\bar\mu, B]_t. \tag{6.13}$$

A lower bound can be obtained in a similar way. Hence, after taking union over $0 \le t \in \mathbb{Q}$,

$$\tilde{\mathbb{P}}[\lim_{\varepsilon\to 0} q(\varepsilon)\bar\phi^\varepsilon_{\varepsilon^{-2}t} = \phi[\bar\mu, B]_t, \forall 0 \le t \in \mathbb{Q}] = 1. \tag{6.14}$$

Since $\bar\phi^\varepsilon_t$ and $\phi[\bar\mu, B]$ are non-decreasing in t, $(q(\varepsilon)\bar\phi^\varepsilon_{\varepsilon^{-2}t})_{t\ge 0}$ converges to $(\phi[\bar\mu, B]_t)_{t\ge 0}$, $\tilde{\mathbb{P}}$-a.s. on $(D(\mathbb{R}_+), M_1)$, finishing the proof of the lemma. □

Theorem 6.1 then follows from Lemma 6.4 by repeating the arguments of the last paragraph in the proof of Theorem 6.2. □

6.2. *Convergence of trapped processes.* The class of time changed random walks is very large, and the associated convergence criteria rather general. Applying these criteria, however, requires to check the convergence of the underlying random measures, which might be complicated in many situations.

As we have seen, the underlying random measures of trapped processes (TRW, TBM) satisfy additional assumptions. This will make checking their convergence easier than in the general case.

PROPOSITION 6.5. *(i) Let μ^ε, μ be Lévy trap measures with independent increments (i.e. they are trap measures of some TBM's). Then μ^ε converges vaguely in distribution to μ, iff $\mu^\varepsilon(I \times [0,1])$ converges in distribution to*

$\mu(I\times[0,1])$ for every compact interval $I=[a,b]$ such that $\mu(\{a,b\}\times\mathbb{R}_+)=0$, $\tilde{\mathbb{P}}$-a.s.

(ii) The same holds true if $\mu^\varepsilon=\mathfrak{S}_\varepsilon(\nu^\varepsilon)$ for a family of trap measures ν^ε of some TRWs.

PROOF. We will use the well known criteria for the convergence of random measures recalled in Proposition A.2 in the appendix. When μ is a Lévy trap measure with independent increments, the distribution of $\mu([a,b]\times[c,d])$, $a,b\in\mathbb{R}$, $c,d\in\mathbb{R}_+$, is determined by the distribution of $\mu([a,b]\times[0,1])$, since by definition $\mu\langle[a,b]\rangle$ is a Lévy process. In particular, the assumptions of the proposition imply the convergence in distribution of $\mu^\varepsilon(A)$ to $\mu(A)$ for every $A\in\mathcal{A}$ where $\mathcal{A}$ is the set of all rectangles $I\times[c,d]$ with I as in the statement of the proposition and $d\geq c\geq 0$.

As $\mu\langle I\rangle$ is a Lévy process, we have $\mathcal{A}\subset\mathcal{T}_\mu$ (see (A.5) for the notation). Moreover, it is easy to see that $\mathcal{A}$ is a DC semiring. The fact that μ^ε are measures with independent increments combined with the well known criteria for vague convergence in distributions of random measures, see Proposition A.2 in the appendix, then implies claim (i).

The proof of claim (ii) is analogous. It suffices to observe that the distribution of ν^ε is determined by distributions of $\mu^\varepsilon([a,b]\times[0,1])$, $a,b\in\mathbb{R}$, as well. □

We apply this proposition in few examples.

EXAMPLE 6.6 (Stone's theorem). Let $\rho_\varepsilon\in M(\mathbb{R})$, $\varepsilon>0$, be a family of measures on $\mathbb{R}$. Assume that, as $\varepsilon\to 0$, ρ_ε converges vaguely to a measure $\rho\in M(\mathbb{R})$ whose support is $\mathbb{R}$. Set $\mu_\varepsilon=\rho_\varepsilon\otimes\mathrm{Leb}_+$, $\mu=\rho\otimes\mathrm{Leb}_+$. We have seen in Example 4.12 that μ_ε and μ are Lévy trap measures with independent increments, and that $B[\mu_\varepsilon]$ and $B[\mu]$ are a time changes of Brownian motion with speed measure ρ_ε and ρ, respectively. Let a,b be such that $\rho(\{a,b\})=0$ and thus $\mu(\{a,b\}\times\mathbb{R}_+)=0$. By vague convergence of ρ_ε to ρ, $\mu_\varepsilon([a,b]\times[0,1])\to\mu([a,b]\times[0,1])$. Also μ is a dispersed, infinite and dense random measure (because the support of ρ is $\mathbb{R}$) Therefore, by Proposition 6.5, μ_ε converges vaguely to μ, and thus, by Theorem 6.2, $B[\mu_\varepsilon]$ converges in distribution to $B[\mu]$ in $D(\mathbb{R}_+,J_1)$.

This result is well known and was originally obtained by Stone [26]. His result states that convergence of speed measures implies convergence of the corresponding time-changed Brownian Motions. Thus, Theorem 6.2 can be viewed a generalization of Stone's result.

EXAMPLE 6.7. Let μ, $Z[\mu]$ be as in Example 4.4 (a continuous-time

random walk à la Montroll-Weiss). Then, using Theorem 6.1 and Proposition 6.5, we can prove that $(\varepsilon Z[\mu]_{\varepsilon^{-2/\gamma}t})_{t\geq 0}$ converges in distribution to the FK process. (This result was previously obtained in [23].)

Indeed, let K_γ be a positive stable law of index γ. It is easy to see that μ is a trap measure corresponding to a TRW. Example 4.13 implies that FK process is a trapped Brownian motion whose corresponding trap measure μ_{FK} is Lévy. Moreover, from the fact that μ_{FK} is defined via Poisson point process whose intensity has no atoms, we see that for every $a \in \mathbb{R}$, $\mu_{\mathrm{FK}}(a \times \mathbb{R}_+) = 0$, $\tilde{\mathbb{P}}$-a.s.

To apply Proposition 6.5 we should check that $\varepsilon^{2/\gamma}\mathfrak{S}_\varepsilon(\mu)([a,b]\times[0,1])$ converges in distribution to $(b-a)^{1/\gamma}K_\gamma$. However,

$$\varepsilon^{2/\gamma}\mathfrak{S}_\varepsilon(\mu)([a,b]\times[0,1]) = \varepsilon^2 \sum_{x=a\varepsilon^{-1}}^{b\varepsilon^{-1}} \sum_{j=1}^{\varepsilon^{-1}} s_x^j, \tag{6.15}$$

where, by their definition in Example 4.4, the $(s_x^j)_{x\in\mathbb{Z},j\in\mathbb{N}}$ are i.i.d. random variables variables in the domain of attraction of the γ-stable law. The classical result on convergence of i.i.d. random variables (see e.g. [16]) yields that (6.15) converges in distribution to $(b-a)^{1/\gamma}K_\gamma$. On the other hand, it is easy to see that μ_{FK} is an infinite, dispersed and dense random measure. The convergence of processes then follows from Theorem 6.1.

We finish this section with a lemma that shows that the trap measures of TBM's are always dispersed, which simplifies checking the assumptions of Theorem 6.1

LEMMA 6.8. *Let μ be a Lévy trap measure with independent increments defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and $f \in C_0(\mathbb{R},\mathbb{R}_+)$. Then, $\mathbb{P}$-a.s.*

$$\mu(\{(x,y)\in\mathbb{H} : y = f(x)\}) = 0, \tag{6.16}$$

that is μ is a dispersed trap measure.

PROOF. Let $I^{n,i} = [(i-1)2^{-n}, i2^{-n})$, and set $m^{n,i} = \inf\{f(x) : x \in I^{n,i}\}$, $M^{n,i} = \sup\{f(x) : x \in I^{n,i}\}$. Let $B^{n,i}$ be the boxes

$$B^{n,i} := I^{n,i} \times [m^{n,i}, M^{n,i}]. \tag{6.17}$$

Then for all n, we have

$$\{(x,y)\in\mathbb{H} : y = f(x)\} \subset \bigcup_i B^{n,i}, \tag{6.18}$$

and $B^{n+1,2i-1} \cup B^{n+1,2i} \subset B^{n,i}$, which implies that $\mu(\cup_i B^{n,i})$ is nonincreasing in n.

The uniform continuity of f implies that for each $\delta > 0$, there exists n_δ, such that for each $n > n_\delta$ and $i \in \mathbb{Z}$, $M^{n,i} - m^{n,i} < \delta$. Since $\mu(B^{n,i})$ is distributed as $\mu(I^{n,i} \times [0, M^{n,i} - m^{n,i}])$, $\mu(B^{n,i})$ is stochastically dominated by $\mu(I^{n,i} \times [0, \delta])$. If $I^{n,i} \cap \operatorname{supp} f = \emptyset$, then $M^{n,i} = m^{n,i} = 0$. Hence, writing J for the 1-neighborhood of $\operatorname{supp} f$, in the stochastic domination sense,

$$\mu\Big(\bigcup_i B^{n,i}\Big) \le \mu(J \times [0, \delta]), \tag{6.19}$$

Since $\mu(J \times [0, \delta]) \xrightarrow{\delta \to 0} 0$, $\mathbb{P}$-a.s., we see that $\mu(\cup_i B^{n,i}) \xrightarrow{n \to \infty} 0$ in distribution. Together with the monotonicity of $\mu(\cup_i B^{n,i})$, this implies that the convergence holds $\mathbb{P}$-a.s. The lemma follows using (6.18). □

7. Convergence of RTRW to RTBM. In this section we give the proofs of the convergence theorems stated in Section 2. First we prove Theorem 2.7, which gives a complete characterization of the set of processes that appear as the scaling limit of such RTRWs. We then provide the proofs of Theorems 2.9, 2.11, 2.15 and 2.17. We recall that these theorems formulate criteria implying the convergence of RTRWs to several limiting processes. Remark, however, that our goal is not to characterize completely their domains of attraction. Instead of this we try to state natural criteria which can be easily checked in applications.

7.1. *Set of limiting processes.* This section contains the proof of Theorem 2.7. We need a simple lemma first.

LEMMA 7.1. *Let X be a RTRW with i.i.d. trapping landscape $\boldsymbol{\pi}$ and random trap measure μ. Assume that for some non-decreasing function $\rho(\varepsilon)$ satisfying $\lim_{\varepsilon \to 0} \rho(\varepsilon) = 0$, the processes $X^\varepsilon_\cdot := \varepsilon X_{\rho(\varepsilon)^{-1} \cdot}$ converge in distribution on $D(\mathbb{R}_+, J_1)$ to some process U satisfying the non-triviality assumption $\limsup_{t \to \infty} |U_t| = \infty$ a.s. Then the family of measures $\mu^\varepsilon := \rho(\varepsilon) \mathfrak{S}_\varepsilon(\mu)$ is relatively compact for the vague convergence in distribution.*

PROOF. By [20, Lemma 16.15], a sequence μ^ε of random measures on $\mathbb{H}$ is relatively compact for the vague convergence in distribution iff for every compact $A \subset \mathbb{H}$ the family of random variables $(\mu^\varepsilon(A))_{\varepsilon > 0}$ is tight in the usual sense.

Assume now, by contradiction, that (μ^ε) is not relatively compact. Then there exists $A \subset \mathbb{H}$ compact and $\delta > 0$ such that

$$\limsup_{\varepsilon \to 0} \mathbb{P}[\mu^\varepsilon(A) > K] \ge \delta \qquad \text{for all } K > 0. \tag{7.1}$$

Let Z be a simple random walk on $\mathbb{Z}$ independent of μ such that $X = Z[\mu]$, and let $L(\cdot,\cdot)$ be its local time. Since, uniformly in $\varepsilon \in (0,1)$, $\varepsilon U_{L(\cdot,\varepsilon^{-2}t)} \xrightarrow{t\to\infty} \mathbb{H}$, it is possible to choose t and M large such that

$$\liminf_{\varepsilon\to 0} \mathbb{P}\Big[A \subset U_{\varepsilon L(\cdot,\varepsilon^{-2}t)}, \sup_{s\le\varepsilon^{-2}t} |\varepsilon Z(s)| \le M\Big] \ge 1/2. \tag{7.2}$$

Since the simple random walk Z and μ are independent, this implies, using the identity $\rho(\varepsilon)\phi[\mu,Z]_{t\varepsilon^{-2}} = \mu^\varepsilon(U_{\varepsilon L(\cdot,\varepsilon^{-2}t)})$,

$$\limsup_{\varepsilon\to 0} \mathbb{P}\Big[\rho(\varepsilon)\phi[\mu,Z]_{t\varepsilon^{-2}} \ge K, \sup_{s\le\varepsilon^{-2}t} |\varepsilon Z(s)| \le M\Big] \ge \varepsilon/2. \tag{7.3}$$

and thus

$$\limsup_{\varepsilon\to 0} \mathbb{P}\Big[\psi[\mu,Z]_{K\rho(\varepsilon)^{-1}} \le t\varepsilon^{-2}, \sup_{s\le\varepsilon^{-2}t} |\varepsilon Z(s)| \le M\Big] \ge \varepsilon/2. \tag{7.4}$$

As $X^\varepsilon = \varepsilon Z_{\psi[\mu,Z]_{t\rho(\varepsilon)^{-1}}}$, and K is arbitrary

$$\limsup_{\varepsilon\to 0} \mathbb{P}\Big[\sup_{s<\infty} |X^\varepsilon(s)| \le M\Big] \ge \varepsilon/2, \tag{7.5}$$

which contradicts the non-triviality assumption on the limit U. □

Proof of Theorem 2.7. Let μ be the random trap measure of the RTRW X, and $\mu^\varepsilon = \rho(\varepsilon)\mathfrak{S}_\varepsilon(\mu)$. In view of Lemma 7.1 and the assumptions of the theorem, the family (μ^ε) is relatively compact. Therefore, there is a sequence ε_k tending to 0 as $k\to\infty$ such that μ^{ε_k} converges vaguely in distribution.

To show the theorem we should thus first characterize all possible limit points of random trap measures of RTRW's with i.i.d. trapping landscape.

Lemma 7.2. *Assume that μ^ε converges as $\varepsilon\to 0$ vaguely in distribution to a non-trivial random measure ν. Then one of the two following possibilities occurs:*

(i) $\rho(\varepsilon) = \varepsilon^2 L(\varepsilon)$ *for a function L slowly varying at 0, and $\nu = c\,\mathrm{Leb}_{\mathbb{H}}$, $c\in(0,\infty)$.*

(ii) $\rho(\varepsilon) = \varepsilon^\alpha L(\varepsilon)$ *for $\alpha > 2$ and a function L slowly varying at 0, and ν can be written as*

$$\nu = c_1\mu_{\mathrm{FK}}^{2/\alpha} + \mu_{\mathrm{SSBM}}^{\mathbb{F}}, \tag{7.6}$$

where $c_1 \in [0,\infty)$, $\mu_{\mathrm{FK}}^{2/\alpha}$ is the random measure corresponding to the FK process defined in Example 4.13, and $\mu_{\mathrm{SSBM}}^{\mathbb{F}}$ is the random measure of SSBM process given in Example 5.10, $\mu_{\mathrm{FK}}^{2/\alpha}$ and $\mu_{\mathrm{SSBM}}^{\mathbb{F}}$ are mutually independent. Moreover, the intensity measure $\mathbb{F}$ determining the law of $\mu_{\mathrm{SSBM}}^{\mathbb{F}}$ satisfies the scaling relation (2.8).

In the both cases the limit measure ν is dense, infinite and dispersed.

We first use this lemma to finish the proof of Theorem 2.7. By Lemmas 7.1 and 7.2, we can find a sequence ε_k tending to 0 as $k \to \infty$ such that $\mu^{\varepsilon} \to \nu$ vaguely in distribution and ν is as in (i) or (ii) of Lemma 7.2, and ν is dense, dispersed and infinite. Therefore, by Theorem 6.1, the family X^{ε} of processes converges in distribution on $D(\mathbb{R}_+, J_1)$ along the subsequence ε_k to a RTBM $X[\nu]$. As we assume that the limit $\lim_{\varepsilon\to 0} X^{\varepsilon} = U$ exists, we see that $U = X[\mu]$.

The theorem then follows from the fact, that if (i) of Lemma 7.2 occurs, then $U = X[\nu]$ is a multiple of Brownian motion and thus (i) of the theorem occurs. On the other hand, if (ii) of Lemma 7.2 occurs, then $U = X[\nu]$ is a FK-SSBM mixture with the claimed properties. □

It remains to show Lemma 7.2.

Proof of Lemma 7.2. The proof that $\rho(\varepsilon)$ must be a regularly varying function is standard: For $a > 0$, $A \in \mathcal{B}(\mathbb{H})$ bounded, observe that $\mathfrak{S}_{\varepsilon a}(\mu)(aA) = \mathfrak{S}_{\varepsilon}(\mu)(A)$. Therefore,

$$\begin{aligned} \nu(A) &= \lim_{\varepsilon\to 0} \rho(\varepsilon)\mathfrak{S}_{\varepsilon}(\mu)(A) \\ &= \lim_{\varepsilon\to 0} \frac{\rho(\varepsilon)}{\rho(a\varepsilon)}\rho(a\varepsilon)\mathfrak{S}_{a\varepsilon}(\mu)(aA) \\ &= \nu(aA)\lim_{\varepsilon\to 0} \frac{\rho(\varepsilon)}{\rho(a\varepsilon)}. \end{aligned} \tag{7.7}$$

As both $\nu(A)$ and $\nu(aA)$ are nontrivial random variables, this implies that the limit $\lim_{\varepsilon\to 0} \frac{\rho(\varepsilon)}{\rho(a\varepsilon)} = c_k$ exists and is non-trivial. The theory of regularly varying functions then yields

$$\rho(\varepsilon) = \varepsilon^{\alpha} L(\varepsilon) \tag{7.8}$$

for $\alpha > 0$ and a slowly varying function L. Inserting (7.8) into (7.7) also implies the scaling invariance of ν,

$$a^{\alpha}\nu(A) \overset{\text{law}}{=} \nu(aA), \qquad A \in \mathcal{B}(\mathbb{H}), a > 0. \tag{7.9}$$

We now need to show that ν is as in (i) or (ii). To this end we use the theory of 'random measures with symmetries' developed by Kallenberg in [19, 21]. We recall from [21, Chapter 9.1] that random measure ξ on $\mathbb{H}$ is said separately exchangeable iff for any measure preserving transformations f_1 of $\mathbb{R}$ and f_2 of $\mathbb{R}_+$

$$\xi \circ (f_1 \otimes f_2)^{-1} \overset{\text{law}}{=} \xi. \tag{7.10}$$

Moreover by [21, Proposition 9.1], to check separate exchangeability it is sufficient to restrict f_1, f_2 to transpositions of dyadic intervals in $\mathbb{R}$ or $\mathbb{R}_+$, respectively.

We claim that the limiting measure ν is separately exchangeable. Indeed, restricting ε to the sequence $\varepsilon_n = 2^{-n}$, taking $I_1, I_2 \subset \mathbb{R}$ and $J_1, J_2 \subset \mathbb{R}_+$ disjoint dyadic intervals of the same length and defining f_1, f_2 to be transposition of I_1, I_2, respectively J_1, J_2, it is easy to see, using the i.i.d. property of the trapping landscape $\boldsymbol{\pi}$ and independence of s_z^i's, that for all n large enough.

$$\rho(\varepsilon_n)\mathfrak{S}_{\varepsilon_n}(\mu) \circ (f_1 \otimes f_2)^{-1} \overset{\text{law}}{=} \rho(\varepsilon_n)\mathfrak{S}_{\varepsilon_n}(\mu). \tag{7.11}$$

Taking the limit $n \to \infty$ on both sides proves the separate exchangeability of ν.

The set of all separately exchangeable measures on $\mathbb{H}$ is known and given in [21, Theorem 9.23] which we recall ([21] treats exchangeable measures on the quadrant $\mathbb{R}_+ \times \mathbb{R}_+$, the statement and proof however adapt easily to $\mathbb{H}$).

THEOREM 7.3. *A random measure ξ on $\mathbb{H}$ is separately exchangeable iff almost surely*

$$\begin{aligned} \xi = \gamma \,\mathrm{Leb}_{\mathbb{H}} &+ \sum_k l(\alpha, \eta_k)\delta_{\rho_k, \rho'_k} + \sum_{i,j} f(\alpha, \theta_i, \theta'_j, \zeta_{ij})\delta_{\tau_i, \tau'_j} \\ &+ \sum_{i,k} g(\alpha, \theta_i, \chi_{ik})\delta(\tau_i, \sigma_{ik}) + \sum_i h(\alpha, \theta_i)(\delta_{\tau_i} \otimes \mathrm{Leb}_+) \\ &+ \sum_{j,k} g'(\alpha, \theta'_j, \chi'_{jk})\delta(\sigma'_{jk}, \tau'_j) + \sum_j h'(\alpha, \theta'_j)(\mathrm{Leb} \otimes \delta_{\tau'_j}), \end{aligned} \tag{7.12}$$

for some measurable functions $f \geq 0$ on $\mathbb{R}_+^4$, $g, g' \geq 0$ on $\mathbb{R}_+^3$, and $h, h', l \geq 0$ on $\mathbb{R}_+^2$, an array of i.i.d. uniform random variables $(\zeta_{i,j})_{i,j \in \mathbb{N}}$, some independent unit rate Poisson processes $(\tau_j, \theta_j)_j$, $(\sigma'_{ij}, \chi'_{ij})_j$, $i \in \mathbb{N}$, on $\mathbb{H}$, $(\tau'_j, \theta'_j)_j$, $(\sigma_{ij}, \chi_{ij})_j$, $i \in \mathbb{N}$ on $\mathbb{R}_+^2$, and $(\rho_j, \rho'_j, \eta_j)_j$ on $\mathbb{H} \times \mathbb{R}_+$, and an independent pair of random variables $\alpha, \gamma \geq 0$.

Ignoring for the moment the issue of convergence of the above sum, let us describe in words various terms in (7.12) to make a link to our result. For this discussion, we ignore the random variable α and omit it from the notation (later we will justify this step).

The term $\sum_k l(\eta_k)\delta_{\rho_k,\rho'_k}$ has the same law as the random measure $\sum_k z_k\delta_{x_k,y_k}$ for a Poisson point process (x_k, y_k, z_k) on $\mathbb{H}\times\mathbb{R}_+$ with intensity $dx\,dy\,\Pi_l(dz)$ where the measure Π_l is given by

$$\Pi_l(A) = \mathrm{Leb}_+(l^{-1}(A)), \qquad A \in \mathcal{B}(\mathbb{R}_+). \tag{7.13}$$

Recalling Example 4.13, this term resembles to the random measure driving the FK process, the z-component of the intensity measure being more general here.

Similarly, the terms $\sum_{i,k} g(\theta_i, \xi_{ik})\delta(\tau_i, \sigma_{ik}) + \sum_i h(\theta_i)(\delta_{\tau_i} \otimes \mathrm{Leb}_+)$ can be interpreted as the random measure $\mu^{\mathbb{F}}_{\mathrm{SSBM}}$ defined in Example 5.10: τ_i's correspond to x_i's, and $f_i = f_{h(\theta_i),\Pi_{g(\theta_i,\cdot)}}$ (recall (2.1), (7.13) for the notation). The intensity measure $\mathbb{F}$ used in the definition of SSBM is thus determined by functions h and g.

The terms with g', h' can be interpreted analogously, with the role of x-, and y-axis interchanged. Term $\gamma\,\mathrm{Leb}_{\mathbb{H}}$ will correspond to Brownian Motion component of ν (recall Example 4.12). Finally, the term containing f can be viewed as a family of atoms placed on the grid $(\tau_i)_i \times (\tau'_j)_j$; we will not need it later.

We now explain why the limiting measure ν appearing in Theorem 2.7 is less general than (7.12). The first reason comes from the fact that the trapping landscape is i.i.d. This implies that ν is not only exchangeable in the x-direction, but also that for every disjoint sets A_1, $A_2 \subset \mathbb{R}$ the processes $\nu\langle A_1\rangle$, $\nu\langle A_2\rangle$ are independent. As the consequence of this property, we see that α and γ must be a.s. constant (or f, h, h', g, g', l independent of α). We can thus omit α from the notation.

Further, this independence implies that $h' = g' = f \equiv 0$. Indeed, assume that it is not the case. Then it is easy to see that, for A_1, A_2 disjoint, the processes $\nu\langle A_1\rangle$, $\nu\langle A_2\rangle$ have a non-zero probability to have a jump at the same time. On the other hand, for every ω fixed, $\nu\langle A_1\rangle(\omega)$ and $\nu\langle A_2\rangle(\omega)$ are Lévy processes (they are limits of i.i.d. sums), and therefore, for every ω, $\tilde{\mathbb{P}}$-a.s, they do not jump at the same time, contradicting the assumption.

The previous reasoning implies that $\nu = \nu_1 + \nu_2 + \nu_3 + \nu_4$ where $\nu_1, \dots, \nu_4$ are the Brownian, FK, FIN and 'pure SSBM' component, respectively (by pure SSBM we understand SSBM with $\mathbb{F}$ supported on Laplace exponents

with $\mathtt{d} = 0$, see (2.1), cf. also Definition 2.4)

$$\begin{aligned} \nu_1 &= \gamma \operatorname{Leb}_{\mathbb{H}}, & \nu_3 &= \sum_i h(\theta_i)(\delta_{\tau_i} \otimes \operatorname{Leb}_+), \\ \nu_2 &= \sum_k l(\eta_k)\delta_{\rho_k,\rho'_k}, & \nu_4 &= \sum_{i,k} g(\theta_i, \xi_{ik})\delta(\tau_i, \sigma_{ik}). \end{aligned} \tag{7.14}$$

Observe that the functions l, g, and h are not determined uniquely by the law of ν. In particular for any measure preserving transformation f of $\mathbb{R}_+$, l and $l \circ f^{-1}$ give rise to the same law of ν, and similarly for h and $g(\theta, \cdot)$. Hence we may assume that l, h are non-increasing, and g is non-increasing in the second coordinate.

The final restriction on ν comes from its scaling invariance (7.9) and the local finiteness. To finish the proof, we should thus explore scaling properties of various components of ν.

The Brownian component ν_1 is trivial. It is scale-invariant with $\alpha = 2$. To find the conditions under which the FK component ν_2 is scale-invariant, we set $A = [0, x] \times [0, y]$ and compute the Laplace transform of $\nu_2 A$. To this end we use the formula

$$\mathbb{E}[e^{\pi f}] = \exp\big\{ -\lambda(1 - e^{-f})\big\}, \tag{7.15}$$

which holds for any Poisson point process π on a measurable space E with intensity measure $\lambda \in M(E)$ and $f : E \to \mathbb{R}$ measurable. Using this formula with $\pi = (\rho_i, \rho'_i, \eta_i)$ and $f(\rho, \rho', \eta) = \mathbb{1}_A(\rho, \rho')\lambda l(\eta)$ we obtain that

$$\mathbb{E}[e^{-\lambda \nu_2 A}] = \exp\Big\{ -xy \int_0^\infty (1 - e^{-\lambda l(\eta)}) d\eta \Big\}. \tag{7.16}$$

The scaling invariance (7.9) then yields

$$a^2 \int_0^\infty (1 - e^{-\lambda l(\eta)}) d\eta = \int_0^\infty (1 - e^{-\lambda a^\alpha l(\eta)}) d\eta, \qquad \forall \lambda, a > 0, \tag{7.17}$$

implying (together with the fact that l is non-increasing) that $l(\eta) = c' \eta^{-\alpha/2}$, for a $c' \geq 0$, $\alpha > 0$. By [21, Theorem 9.25], ν_2 is locally finite iff $\int_0^\infty (1 \wedge l(\eta)) d\eta < \infty$, yielding $\alpha > 2$. Finally, using the observation from the discussion around (7.13), we see that $\nu_2 = c \mu_{\mathrm{FK}}^{2/\alpha}$.

The component ν_3 can be treated analogously. Using formula (7.16) with $\pi = (\tau_i, \theta_i)$ and $f = \lambda y h(\theta) \mathbb{1}_{[0,x]}(\tau)$, we obtain

$$\mathbb{E}[e^{-\lambda \nu_3 A}] = \exp\Big\{ -x \int_0^\infty (1 - e^{-\lambda y h(v)}) dv \Big\}. \tag{7.18}$$

The scaling invariance and the fact that h is non-increasing then yields $h(\theta) = c\theta^{1-\alpha}$, for $c \geq 0$, $\alpha \geq 1$. Using [21, Theorem 9.25] again, ν_3 is locally finite iff $\int_0^\infty (1 \wedge h(\theta))d\theta < \infty$, implying $\alpha > 2$.

The component ν_4 is slightly more difficult as we need to deal with many Poisson point processes. Using formula (7.16) for the processes $(\sigma_{ij})_j$ and $(\chi_{ij})_j$ we get

$$\mathbb{E}[e^{-\lambda\nu_4 A}|(\theta_i),(\tau_i)] = \exp\Big\{ -\sum_i \mathbb{1}_{[0,x]}(x_i) y \int_0^\infty (1 - e^{-\lambda g(\theta_i,\chi)})d\chi \Big\}. \tag{7.19}$$

Applying (7.16) again, this time for processes (τ_i), (θ_i), then yields

$$\mathbb{E}[e^{-\lambda\nu_4 A}] = \exp\Big\{ -x \int_0^\infty \big(1 - e^{-y\int_0^\infty (1-e^{-\lambda g(\theta,\chi)})d\chi}\big)d\theta \Big\}. \tag{7.20}$$

Hence, by scaling invariance and trivial substitutions, g should satisfy

$$\begin{aligned} &\int_0^\infty \big(1 - e^{-y\int_0^\infty (1-e^{-\lambda g(\theta,\chi)})d\chi}\big)d\theta \\ &= \int_0^\infty \big(1 - e^{-y\int_0^\infty (1-e^{-\lambda a^{-\alpha} g(\theta/a,\chi/a)})d\chi}\big)d\theta \end{aligned} \tag{7.21}$$

for every $a, y, \lambda > 0$.

By [21, Theorem 9.25] once more, ν_4 is locally finite iff

$$\int \Big\{ 1 \wedge \int (1 \wedge g(\theta,\chi))d\chi \Big\} d\theta < \infty. \tag{7.22}$$

We use this condition to show that for ν_4 the scaling exponent must satisfy $\alpha > 2$. As $\alpha \geq 1$ is obvious, we should only exclude $\alpha \in [1,2]$. By (7.21) and the fact that Laplace transform determines measures on $\mathbb{R}_+$,

$$\begin{aligned} &\mathrm{Leb}_+\Big\{\theta : \int (1 - e^{-g(\theta,\chi)})d\chi \geq u\Big\} \\ &\qquad = \mathrm{Leb}_+\Big\{\theta : \int (1 - e^{-a^{-\alpha} g(\theta/a,\chi/a)})d\chi \geq u\Big\}. \end{aligned} \tag{7.23}$$

For some $c > 1$, $c^{-1}(1 \wedge x) \le 1 - e^{-x} \le 1 \wedge x$, therefore for $u \in (0,1)$

$$
\begin{aligned}
K(u) &:= \mathrm{Leb}_+\{\theta : \int (1 \wedge g(\theta,\chi))d\chi \ge u\} \\
&\ge \mathrm{Leb}_+\{\theta : \int (1 - e^{-g(\theta,\chi)})d\chi \ge u\} \\
&= \mathrm{Leb}_+\{\theta : \int (1 - e^{-a^{-\alpha} g(\theta/a,\chi/a)})d\chi \ge u\} \\
&\ge a\,\mathrm{Leb}_+\{\theta : \int (a^\alpha \wedge g(\theta,\chi))d\chi \ge c a^{\alpha-1} u\} \\
&\ge u^{-1/(\alpha-1)}\,\mathrm{Leb}_+\{\theta : \int (1 \wedge g(\theta,\chi))d\chi \ge c\} \\
&= u^{-1/(\alpha-1)} K(c),
\end{aligned}
\tag{7.24}
$$

where for the last inequality we set $a \ge 1$ so that $a^{\alpha-1}u = 1$. Using (7.24), it can be checked easily that the integral over θ in (7.22) is not finite when $\alpha \in [1,2]$, implying $\alpha > 2$.

To complete the proof of Theorem 2.7, it remains to show the scaling relation (2.8). This is easy to be done using the correspondence of $\nu_3 + \nu_4$ and $\mu^{\mathbb{F}}_{\mathrm{SSBM}}$. Indeed, let $\mu^{\mathbb{F}}_{\mathrm{SSBM}}$, $\mu_{(x_i,f_i)}$ be as in Example 5.10. By scaling considerations,

$$
a^{-\alpha}\mathfrak{S}_{a^{-1}}\mu_{(x_i,f_i)} \overset{\text{law}}{=} \mu_{(x_i/a,\sigma_a^\alpha f_i)}, \tag{7.25}
$$

from which (2.8) follows immediately.

The fact that ν is dispersed follows from Lemma 6.8, as in the both cases, (i) and (ii), ν is a trap measure of RTBM. Density of ν can be easily deduced from its scaling invariance and infinitness of ν is obvious. □

7.2. *Convergence to the Brownian Motion.* Here we present the proof of the convergence to Brownian Motion stated in Theorem 2.9. For reading the proof it is useful to recall the notation introduced when defining RTRW in Section 5.1.

Proof of Theorem 2.9. Let μ be the random trap measure of the RTRW X under consideration. We recall that s^i_z stands for the duration of the i-th visit of $X = Z[\mu]$ to $z \in \mathbb{Z}$.

We use the multidimensional individual ergodic theorem, which we recall for the sake of completeness in the appendix, Theorem A.3. We apply it for $X = \mathbb{R}_+^{\mathbb{Z}\times\mathbb{Z}}$, Q the distribution of $(s^i_z)_{z,i\in\mathbb{Z}}$ under $\mathbb{P}\otimes\tilde{\mathbb{P}}$, and $\mathcal{G}$ the cylinder field (here we extend s^i_z to negative i's in the natural way). We define

$(\theta_{i,j})_{i,j\in\mathbb{Z}} : \mathbb{R}_+^{\mathbb{Z}\times\mathbb{Z}} \to \mathbb{R}_+^{\mathbb{Z}\times\mathbb{Z}}$ via $\theta_{x,j}((s_z^i)_{z,i\in\mathbb{Z}}) = (s_{x+z}^{i+j})_{z,i\in\mathbb{Z}}$. It is clear from the construction that Q is stationary under $\theta_{x,j}$. As the trapping landscape and $(s_z^i)_i$, $z \in \mathbb{Z}$, are i.i.d., Q is ergodic with respect to every $\theta_{x,j}$ with $x \neq 0$. Hence, the invariant field is trivial. The multidimensional ergodic theorem then implies that for any two intervals $I, J \subset \mathbb{R}$

$$\frac{1}{n^2} \sum_{z:\frac{z}{n}\in I} \sum_{i:\frac{i}{n}\in J} s_z^i \xrightarrow{n\to\infty} |I||J|(\mathbb{E}\otimes\tilde{\mathbb{E}})[s_z^i] = |I||J|M, \qquad Q\text{-a.s.} \tag{7.26}$$

Therefore, $\varepsilon^2\mathfrak{S}_\varepsilon(\mu)(I\times J) \to |I||J|M$, and thus $\varepsilon^2\mathfrak{S}_\varepsilon(\mu)$ converges to $M \times \mathrm{Leb}_{\mathbb{H}}$, $\mathbb{P}\times\tilde{\mathbb{P}}$-a.s. This together with Theorem 6.1 completes the proof. □

7.3. *Convergence to the FK process.* Here we present the proof of Theorem 2.11. As usual, μ will stand for the random trap measure of the RTRW under consideration.

Proof of Theorem 2.11. To show the convergence in $P^{\boldsymbol{\pi}}$-distribution, in $\mathbb{P}$-probability, we will show the equivalent statement, see [20, Lemma 4.2]:

$$\begin{aligned}&\text{For every sequence } \varepsilon_n \text{ there exists a subsequence } \varepsilon_{n_k} \text{ such that}\\ &\text{as } k\to\infty,\ (\varepsilon X_{q_{\mathrm{FK}}(\varepsilon_{n_k})^{-1}t})_{t\geq 0} \text{ converges to the FK process with}\\ &\text{parameter } \gamma = 2/\alpha, \text{ in } P^\pi\text{-distribution, } \mathbb{P}\text{-a.s.}\end{aligned} \tag{7.27}$$

We thus fix a sequence $\varepsilon_n \to 0$ and check (7.27) for a subsequence $\varepsilon_{n_k} =: \tilde{\varepsilon}_k$ satisfying

$$\sum_{k=1}^{\infty} \tilde{\varepsilon}_k^{-3}\mathbb{E}\big[\big(1-\hat{\pi}(q_{\mathrm{FK}}(\tilde{\varepsilon}_k))\big)^2\big] < \infty. \tag{7.28}$$

By Theorem 6.1, it is sufficient to show that $\mu_{\tilde{\varepsilon}_k} := q_{\mathrm{FK}}(\tilde{\varepsilon}_k)\mathfrak{S}_{\tilde{\varepsilon}_k}(\mu)$ converges vaguely in distribution to μ_{FK}^γ, $\mathbb{P}$-a.s., where μ_{FK}^γ is the driving measure of the FK process introduced in Example 4.13, and μ is the trap measure of the RTRW X. For every given $\omega\in\Omega$, $\mu = \mu(\omega,\tilde{\omega})$ is the trap measure of a TRW. We also know that μ_{FK}^γ is Lévy and has independent increments. Therefore we can apply Proposition A.2, and only check that for every rectangle $A = [x_1,x_2]\times[y_1,y_2]$ with rational coordinates, $\mathbb{P}$-a.s, $\mu_{\tilde{\varepsilon}_k}(A) \xrightarrow{k\to\infty} \mu_{\mathrm{FK}}^\gamma(A)$ (it is easy to see that such rectangles form a DC semiring and are in $\mathcal{T}_{\mu_{\mathrm{FK}}^\gamma}$). $\mu_{\mathrm{FK}}^\gamma(A)$ has a γ-stable distribution with scaling parameter proportional to $\mathrm{Leb}_{\mathbb{H}}(A)$ and thus its Laplace exponent is $(x_2-x_1)(y_2-y_1)\lambda^\gamma$. The Laplace

transform of $\mu_\varepsilon(A)$ given ω (and thus given the trapping landscape $(\pi_z)_{z\in\mathbb{Z}}$) is easy to compute. By the independence of s_z^i's,

$$\tilde{\mathbb{E}}[e^{-\lambda\mu_\varepsilon(A)}] = \prod_{x=x_1\varepsilon^{-1}}^{x_2\varepsilon^{-1}} \hat{\pi}_x(\lambda q_{\mathrm{FK}}(\varepsilon))^{\varepsilon^{-1}(y_2-y_1)}. \tag{7.29}$$

Hence, taking the $-\log$ to obtain the Laplace exponent, we shall show that $\mathbb{P}$-a.s, for every $x_1 < x_2$, $y_1, y_2 \in \mathbb{Q}$, $0 \le \lambda \in \mathbb{Q}$,

$$\tilde{\varepsilon}_k^{-1}(y_2-y_1) \sum_{x=x_1\tilde{\varepsilon}_k^{-1}}^{x_2\tilde{\varepsilon}_k^{-1}} \big(-\log \hat{\pi}_z(\lambda q_{\mathrm{FK}}(\tilde{\varepsilon}_k))\big) \xrightarrow{k\to\infty} (y_2-y_1)(x_2-x_1)\lambda^\gamma. \tag{7.30}$$

As $\mathbb{Q}$ is countable, it is sufficient to show this for fixed x's, y's and λ. This will follow by a standard law-of-large-numbers argument as π_z's are i.i.d. under $\mathbb{P}$. To simplify the notation we set $x_1 = 0$, $x_2 = 1$; y's can be omitted trivially.

We first consider $\lambda \le 1$ and truncate. Using the monotonicity of $\hat{\pi}$, $\lambda \le 1$, and the Chebyshev inequality

$$\mathbb{P}[\sup_{0\le z\le\tilde{\varepsilon}_k^{-1}} (1-\hat{\pi}(q_{\mathrm{FK}}(\lambda\tilde{\varepsilon}_k))) \ge \tilde{\varepsilon}_k] \le \tilde{\varepsilon}_k^{-3}\mathbb{E}[(1-\hat{\pi}(q_{\mathrm{FK}}(\tilde{\varepsilon}_k)))^2]. \tag{7.31}$$

(7.28) then implies that the above supremum is smaller than $\tilde{\varepsilon}_k$ for all k large enough, $\mathbb{P}$-a.s. Hence, for all k large

$$\begin{aligned} &\tilde{\varepsilon}_k^{-1} \sum_{x=0}^{\tilde{\varepsilon}_k^{-1}} \big(-\log \hat{\pi}_z(\lambda q_{\mathrm{FK}}(\tilde{\varepsilon}_k))\big) \\ &= \tilde{\varepsilon}_k^{-1} \sum_{x=0}^{\tilde{\varepsilon}_k^{-1}} \big(-\log\big((1-\tilde{\varepsilon}_k) \vee \hat{\pi}_z(\lambda q_{\mathrm{FK}}(\tilde{\varepsilon}_k))\big)\big). \end{aligned} \tag{7.32}$$

For any $\delta > 0$ there is ε small so that

$$(1-x) \le -\log x \le (1-x) + (\tfrac{1}{2}+\delta)(1-x)^2, \quad x \in (1-\varepsilon, 1]. \tag{7.33}$$

The expectation of the right-hand side of (7.32) is bounded from above by

$$\begin{aligned} &\tilde{\varepsilon}_k^{-2}\mathbb{E}[\tilde{\varepsilon}_k \wedge (1-\hat{\pi}_z(\lambda q_{\mathrm{FK}}(\tilde{\varepsilon}_k)))] + c\tilde{\varepsilon}_k^{-2}\mathbb{E}\big[\big(\tilde{\varepsilon}_k \wedge (1-\hat{\pi}_z(\lambda q_{\mathrm{FK}}(\tilde{\varepsilon}_k)))\big)^2\big] \\ &\quad \le \tilde{\varepsilon}_k^{-2}\mathbb{E}[1-\hat{\pi}_z(\lambda q_{\mathrm{FK}}(\tilde{\varepsilon}_k))] + o(1), \end{aligned} \tag{7.34}$$

as $k \to \infty$, by (2.16). The lower bound for the expectation is then

$$\begin{aligned} &\tilde{\varepsilon}_k^{-2}\mathbb{E}[\tilde{\varepsilon}_k \wedge (1-\hat{\pi}_z(\lambda q_{\mathrm{FK}}(\tilde{\varepsilon}_k)))] \\ &\geq \tilde{\varepsilon}_k^{-2}\mathbb{E}[1-\hat{\pi}_z(\lambda q_{\mathrm{FK}}(\tilde{\varepsilon}_k))] + \tilde{\varepsilon}_k^{-2}\mathbb{P}[\hat{\pi}_z(\lambda q_{\mathrm{FK}}(\tilde{\varepsilon}_k)) \leq 1-\tilde{\varepsilon}_k]. \end{aligned} \tag{7.35}$$

The second term is again $o(1)$ as $k \to \infty$ by a similar estimate as in (7.31). Moreover,

$$\tilde{\varepsilon}_k^{-2}\mathbb{E}[1-\hat{\pi}_z(\lambda q_{\mathrm{FK}}(\tilde{\varepsilon}_k))] = \frac{\Gamma(\lambda q_{\mathrm{FK}}(\tilde{\varepsilon}_k))}{\Gamma(q_{\mathrm{FK}}(\tilde{\varepsilon}_k))} \xrightarrow{k\to\infty} \lambda^\gamma, \tag{7.36}$$

by the fact that Γ is regularly varying. Therefore the expectation of the right-hand side of (7.32) equals λ^γ.

To compute the variance of the right-hand side of (7.32), we observe that the second moment of one term is, for k large, bounded by

$$2\mathbb{E}\big[\big(\tilde{\varepsilon}_k \wedge (1-\hat{\pi}_z(\lambda q_{\mathrm{FK}}(\tilde{\varepsilon}_k)))\big)^2\big] \leq 2\mathbb{E}\big[\big(1-\hat{\pi}_z(q_{\mathrm{FK}}(\tilde{\varepsilon}_k))\big)^2\big] = o(\tilde{\varepsilon}_k^3), \tag{7.37}$$

as $k \to \infty$, by (2.16). Since the first moment of one term is $O(\tilde{\varepsilon}_k^2)$, by the previous computation, we see that the variance of the right-hand side of (7.32) is bounded by

$$C\tilde{\varepsilon}_k^{-3}\mathbb{E}\big[\big(1-\hat{\pi}_z(q_{\mathrm{FK}}(\tilde{\varepsilon}_k))\big)^2\big], \tag{7.38}$$

which is summable over k, by (7.28). This implies the strong law of large numbers for (7.32) and thus (7.30) for $\lambda \leq 1$. For $\lambda \geq 1$ (7.30) follows from the analyticity of Laplace transform. This proves (7.30) and thus the first claim of the theorem.

To prove the second claim of the theorem, it is sufficient to repeat the previous argument with $\tilde{\varepsilon}_k = k^{-1+\frac{\delta}{2}}$. From the assumption of the theorem then follows that $\varepsilon^{-4-\delta}\mathbb{E}\big[\big(1-\hat{\pi}(q_{\mathrm{FK}}(\varepsilon))\big)^2\big] = o(1)$, and thus

$$\tilde{\varepsilon}_k^{-3}\mathbb{E}\big[\big(1-\hat{\pi}(q_{\mathrm{FK}}(\tilde{\varepsilon}_k))\big)^2\big] = o(\tilde{\varepsilon}_k^{1+\delta}) = o\big(k^{(1+\delta)(1-\frac{\delta}{2})}\big), \tag{7.39}$$

and hence (7.28) holds. Therefore $\mathbb{P}$-a.s. holds along $\tilde{\varepsilon}_k$. To pass from the convergence along $\tilde{\varepsilon}_k$ to the convergence as $\varepsilon \to 0$, it is sufficient to observe that, since $\tilde{\varepsilon}_{k+1}^{-1} - \tilde{\varepsilon}_k^{-1} \xrightarrow{k\to\infty} 0$, for any rectangle A and ε small enough there is k such that $\mathfrak{S}_\varepsilon(\mu)(A) = \mathfrak{S}_{\tilde{\varepsilon}_k}(\mu)(A)$. □

7.4. *Convergence to the SSBM process.* Next we prove Theorem 2.15. Again, μ stands for the random trap measure of the RTRW X under consideration.

PROOF OF THEOREM 2.15. The proof is based on the following lemma.

LEMMA 7.4. *There exists a probability space $(\bar\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ and a family of trap measures $(\bar\mu^\varepsilon_{\bar\omega})_{\varepsilon\geq 0, \bar\omega\in\bar\Omega}$ on another probability space $(\tilde\Omega, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$ indexed by $\bar\omega \in \bar\Omega$, such that, when $\bar\mu^\varepsilon$, $\varepsilon \geq 0$, denotes the mixture of $\bar\mu^\varepsilon_{\bar\omega}$ w.r.t. $\bar{\mathbb{P}}$, the following conditions holds:*

(a) For every $\bar\omega \in \bar\Omega$ and $\varepsilon > 0$, $\bar\mu^\varepsilon_{\bar\omega}$ is a trap measure of a TRW, and $\bar\mu^0_{\bar\omega}$ is a trap measure of a TBM.
(b) For every $\varepsilon > 0$, $\bar\mu^\varepsilon$ is distributed as μ.
(c) $\bar\mu^0$ is distributed as $\mu^{\mathbb{F}}_{\mathrm{SSBM}}$.
(d) $q(\varepsilon)\mathfrak{S}_\varepsilon(\bar\mu^\varepsilon_{\bar\omega})$ converges vaguely in $\tilde{\mathbb{P}}$-distribution to $\bar\mu^0_{\bar\omega}$ as $\varepsilon \to 0$, for $\bar{\mathbb{P}}$-a.e. $\bar\omega$.

We first finish the proof of Theorem 2.15 using the previous lemma. As, by (a), $\bar\mu^0_{\bar\omega}$ is a Lévy trap measure for every $\bar\omega$, it is dispersed trap measure for every $\bar\omega$, by Lemma 6.8. By Assumption (L), μ and $\mu^{\mathbb{F}}_{\mathrm{SSBM}}$ are $\mathbb{P}\otimes\tilde{\mathbb{P}}$-infinite. Hence, due to (a)–(c) of the last lemma, $(\bar\mu^\varepsilon_{\bar\omega})_{\varepsilon\geq 0}$ are infinite measures, $\bar{\mathbb{P}}$-a.s. From the scaling relation (2.25) one further deduces that $\mathbb{F}$ is not a finite measure, so $\bar\mu^0$ is $\bar{\mathbb{P}}$-a.s. dense. Thus, we can apply Theorem 6.1 and deduce from (d) the $\bar{\mathbb{P}}$-a.s. convergence in $\tilde{\mathbb{P}}$-distribution of $(\varepsilon Z[\bar\mu^\varepsilon_{\bar\omega}]_{q(\varepsilon)^{-1}t})_{t\geq 0}$ to $(B[\bar\mu^0_{\bar\omega}]_t)_{t\geq 0}$. By (b),(c) of the last lemma, for every $\varepsilon > 0$, $Z[\mu]$ is distributed as $Z[\bar\mu^\varepsilon]$, and $B[\mu^{\mathbb{F}}_{\mathrm{SSBM}}]$ is distributed as $B[\bar\mu^0]$, this implies the claim of the theorem. □

PROOF OF LEMMA 7.4. The proof of Lemma 7.4 is split to two parts. In the first, we construct the coupling that satisfies (a)–(c) of the lemma. In the second part, we prove that this coupling satisfies the convergence claim (d).

Construction of the coupling. We consider a probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ on which we construct a Poisson point process $(x_i, v_i)_{i\in\mathbb{N}}$ on $\mathbb{R}\times(0,\infty)$ with intensity $\gamma v^{-\gamma-1}dx\,dv$. For $\omega \in \Omega_1$, we define $\rho(\omega) = \sum_{i>0} v_i\delta_{x_i}$, and $V(\omega) \in D(\mathbb{R})$ by $V_0(\omega) = 0$ and $V_b(\omega) - V_a(\omega) = \rho((a,b])(\omega)$, $a < b$, so that V is a two-sided γ-stable subordinator.

On the same probability space, we construct for every $\varepsilon > 0$ a families of non-negative random variables $(m^\varepsilon_z)_{z\in\mathbb{Z}}$, such that $(m^\varepsilon_z)_{z\in\mathbb{Z}}$ has the same distribution as $(m(\pi_z))_{z\in\mathbb{Z}}$. Similarly as in (2.19), we define $V^\varepsilon \in D(\mathbb{R})$ by

$$V^\varepsilon_x = \begin{cases} \sum_{i=1}^{\lfloor x\rfloor} m^\varepsilon_i, & x \geq 1, \\ 0, & x \in [0,1), \\ \sum_{\lfloor x\rfloor+1}^{0} m^\varepsilon_i, & x < 0. \end{cases} \tag{7.40}$$

By Assumption (HT), using Remark 2.13, $d(\varepsilon)^{-1}V^{\varepsilon}_{\varepsilon^{-1}x}$ converges in distribution on $(D(\mathbb{R}), J_1)$ to V. By Skorokhod representation theorem we may choose $(m^{\varepsilon}_z)_{z,\varepsilon}$ so that this convergence holds $\mathbb{P}_1$-a.s., and we do so.

For $\omega \in \Omega_1$ for which $\varepsilon^{1/\gamma} V^{\varepsilon}_{\varepsilon^{-1}\cdot}(\omega) \to V(\omega)$, we fix an injective mapping $I^{\varepsilon}_{\omega}(z) : \mathbb{Z} \to \mathbb{N}$ which satisfies

$$\varepsilon z^{\varepsilon}_i \xrightarrow{\varepsilon\to 0} x_i, \qquad d(\varepsilon)^{-1} m^{\varepsilon}_{z^{\varepsilon}_i} \xrightarrow{\varepsilon\to 0} v_i, \qquad \text{for every } i \in \mathbb{N}, \tag{7.41}$$

with $z^{\varepsilon}_i := (I^{\varepsilon}_{\omega})^{-1}(i)$, $i \in \mathbb{N}$, $\varepsilon > 0$. This is possible by the matching of jumps property of the J_1-topology (see e.g. [27, Section 3.3]). Remark that, as I^{ε}_{ω} is not necessarily surjective, z^{ε}_i is not defined for all i and ε. On the other hand, (7.41) implicitly requires that, for every $i \in \mathbb{N}$, z^{ε}_i is defined for all ε small enough.

To proceed with the construction we need a simple lemma.

LEMMA 7.5. *Let $(v_\varepsilon)_{\varepsilon>0}$ be such that $v_\varepsilon \to v$ as $\varepsilon \to 0$. Then*

$$\Psi_\varepsilon(\pi^{d(\varepsilon)v_\varepsilon}) \xrightarrow{\varepsilon\to 0} \mathbb{F}_v. \tag{7.42}$$

PROOF. Let $t(\varepsilon)$ be defined by $d(t(\varepsilon)) = d(\varepsilon)v_\varepsilon$ or equivalently $t(\varepsilon) := d^{-1}(d(\varepsilon)v_\varepsilon)$ (recall that d is strictly decreasing and continuous). Then, using the function σ^{α}_{a} introduced in (2.9),

$$\begin{aligned}
\Psi_\varepsilon(\pi^{d(\varepsilon)v_\varepsilon}) &= \varepsilon^{-1}\Big(1 - \int_{\mathbb{R}_+} e^{-q(\varepsilon)\lambda u}\pi^{d(\varepsilon)v_\varepsilon}(du)\Big) \\
&= \varepsilon^{-1}\Big(1 - \int_{\mathbb{R}_+} e^{-\varepsilon d(\varepsilon)^{-1}\lambda u}\pi^{d(t(\varepsilon))}(du)\Big) \\
&= \varepsilon^{-1}\Big(1 - \int_{\mathbb{R}_+} e^{-\varepsilon v_\varepsilon t(\varepsilon)^{-1} q(t(\varepsilon))\lambda u}\pi^{d(t(\varepsilon))}(du)\Big) \\
&= \frac{t(\varepsilon)}{\varepsilon}\Big(\frac{\varepsilon v_\varepsilon}{t(\varepsilon)}\Big)^{\frac{\gamma}{1+\gamma}} \sigma^{1+1/\gamma}_{\left(\frac{\varepsilon v_\varepsilon}{t(\varepsilon)}\right)^{-\frac{\gamma}{\gamma+1}}}(\Psi_{t(\varepsilon)}(\pi^{d(t(\varepsilon))})) \\
&= \Big(\frac{t(\varepsilon)}{\varepsilon}\Big)^{\frac{1}{\gamma+1}} v_\varepsilon^{\frac{\gamma}{\gamma+1}} \sigma^{1+1/\gamma}_{v^\gamma\left(\frac{\varepsilon v_\varepsilon}{t(\varepsilon)}\right)^{-\frac{\gamma}{\gamma+1}}}\big(\sigma^{1+1/\gamma}_{v^{-\gamma}}(\Psi_{t_\varepsilon}(\pi^{d(t(\varepsilon))}))\big).
\end{aligned} \tag{7.43}$$

As $d(\varepsilon)$ and thus $d^{-1}(\varepsilon)$ are regularly varying,

$$\frac{t(\varepsilon)}{\varepsilon} = \frac{d^{-1}(v_\varepsilon d(\varepsilon))}{d^{-1}(d(\varepsilon))} \xrightarrow{\varepsilon\to 0} v^{-\gamma}. \tag{7.44}$$

Hence,

$$\Big(\frac{t(\varepsilon)}{\varepsilon}\Big)^{\frac{1}{\gamma+1}} v_\varepsilon^{\frac{\gamma}{\gamma+1}} \xrightarrow{\varepsilon\to 0} 1, \tag{7.45}$$

and similarly

$$v_\varepsilon^\gamma \Big(\frac{\varepsilon v_\varepsilon}{t(\varepsilon)}\Big)^{-\frac{\gamma}{\gamma+1}} \xrightarrow{\varepsilon\to 0} 1, \tag{7.46}$$

and thus $\sigma^{1+1/\gamma}_{v_\varepsilon^\gamma \left(\frac{\varepsilon v_\varepsilon}{t(\varepsilon)}\right)^{-\frac{\gamma}{\gamma+1}}}$ converges to the identity. Assumption (L) together with $t(\varepsilon)\to 0$ and (2.25) then implies the lemma. □

The space $C(\mathbb{R}_+)$, and thus $\mathfrak{F}^* \subset C(\mathbb{R}_+)$, endowed with the topology of uniform convergence over compact sets is separable. It is a known fact that in the space $\mathfrak{F}^*$ the pointwise convergence and the uniform convergence over compact sets coincide. (Recall $\mathfrak{F}^*$ is the space of Laplace exponents. When the Laplace exponents converge pointwise to an element of $\mathfrak{F}^*$, the corresponding probability measures converge weakly, which in turns gives the uniform convergence over compacts.) We deduce that $\mathfrak{F}^*$ with the topology of pointwise convergence is also separable.

We further consider a measurable space $(\Omega_2, \mathcal{F}_2)$ and construct a probability kernel $\mathbb{P}_2$ from Ω_1 to Ω_2, and $\mathfrak{F}^*$-valued random variables $(\psi_z^\varepsilon)_{z\in\mathbb{Z},\varepsilon>0}$, $(f_i)_{i\in\mathbb{N}}$ on Ω_2 such that under $\mathbb{P}_2^\omega$ the random variables $(\psi_z^\varepsilon)_{z\in\mathbb{Z}}$ are independent for every $\varepsilon>0$, ψ_z^ε has the same distribution as $\Psi_\varepsilon(\pi_z^{m_z^\varepsilon(\omega)})$, and f_i, $i\in\mathbb{N}$, are i.i.d. with marginal $\mathbb{F}_1$. As $v_i^\varepsilon := d(\varepsilon)^{-1} m^\varepsilon_{z_i^\varepsilon} \to v_i$, by Lemma 7.5,

$$\psi^\varepsilon_{z_i^\varepsilon} \xrightarrow{\varepsilon\to 0} \sigma^{1+1/\gamma}_{v_i^{-\gamma}} f_i, \qquad \text{for all } i\in\mathbb{N}, \tag{7.47}$$

in distribution on $\mathfrak{F}^*$. Using the separability of $\mathfrak{F}^*$ and thus of $(\mathfrak{F}^*)^{\mathbb{Z}}$, by Skorokhod representation theorem, we may require that ψ_z^ε's are such that this convergence holds $\mathbb{P}_2^\omega$-a.s.

We take $\bar\Omega = \Omega_1\times\Omega_2$, $\bar{\mathcal{F}} = \mathcal{F}_1\otimes\mathcal{F}_2$ and we define $\bar{\mathbb{P}}$ to be a semi-direct product

$$\bar{\mathbb{P}}[A] = \int_{\Omega_1} \mathbb{P}_2^{\omega_1}[\{\omega_2 : (\omega_1,\omega_2)\in A\}]\mathbb{P}_1(d\omega_1). \tag{7.48}$$

For $\bar\omega = (\omega_1,\omega_2)\in\bar\Omega$ we define sequences of probability measures $(\pi_z^\varepsilon(\bar\omega))_{z\in\mathbb{Z}}$, $\varepsilon>0$, by requiring that

$$\Psi_\varepsilon(\pi_z^\varepsilon(\bar\omega)) = \psi_z^\varepsilon(\bar\omega). \tag{7.49}$$

This determines $\pi_z^\varepsilon(\bar\omega)$ uniquely, because Ψ_ε is an affine transformation of the Laplace transform. Since $(m_z^\varepsilon)_z$ has the same distribution as $(m(\pi_z))_z$

and $(\psi_z^\varepsilon)_z$ has the same distribution as $(\Psi(\pi_z^{m_z^\varepsilon}))_z$, it follows that for every $\varepsilon > 0$, $(\pi_z^\varepsilon)_z$ has the same distribution as $(\pi_z)_z$.

Finally, we set $\bar{\mu}_{\bar{\omega}}^\varepsilon$ to be the trap measure on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with the trapping landscape $(\pi_z^\varepsilon(\bar{\omega}))_z$, and define $\bar{\mu}^\varepsilon$ to be mixture of $\bar{\mu}_{\bar{\omega}}^\varepsilon$ w.r.t. $\bar{\mathbb{P}}$. From the previous discussion it is obvious that $\bar{\mu}^\varepsilon$ satisfy (a),(b) of the Lemma 7.4. We define

$$\bar{\mu}_{\bar{\omega}}^0 = \mu_{(x_i(\omega_1), \sigma_{v_i(\omega_1)^{-\gamma}}^{1+1/\gamma} f_i(\omega_1))} \tag{7.50}$$

(see Example 5.10 for the notation) and set $\bar{\mu}^0$ to be mixture of $\bar{\mu}_{\bar{\omega}}^0$ w.r.t. $\bar{\mathbb{P}}$. The measure $\bar{\mu}^0$ clearly satisfies (a),(c) of the lemma.

$\bar{\mathbb{P}}$-a.s. convergence of $\bar{\mu}_{\bar{\omega}}^\varepsilon$. We need to show that $\bar{\mathbb{P}}$-a.s., the trap measures $\rho(\varepsilon)\mathfrak{S}_\varepsilon(\bar{\mu}_{\bar{\omega}}^\varepsilon)$ converge to the Lévy trap measure $\bar{\mu}_{\bar{\omega}}^0$ vaguely in distribution. Using Proposition 6.5, it is sufficient to check that

$$\rho(\varepsilon)\mathfrak{S}_\varepsilon(\bar{\mu}_{\bar{\omega}}^\varepsilon)(I \times [0,1]) \xrightarrow{\varepsilon \to 0} \bar{\mu}_{\bar{\omega}}^0(I \times [0,1]) \tag{7.51}$$

$\bar{\mathbb{P}}$-a.s., in distribution, for every interval $I = [a, b]$ whose boundary points are not in the set $\{x_i : i \in \mathbb{N}\}$. Computing Laplace transforms, and taking $-\log$, the last display is equivalent to

$$-\sum_{z : z\varepsilon \in I} \varepsilon^{-1} \log\left(\hat{\pi}_z^\varepsilon(\bar{\omega})(\rho(\varepsilon)\lambda)\right) \xrightarrow{\varepsilon \to 0} \sum_{i : x_i \in I} \sigma_{v_i(\bar{\omega})^{-\gamma}}^{1+1/\gamma} f_i(\bar{\omega})(\lambda), \tag{7.52}$$

for all $\lambda \geq 0, \bar{\mathbb{P}}$-a.s.

We fix $\delta, \delta' > 0$ (depending on $\bar{\omega}$) such that

$$\sum_{i : x_i \in I} v_i \mathbb{1}\{v_i \leq \delta'\} \leq \delta. \tag{7.53}$$

This is always possible as V is an increasing pure jump process and $V(b) - V(a)$ is $\bar{\mathbb{P}}$-a.s. finite. We define a finite set $J := \{i : x_i \in I, v_i > \delta'\}$. We consider ε small enough so that z_i^ε is defined for all $i \in J$, and set $J^\varepsilon = \{z_i^\varepsilon : i \in J\}$. We consider separately the sum over J and its complement.

We start with the sum over J. Observe that as the boundary points of I are not in $\{x_i\}_i$, $\varepsilon z_i^\varepsilon \in I$ for all ε small enough. By the coupling construction, more precisely by (7.47) and (7.49), using that J is finite and some elementary analysis, we see that for δ, δ' fixed, $\bar{\mathbb{P}}$-a.s.,

$$-\sum_{z \in J^\varepsilon} \varepsilon^{-1} \log\left(\hat{\pi}_z^\varepsilon(\bar{\omega})(\rho(\varepsilon)\lambda)\right) \xrightarrow{\varepsilon \to 0} \sum_{i \in J} \sigma_{v_i(\bar{\omega})^\gamma}^{1+1/\gamma} f_i(\bar{\omega})(\lambda), \qquad \forall \lambda \geq 0, \bar{\mathbb{P}}\text{-a.s.} \tag{7.54}$$

The contribution of $i \notin J$ might be neglected on the right-hand side of (7.52). Indeed, by Remark 2.14 and (7.53),

$$0 \le \sum_{i\notin J: x_i \in I} \sigma_{v_i(\bar\omega)^{-\gamma}}^{1+1/\gamma} f_i(\bar\omega)(\lambda) = \sum_{i\notin J: x_i\in I} v_i^{-\gamma} f_i(v_i^{\gamma+1}\lambda) \le \lambda \sum_{i\notin J: x_i\in I} v_i \le \lambda\delta. \tag{7.55}$$

Finally, the contribution of the sum over $z \notin J^\varepsilon$ on the left-hand side of (7.52) is asymptotically negligible. Indeed, as J is finite, $\varepsilon^{1/\gamma} m^\varepsilon_{z_i^\varepsilon} \to v_i$ for every $i \in J$, and $\varepsilon^{1/\gamma} V^\varepsilon_{\varepsilon^{-1}\cdot}$ converges to V, it follows that for ε small enough

$$d(\varepsilon)^{-1} \sum_{z\in\varepsilon^{-1}I\setminus J^\varepsilon} m_z^\varepsilon \le 2\delta. \tag{7.56}$$

It follows that $m_z^\varepsilon \le 2\delta d(\varepsilon)$ and thus $m_z^\varepsilon \rho(\varepsilon) \xrightarrow{\varepsilon\to 0} 0$, for every $z \notin J^\varepsilon$. From $m(\pi_z^\varepsilon) = m_z^\varepsilon$, it follows that $\hat\pi_z^\varepsilon(\rho(\varepsilon)\lambda) \ge 1 - m_z^\varepsilon\rho(\varepsilon)\lambda$. Using the inequality $-\log x \le 2(1-x)$ which holds in some interval $(c, 1]$, we obtain

$$\begin{aligned} 0 &\le -\sum_{z\in\varepsilon^{-1}I\setminus J^\varepsilon} \varepsilon^{-1} \log\big(\hat\pi_z^\varepsilon(\bar\omega)(\rho(\varepsilon)\lambda)\big) \le 2 \sum_{z\in\varepsilon^{-1}I\setminus J^\varepsilon} \varepsilon^{-1} m_z^\varepsilon \rho(\varepsilon)\lambda \\ &= 2\lambda\varepsilon^{1/\gamma} \sum_{z\in\varepsilon^{-1}I\setminus J^\varepsilon} m_z^\varepsilon \le 4\lambda\delta, \end{aligned} \tag{7.57}$$

by (7.53) again. This finishes the proof. □

7.5. *Convergence to FIN.* Since the FIN diffusion is a special case of the SSBM (see Definition 2.4), we can specialize Theorem 2.15 to obtain criteria for the convergence of a rescaled RTRW with i.i.d. trapping landscape is the FIN diffusion. Here we present the proof of such convergence as stated in Theorem 2.17. We recall that μ is a trapping measure of a RTRW $X = Z[\mu]$ with an i.i.d. random trapping landscape $\boldsymbol{\pi}$ whose marginal is P.

Proof of Theorem 2.17. Due to Definition 2.4 and the scaling property (2.24), we only need to verify Assumption (L) with $\mathbb{F}_1 = \delta_{\lambda\mapsto\lambda}$. For all positive x, it holds that $x - \frac{x^2}{2} \le 1 - e^{-x} \le x$. Inserting this inequality in the definition of Ψ_ε, we obtain

$$\varepsilon^{-1}\big(\lambda q(\varepsilon) m(\pi^{d(\varepsilon)}) - \tfrac12 q(\varepsilon)^2\lambda^2 m_2(\pi^{d(\varepsilon)})\big) \le \Psi_\varepsilon(\pi^{d(\varepsilon)})(\lambda) \le \varepsilon^{-1}\lambda q(\varepsilon) m(\pi^{d(\varepsilon)}). \tag{7.58}$$

Taking the limit $\varepsilon \to 0$ in this inequality, recalling $q(\varepsilon) = \varepsilon d(\varepsilon)^{-1}$, we obtain using the assumptions of the theorem,

$$\lim_{\varepsilon\to 0} \Psi_\varepsilon(\pi^{d(\varepsilon)})(\lambda) = \lambda \tag{7.59}$$

in distribution. This completes the proof. □

8. Applications. In this section we make use of the previously developed theory to prove Theorems 3.2 and 3.5.

8.1. *The simplest case of a phase transition.* Recall from Definition 3.1, that the trap model with transparent traps is defined using two positive parameters α, β, a family $(\tau_x)_{x\in\mathbb{Z}}$ of i.i.d. random variables satisfying $\tau_x > 1$ and

$$\lim_{u\to\infty} u^{\alpha}\mathbb{P}(\tau_0 > u) = c \in (0,\infty), \tag{8.1}$$

and its i.i.d. trapping landscape $\boldsymbol{\pi} = (\pi_x)_{x\in\mathbb{Z}}$, where

$$\pi_x(\omega) := (1-\tau_x(\omega)^{-\beta})\delta_1 + \tau_x(\omega)^{-\beta}\delta_{\tau_x(\omega)}. \tag{8.2}$$

In words, given τ_x's, at site x the walk is trapped for time τ_x with probability $\tau_x^{-\beta}$, otherwise it spends just a unit time at x. Here we present the proof of Theorem 3.2.

REMARK 8.1. For the sake of simplicity, during the computations we will replace the traps $\pi_x := (1-\tau_x^{-\beta})\delta_1 + \tau_x^{-\beta}\delta_{\tau_x}$ by $(1-\tau_x^{-\beta})\delta_0 + \tau_x^{-\beta}\delta_{\tau_x}$. It should be clear that the asymptotics should be the same in both cases.

PROOF. Directly from the definition of the model, $m(\pi_z(\omega)) = \tau_z(\omega)^{1-\beta}$, and thus

$$\lim_{x\to\infty} x^{\frac{\alpha}{1-\beta}}\mathbb{P}[m(\pi_z) \geq x] = 1. \tag{8.3}$$

When $\alpha+\beta > 1$, $m(\pi_z)$ has finite expectation, and Theorem 2.9 yields claim (i).

For claims (ii) and (iv), condition (HT) is verified due to (8.3). The function $d(\varepsilon)$ introduced in Remark 2.13 may be chosen to be $d(\varepsilon) = \varepsilon^{-1/\gamma}$. Conditioning on $m(\pi_0) = d(\varepsilon)$ is equivalent to conditioning on $\tau_0^{1-\beta} = \varepsilon^{-1/\gamma}$, which, in turn, is equivalent to $\tau_0 = \varepsilon^{-1/\alpha}$. Hence, conditionally on $m(\pi_0) = d(\varepsilon)$, π_0 is deterministic probability measure $\pi_z^{d(\varepsilon)} = (1-\varepsilon^{\beta/\alpha})\delta_0 + \varepsilon^{\beta/\alpha}\delta_{\varepsilon^{-1/\alpha}}$, and

$$\hat{\pi}^{d(\varepsilon)}(\lambda) = 1 - \varepsilon^{\beta/\alpha} + \varepsilon^{\beta/\alpha}\exp(-\lambda\varepsilon^{-1/\alpha}). \tag{8.4}$$

Therefore, $\Psi_\varepsilon(\hat{\pi}^{d(\varepsilon)})$ is deterministic,

$$\Psi_\varepsilon(\hat{\pi}^{d(\varepsilon)})(\lambda) = \varepsilon^{(\beta-\alpha)/\alpha}(1-\exp(-\lambda\varepsilon^{(\alpha-\beta)/\alpha})). \tag{8.5}$$

When $\alpha+\beta<1$ and $\alpha>\beta$, this implies $\lim_{\varepsilon\to 0}\Psi_\varepsilon(\hat\pi^{d(\varepsilon)})(\lambda)=\lambda$. Hence condition (L) is verified, and Theorem 2.15 together with Definition 2.4 yields claim (ii).

Similarly, when $\alpha+\beta<1$ and $\alpha=\beta$, $\lim_{\varepsilon\to 0}\Psi_\varepsilon(\hat\pi^{d(\varepsilon)})(\lambda)=1-\exp(-\lambda)$, which implies (iv). Observe that in this case, the traps are "Poissonian" in the sense that $\mathbb{F}_1$ is concentrated on $\lambda\mapsto 1-\exp(-\lambda)$, which is the Laplace exponent of a Poisson process.

When $\alpha+\beta<1$ and $\alpha<\beta$, $\Psi_\varepsilon(\hat\pi^{d(\varepsilon)})$ converges to 0, indicating that Theorem 2.11 should be used instead of Theorem 2.15. Recall that $\Gamma(\varepsilon)=\mathbb{E}(1-\hat\pi(\varepsilon))$. We will first show that $\Gamma(\varepsilon)$ is regularly varying of index κ at $\varepsilon=0$. Let ν be the distribution of τ_0. Then

$$\mathbb{E}(1-\hat\pi(\varepsilon))=\int_0^\infty t^{-\beta}(1-\exp(-\varepsilon t))\nu(dt). \tag{8.6}$$

Changing variables we obtain

$$\mathbb{E}(1-\hat\pi(\varepsilon))=\varepsilon^\beta\int_0^\infty t^{-\beta}(1-\exp(-t))\nu(\varepsilon^{-1}dt). \tag{8.7}$$

By (8.1), $\varepsilon^{-\alpha}\nu(\varepsilon^{-1}dt)$ converges weakly to $c\alpha t^{-1-\alpha}dt$. After a simple calculation this yields

$$\mathbb{E}(1-\hat\pi(\varepsilon))=c\alpha\varepsilon^{\alpha+\beta}\int_0^\infty t^{-1-\alpha-\beta}(1-\exp(-u))du(1+o(1)). \tag{8.8}$$

The integral on the right hand side is finite, so the condition (2.15) (cf. also (2.17)) of Theorem 2.11 is verified with $q_{\mathrm{FK}}(\varepsilon)=\varepsilon^{2/\kappa}$. Similarly,

$$\begin{aligned}\mathbb{E}((1-\hat\nu(\varepsilon))^2)&=\alpha\int_0^\infty t^{-2\beta}(1-\exp(-\varepsilon t))^2\nu(dt)\\&\sim\alpha\varepsilon^{2\beta+\alpha}\int_0^\infty u^{-2\beta-1-\alpha}(1-\exp(-u))^2\,du\end{aligned} \tag{8.9}$$

Leading to

$$\varepsilon^{-3}\mathbb{P}((1-\hat\pi(q_{\mathrm{FK}}(\varepsilon)))^2)\to 0. \tag{8.10}$$

Hence, the assumptions of Theorem 2.11 are fulfilled, and claim (iii) holds. □

8.2. *The comb model.* In this section we give the proof of Theorem 3.5.

PROOF OF THEOREM 3.5. To proof the theorem we need to control the distribution of the time that the simple random walk Y^{comb} spends in the teeth of the comb. Therefore, for $N \geq 1$, we let $V^N = (V_k^N)_{k\geq 0}$ to stand for a random walk on $0, \dots, N$ with drift $g(N)$, reflection on N, started from $V_0^N = 1$. Let $\tau^N = \inf\{n \geq 0, V_n = 0\}$ be the hitting time of 0 by V^N, and let θ^N be the law of τ^N.

It is easy to see that the distribution π_z of the time that X^{comb} spends on one visit to z coincides with the law of $\sum_{i=0}^{G}(1+\xi_i^z)$, where ξ_i^z are i.i.d. with distribution θ^{N_z}, and G is a geometric random variable with parameter $\frac{2}{3}$, $\mathbb{P}[G = k] = \frac{2}{3}(\frac{1}{3})^k$, $k \geq 0$. In particular,

$$m(\pi_z) = (1 + m(\theta^{N_z}))/2, \qquad \text{and} \qquad \hat{\pi}_z(\lambda) = \frac{2e^{-\lambda}}{3 - \hat{\theta}^{N_z}(\lambda)}, \tag{8.11}$$

and thus

$$1 - \hat{\pi}_z(\lambda) = (\lambda + \big(1 - \hat{\theta}^{N_z}(\lambda)\big)/2\big)(1 + o(1)), \qquad \text{as } \lambda \to 0. \tag{8.12}$$

The distribution θ^N is characterized by the following lemma.

LEMMA 8.2. *Let $p = (1 + g(N))/2$, $\xi = (1 - p)/p$, and*

$$\chi = \frac{1 + \sqrt{1 - 4s^2p(1-p)}}{2sp}. \tag{8.13}$$

Then, the generating function of θ^N is given by

$$\hat{\theta}^N(-\log s) = \mathbb{E}[s^{\tau^N}] = \frac{\xi\chi^{2N-2}(\chi - s) + \xi^{N-1}\chi(s\chi - \xi)}{\chi^{2N-1}(\chi - s) + \xi^{N-1}(s\chi - \xi)}. \tag{8.14}$$

PROOF. The proof is a standard one-dimensional random walk computation. Writing $f_x(s) = \mathbb{E}[s^{\tau^N}|V_0 = x]$ for the generating function of τ^N for the random walk starting at x (i.e. $\hat{\theta}^N(-\log s) = f_1(s)$), we have the equation

$$f_x(s) = spf_{x+1}(s) + s(1-p)f_{x-1}(s), \qquad \text{for } 1 \leq x \leq N-1, \tag{8.15}$$

with the boundary conditions $f_0(s) = 1$, and $f_N(s) = sf_{N-1}(s)$. Solving this system we obtain

$$f_x(s) = A_+(s)\lambda_+(s)^x + A_-(s)\lambda_-(s)^x, \tag{8.16}$$

with $\lambda_+(s) = \chi$, $\lambda_-(s) = \xi/\chi$ and

$$A_+(s) = \frac{-\lambda_-(s)^{N-1}(\lambda_-(s) - s)}{\lambda_+(s)^{N-1}(\lambda_+(s) - s) - \lambda_-(s)^{N-1}(\lambda_-(s) - s)}, \tag{8.17}$$

$$A_-(s) = \frac{\lambda_+(s)^{N-1}(\lambda_+(s)-s)}{\lambda_+(s)^{N-1}(\lambda_+(s)-s) - \lambda_-(s)^{N-1}(\lambda_-(s)-s)}. \tag{8.18}$$

A simple rearrangement yields the claim. □

Knowing the generating function, the moments of θ^N can be obtained easily. We collect the asymptotic behavior of the first and second moments in the following lemma. Its proof is an easy asymptotic analysis of the derivatives of the generating function of θ^N and is omitted.

LEMMA 8.3. *When $\beta > 0$, as $N \to \infty$, the first and second moment of θ^N satisfy*

$$m(\theta^N) \sim \frac{N^{2\beta+1}}{\beta \log(N)}, \qquad m_2(\theta^N) \sim \frac{N^{3+4\beta}}{\beta^3 \log^3(N)}, \tag{8.19}$$

where $f \sim g$ as $N \to \infty$ means $\lim_{N\to\infty} f/g = 1$. Moreover, when $\beta = 0$, then $m(\theta^N) \sim 2N$.

We further need asymptotics of $1 - \hat{\theta}^N(\varepsilon)$ as $\varepsilon \to 0$ for large (possibly diverging) N. This is the content of the next two lemmas.

LEMMA 8.4. *When $\beta = 0$, then there is $c > 0$, such that for all $N \geq 1$ and $\varepsilon \in (0, 1/2)$*

$$1 - \hat{\theta}^N(\varepsilon) \leq c\big(1 \wedge (e^{N\sqrt{2\varepsilon}} - 1)\big). \tag{8.20}$$

Moreover, for $y > 0$,

$$\frac{(1 - \hat{\theta}^{\lfloor y/\sqrt{2\varepsilon}\rfloor}(\varepsilon))}{\sqrt{2\varepsilon}} \xrightarrow{\varepsilon\to 0} \tanh(y). \tag{8.21}$$

PROOF. From (8.14), we obtain

$$1 - \hat{\theta}^N(-\log s) = \frac{(\chi-\xi)\chi^{2N-2}(\chi - s) + \xi^{N-1}(1-\chi)(s\chi - \xi)}{\chi^{2N-1}(\chi - s) + \xi^{N-1}(s\chi - \xi)}. \tag{8.22}$$

When $\beta = 0$, then $\xi = 1$ and $\chi = (1 + \sqrt{1-s^2})/s$. Therefore, setting $s = e^{-\varepsilon} \sim 1 - \varepsilon$, we find as $\varepsilon \to 0$,

$$\chi - 1 \sim \sqrt{2\varepsilon}. \tag{8.23}$$

This together with (8.22), implies

$$1 - \hat{\theta}^N(\varepsilon) \sim \sqrt{2\varepsilon}\, \frac{(1+\sqrt{2\varepsilon})^{2N} - 1}{(1+\sqrt{2\varepsilon})^{2N} + 1}. \tag{8.24}$$

This yields the both claims of the lemma. □

LEMMA 8.5. *When $\beta > 0$, set*

$$u(\varepsilon) = \varepsilon^{-1/(2+2\beta)} \log^{1/(1+\beta)}(\varepsilon^{-1}). \tag{8.25}$$

Then, for a constant $c < \infty$,

$$1 - \hat{\theta}^N(\varepsilon) \leq \begin{cases} \varepsilon m(\theta^N), & N < u(\varepsilon)^{1/2}, \\ cg, & u(\varepsilon)^{1+\frac{\beta}{2}} < N < u(\varepsilon)^{1+\beta}, \\ c\sqrt{\varepsilon}, & N > u(\varepsilon)^{1+\beta}, \end{cases} \tag{8.26}$$

Moreover, setting

$$v(N, \varepsilon) = \frac{2\beta N^{1+2\beta} \log N}{N^{2+2\beta} + 2\beta^2 \varepsilon^{-1} \log^2 N} \tag{8.27}$$

we have

$$\sup_{u(\varepsilon)^{1/2} \leq N \leq u(\varepsilon)^{1+\frac{\beta}{2}}} \left| \frac{1 - \hat{\theta}^N(\varepsilon)}{v(N, \varepsilon)} - 1 \right| \xrightarrow{\varepsilon \to 0} 0. \tag{8.28}$$

PROOF. The first line of (8.26) follows from the fact that $1 - \hat{\nu}(\lambda) \leq \lambda m(\nu)$ for every probability distribution ν supported on $[0, \infty)$.

For the remaining parts of (8.26), observe that

$$\hat{\theta}^N(-\log s) \geq \xi/\chi. \tag{8.29}$$

To see that this inequality holds, it is sufficient to replace $\hat{\theta}^N(-\log s)$ by the right-hand side of (8.14), multiply the inequality by the denominator (which is always positive) and observe that $\chi \geq 1 \geq \xi$. Using (8.29),

$$1 - \hat{\theta}^N(-\log s) \leq (\chi - \xi)/\chi \leq \chi - \xi. \tag{8.30}$$

Moreover, for $s = e^{-\varepsilon} \sim 1 - \varepsilon$ and $\beta > 0$ that is $g = g(N) \neq 0$, we have $1 - \xi \sim 2g$ as $g \to 0$, and

$$\chi - 1 = \frac{\varepsilon - g + \varepsilon g + \sqrt{2\varepsilon - \varepsilon^2 + s^2 g^2}}{s(1+g)}. \tag{8.31}$$

Therefore, after some computations, as $\varepsilon \to 0$,

$$\chi - 1 \sim \begin{cases} \sqrt{2\varepsilon}, & \text{when } g^2 \ll \varepsilon, \\ \frac{\varepsilon}{g} & \text{when } 1 \gg g^2 \gg \varepsilon, \end{cases} \tag{8.32}$$

and in general for some $c < \infty$

$$\chi - 1 \le c\big(\sqrt{\varepsilon} + \tfrac{\varepsilon}{g}\big). \tag{8.33}$$

Going back to (8.30), this implies that

$$1 - \hat{\theta}^N(-\log s) \le c(\sqrt{\varepsilon} + \tfrac{\varepsilon}{g} + g). \tag{8.34}$$

Observing further that when $N = u(\varepsilon)^{1+\beta}$, then g^2 is comparable with ε, the rest of (8.26) follows.

Finally, to show (8.28), observe that uniformly over N in the considered regime (i.e. in the same sense as in (8.28)), $\chi - 1 \sim \varepsilon/g$ by (8.31), $1 - \xi \sim 2g$, and thus $\xi^{N-1} \sim N^{-2\beta}$, $\chi^{2N-1} \sim 1$ as $\varepsilon/g \ll N^{-1}$. Inserting these observations into (8.22), (8.28) follows. □

We can now proceed with the proof of Theorem 3.5. From (8.11) and Lemma 8.3, it follows that for $\beta \ge 0$,

$$\mathbb{P}[m(\pi_0) \ge x] = x^{-\gamma} L(x), \tag{8.35}$$

for $\gamma = \alpha/(1 + 2\beta)$ and a slowly varying function L. This implies that $\mathbb{E}[m(\pi_0)]$ is finite for $\alpha > 1 + 2\beta$, and claim (i) follows by applying Theorem 2.9.

To show claim (ii), we observe that Lemma 8.3 implies that $m(\theta^N)^{2+\gamma} \gg m_2(\theta^N)$ as $N \to \infty$, which is sufficient to check the assumptions of Theorem 2.17.

For claim (iii), we need to check the assumptions of Theorem 2.11. Using (8.11), and dominated convergence

$$\Gamma(\varepsilon) = \mathbb{E}(1 - \hat{\pi}_0(\varepsilon)) \sim \varepsilon + \frac{1}{2\mathcal{Z}} \sum_{N=1}^{\infty} N^{-1-\alpha}(1 - \hat{\theta}^N(\varepsilon)). \tag{8.36}$$

We now discuss separately the cases $\beta = 0$ and $\beta > 0$.

When $\beta = 0$, choosing $\delta > 0$ small, using the second claim of Lemma 8.4, and the change of variables $y = \sqrt{2\varepsilon} N$ we obtain

$$\begin{aligned} \sum_{N=\delta\sqrt{2\varepsilon}}^{\delta^{-1}\sqrt{2\varepsilon}} N^{-1-\alpha}(1 - \hat{\theta}^N(\varepsilon)) &\sim \int_{\delta}^{\delta^{-1}} (2\varepsilon)^{\frac{1+\alpha}{2}} y^{-1-\alpha}(1 - \hat{\theta}^{\lfloor y/\sqrt{2\varepsilon}\rfloor}(\varepsilon))(2\varepsilon)^{-1/2} dy \\ &\sim (2\varepsilon)^{\frac{1+\alpha}{2}} \int_{\delta}^{\delta^{-1}} y^{-1-\alpha} \tanh(y) dy. \end{aligned} \tag{8.37}$$

The first claim of Lemma 8.4 can be then used to justify that the remaining part of the sum is bounded from above by $c(\delta)\varepsilon^{\frac{1+\alpha}{2}}$ for some $c(\delta) \to 0$ as $\delta \to 0$. As the integral on the right-hand side of (8.37) converges, we have proved that $\Gamma(\varepsilon)$ is regularly varying with index $\kappa = (1+\alpha)/2$, that is (2.17) and thus (2.15) holds for $q(\varepsilon) = \varepsilon^{2/\kappa}$.

Repeating the same line of reasoning we obtain

$$\varepsilon^{-3}\mathbb{E}((1-\hat{\nu}(q(\varepsilon)))^2) \sim c\varepsilon^{(1-\kappa)/\kappa}. \tag{8.38}$$

Hence, the second assumption of Theorem 2.11 is verified and claim (iii) is proved for $\beta = 0$.

We follow similar steps in the case $\beta > 0$, using the estimates from Lemma 8.5. We first get using the first part of (8.26) and Lemma 8.3

$$\begin{aligned}\sum_{N=1}^{u(\varepsilon)^{1/2}} N^{-1-\alpha}(1-\hat{\theta}^N(\varepsilon)) &\le c\sum_{N=1}^{u(\varepsilon)^{1/2}} N^{-1-\alpha}\varepsilon\frac{N^{2\beta+1}}{\beta\log N}\\ &\le \varepsilon^{(2\beta+\alpha+3)/(2\beta+2)}L(\varepsilon) \ll \varepsilon^{\kappa}.\end{aligned} \tag{8.39}$$

where L is a slowly varying function and $\kappa = \frac{1+\alpha}{2\beta+2}$, as in the theorem. Further, by the second part of (8.26),

$$\begin{aligned}\sum_{N=u(\varepsilon)^{1+\frac{\beta}{2}}}^{u(\varepsilon)^{1+\beta}} N^{-1-\alpha}(1-\hat{\theta}^N(\varepsilon)) &\le \sum_{N=u(\varepsilon)^{1+\frac{\beta}{2}}}^{u(\varepsilon)^{1+\beta}} cN^{-2-\alpha}\beta\log N\\ &\le (\varepsilon^{\kappa})^{\frac{2+\beta}{2}}L(\varepsilon) \ll \varepsilon^{\kappa},\end{aligned} \tag{8.40}$$

and by the third part of (8.26),

$$\sum_{N=u(\varepsilon)^{1+\beta}}^{\infty} N^{-1-\alpha}(1-\hat{\theta}^N(\varepsilon)) \le c\sqrt{\varepsilon}\sum_{N=u(\varepsilon)^{1+\beta}}^{\infty} N^{-1-\alpha} \le (\varepsilon^k)^{\frac{1}{1+\beta}}L(\varepsilon) \ll \varepsilon^{\kappa}. \tag{8.41}$$

Using (8.28), we then get for the remaining part of the sum

$$\sum_{N=u(\varepsilon)^{1/2}}^{u(\varepsilon)^{1+\frac{\beta}{2}}} N^{-1-\alpha}(1-\hat{\theta}^N(\varepsilon)) \sim \sum_{N=u(\varepsilon)^{1/2}}^{u(\varepsilon)^{1+\frac{\beta}{2}}} \frac{2\beta N^{2\beta-\alpha}\log N}{N^{2+2\beta}+2\beta^2\varepsilon^{-1}\log^2 N}. \tag{8.42}$$

Substituting $N = u(\varepsilon)y$, an easy analysis yields

$$\sim \int_{u(\varepsilon)^{-1/2}}^{u(\varepsilon)^{\beta}} \frac{2\beta u(\varepsilon)^{2\beta-\alpha+1}y^{2\beta-\alpha}\log(u(\varepsilon)y)}{u(\varepsilon)^{2(1+\beta)}y^{2(1+\beta)}+2\beta^2\varepsilon^{-1}\log(u(\varepsilon)y)} \sim \varepsilon^{\kappa}L(\varepsilon). \tag{8.43}$$

Combining all the parts of the sum yields $\Gamma(\varepsilon) = \varepsilon^{\kappa} L(\varepsilon)$, that is the first assumption of Theorem 2.11 is satisfied with $q(\varepsilon) = \varepsilon^{2/\kappa}\tilde{L}(\varepsilon)$. Analogously it can be shown that (2.16) holds. Claim (iii) for $\beta > 0$ then follows from Theorem 2.11. This completes the proof. □

APPENDIX A: RANDOM MEASURES

In this appendix we collect frequently used notation and recall few known theorems from the theory of random measures.

For any topological space E, $\mathcal{B}(E)$ stands for the Borel σ-field of E. We write $M(E)$ for the set of positive Radon measures on E, that is for the set of positive Borel measures on E that are finite over compact sets. We will endow $M(E)$ with the topology of vague convergence. $M_1(E)$ stands for the space of probability measures over E endowed with the weak convergence.

It is know fact [18, Lemma 1.4, Lemma 4.1] that the σ-field $\mathcal{B}(M(E))$ coincides with the field generated by the functions $\{\mu \mapsto \mu(A) : A \in \mathcal{B}(E) \text{ bounded }\}$, as well as with the with the σ-field generated by the functions $\{\mu \mapsto \int_E f d\mu : f \in C_0(E)\}$.

For every measure $\nu \in M((0,\infty))$, we define its Laplace transform $\hat{\nu} \in C(\mathbb{R}_+)$ as

$$\hat{\nu}(\lambda) := \int_{\mathbb{R}_+} \exp(-\lambda t)\nu(dt). \tag{A.1}$$

We recall that μ is a **random measure on** $\mathbb{H}$ defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ iff $\mu : \tilde{\Omega} \to M(\mathbb{H})$ is a measurable function from the measurable space $(\tilde{\Omega}, \tilde{\mathcal{F}})$ to the measurable space $(M(\mathbb{H}), \mathcal{B}(M(\mathbb{H})))$ (see [18]). Equivalently, μ is a random measure iff $\mu(A) : \tilde{\Omega} \to \bar{\mathbb{R}}_+$ is a measurable function for every $A \in \mathcal{B}(\mathbb{H})$. The law induced by μ on $M(\mathbb{H})$ will be denoted P_μ,

$$P_\mu = \tilde{\mathbb{P}} \circ \mu^{-1}. \tag{A.2}$$

Let μ be a random measure on $\mathbb{H}$ defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and $f : \mathbb{H} \to \mathbb{R}_+$ be a measurable function. We define Laplace transforms

$$L_\mu(f) = \tilde{\mathbb{E}}\Big[\exp\Big\{-\int_{\mathbb{H}} f(t)\mu(dt)\Big\}\Big]. \tag{A.3}$$

The following proposition is well known (see Lemma 1.7 of [18]).

PROPOSITION A.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(\mu_\omega)_{\omega\in\Omega}$ be a family of random measures on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ indexed by $\omega \in \Omega$. Then there*

exists a probability measure $\mathcal{P}$ *on* $M(\mathbb{H})$ *given by (recall* (A.2) *for the notation)*

$$\mathcal{P}(A) = \int_\Omega P_{\mu_\omega}(A)\mathbb{P}(d\omega) \quad \text{for each } A \in \mathcal{B}(M(\mathbb{H})) \tag{A.4}$$

if and only if the mapping $\omega \mapsto L_{\mu_\omega}(f)$ *is* $\mathcal{F}$*-measurable for each* $f \in C_0(\mathbb{H})$. *The random measure* $\mu : \Omega \times \tilde{\Omega} \to M(\mathbb{H})$ *given by* $\mu(\omega, \tilde{\omega}) = \mu_\omega(\tilde{\omega})$ *whose distribution is* $\mathcal{P}$ *is called the* **mixture of** $(\mu_\omega)_{\omega\in\Omega}$ **with respect to** $\mathbb{P}$.

Let μ be a random measure. Denote

$$\mathcal{T}_\mu := \{A \in \mathcal{B}(\mathbb{H}) : \mu(\partial A) = 0 \ \tilde{\mathbb{P}}\text{-a.s.}\}. \tag{A.5}$$

By a **DC semiring** we shall mean a semiring $\mathcal{U} \subset \mathcal{B}(\mathbb{H})$ with the property that, for any given $B \in \mathcal{B}(\mathbb{H})$ bounded and any $\varepsilon > 0$, there exist some finite cover of B by $\mathcal{U}$-sets of diameter less than ε. It is a known fact that

PROPOSITION A.2 (Theorem 4.2 of [18]). *Let* μ *be a random measure and suppose that* $\mathcal{A}$ *is a DC semiring contained in* $\mathcal{T}_\mu$. *To prove vague convergence in distribution of random measures* μ^ε *to* μ *as* $\varepsilon \to 0$, *it suffices to prove convergence in distribution of* $(\mu^\varepsilon(A_i))_{i\le k}$ *to* $(\mu(A_i))_{i\le k}$ *as* $\varepsilon \to 0$ *for every finite family* $(A_i)_{i\le k}$ *of bounded, pairwise disjoints sets in* $\mathcal{A}$.

Finally, we recall here the multidimensional individual ergodic theorem. For its proof for square domanins see e.g. [15, Theorem 14.A5]. The proof can be easily adapted to rectangles.

THEOREM A.3 (Multidimensional ergodic theorem). *Let* $(X, \mathcal{G}, Q)$ *be a probability space and* $\Theta = (\theta_{i,j})_{(i,j)\in\mathbb{Z}^2}$ *be a group of* Q *preserving transformations on* X *such that* $\theta_{(i_1,j_1)} \circ \theta_{(i_2,j_2)} = \theta_{(i_1+i_2,j_1+j_2)}$. *Let* $\mathcal{I}$ *be the field of* Θ*-invariant sets,* $a \le 0 < b$ *and* $c \le 0 < d$ *be real numbers, and* $\Delta_n = [\lfloor an\rfloor, \lfloor bn\rfloor] \times [\lfloor cn\rfloor, \lfloor dn\rfloor]$. *Then, for any* Q*-measurable* f *with* $Q(|f|) < \infty$

$$\lim_{n\to\infty} \frac{1}{|\Delta_n|} \sum_{i\in\Delta_n} f \circ \theta_i = Q(f|\mathcal{I}), Q\text{-a.s.} \tag{A.6}$$

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