On Two Properties of Strongly Disordered Systems, Aging and Critical Path Analysis

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ABSTRACT

This Ph-D Thesis is devoted to the mathematical study of two properties of strongly disordered systems.

In the first part, we investigate a method that was used widely by physicists to estimate the conductivity of disordered systems. This method is called Critical Path Analysis. We construct a model based on continuum percolation in which, in the limit of strong disorder, we prove the validity of Critical Path Analysis. We use homogenisation techniques to estimate the conductivity.

In the second part of the thesis, we study aging in Bouchaud's trap model on $\mathbb{Z}^d$. This model describes a Markov process whose evolution is slowed down by a random environment. The transition rates are given by $w_{ij} = \nu \exp(-\beta((1-a)E_i - aE_j))$ if i, j are neighbours, where E_i are i.i.d. exponential random variables, β is inverse temperature and a determines the influence of neighbouring sites on the dynamics. We study two two-point functions. The first one, $\Pi(t_w, t+t_w)$, is the probability that the system does not jump between the times t_w and $t_w + t$. The second one, $R(t_w, t + t_w)$, is the probability that the system is in the same state at times t_w and $t+t_w$. If $d = 1$ and the disorder is strong enough ($\beta > 1$), we prove, for any $a \in [0, 1]$, that a proper rescaling of process X converges to a singular diffusion. We use this result to prove aging behaviour for the function R and subaging for the function Π. We get the same results in higher dimensions ($d = 2, 3, \dots$), but only in the case $a = 0$.

RÉSUMÉ

Cette thèse est consacrée à l'étude mathématique de deux propriétés des systèmes fortement désordonnés.

Dans la première partie, on étudie une méthode bien connue des physiciens, servant à estimer la conductivité des systèmes désordonnés. Cette méthode s'appelle Critical Path Analysis ("analyse des chemins critiques"). On construit un modèle de percolation continue dans lequel on montre, dans la limite de grand désordre, la validité de CPA. On utilise des techniques d'homogénéisation pour estimer la conductivité.

Dans la deuxième partie du travail, on étudie le vieillissement du modèle de pièges de Bouchaud sur $\mathbb{Z}^d$. Ce modèle décrit un processus Markovien X dont l'évolution est freinée par un environnement aléatoire. Les taux de transition sont donnés par $w_{ij} = \nu \exp(-\beta((1-a)E_i - aE_j))$ si i, j sont plus proches voisins. Les E_i sont des variables aléatoires exponentielles, β est l'inverse de la température et a détermine l'influence des sites voisins sur la dynamique. On considère deux fonctions à deux points. La première, $\Pi(t_w, t+t_w)$, est la probabilité que le système ne change pas d'état entre les temps t_w et $t+t_w$. La seconde, $R(t_w, t+t_w)$, est la probabilité que le système soit dans le même état aux temps t_w et t_w+t. Si $d=1$, on montre que pour tout $a \in [0,1]$, le processus X converge, après un changement d'échelle approprié, vers une diffusion singulière. On utilise ensuite ce résultat pour montrer que R présente le phénomène dit de vieillissement ("aging") et que Π présente le sous-vieillissement. On obtient les mêmes résultats en dimensions supérieures ($d=2,3,\dots$) dans le cas $a=0$.

PREFACE

Disorder plays a fundamental role in many areas of industrial and scientific interest. To cite a few, we mention geological systems (like porous and fractured rocks), semiconductors, polymers, and spin-glasses. All these systems have been studied for quite a long time both experimentally and theoretically. During the last three decades, the development of powerful theoretical methods has allowed to interpret experimental observations. Many of these methods have been put on a rigorous basis, but there are still many open areas.

In the two main parts of this thesis we shed light on two methods that were used in physics literature to describe the properties of disordered systems. Regarding the treated models and the properties which we are interested in, the two parts of the thesis are independent. However, both are connected by one crucial property, the presence of strong disorder. This means that the probability distributions of some characteristics important for the behaviour system are very broad. The presence of such broad distributions implies that some parts of the system have more importance and dominate somehow the properties of the whole system. In this short introduction we describe the kind of problems treated in the sequel and we give a trivial example where it is possible to spot the connection between the two parts of the thesis.

In the first part we will study the conductivity (or water permeability) of disordered systems. Here strong disorder will be present in the distribution of local conductivity. If an electrical potential is applied on such a system, the presence of strong disorder leads to the creation of a few relatively small domains (paths) where the largest amount of the electrical current flows. It is evident that the conductivity of these small domains should influence the conductivity of the whole system. This behaviour is completely different from the one observed in systems where the distribution of local conductivity is relatively narrow, since in these systems the whole volume is important for transport.

We will construct a model (which is a modification of the random chessboard model of [GK99]), where it is possible to show the existence of such strongly conducting paths and to prove that the conductivity of these narrow paths dominates the overall conductivity.

The second part of the thesis is devoted to the study of one model that has been used in the physics literature [Bou92, MB96, RMB01, BCKM98] to explain the aging properties of some glassy materials. In this model the phase space of the system consists of states whose energy distribution is very broad. The system stays (is trapped) in any state a time that is proportional to the energy of this state. The broad distribution of the local properties in this case implies that most of the time is spent in a very small number of states.

The rules governing the dynamics between the different states can be chosen quite arbitrarily. Evidently, some choices have more physical relevance than others. Here we will study the case where the states are placed on the d-dimensional square lattice and the system can jump only between neighbouring sites. If $d \geq 2$, then the system will perform the simple random walk on $\mathbb{Z}^d$, and will be slowed down by the presence of traps. In $d = 1$ we will work with more complicated rules for the dynamics of the system. Namely, we will be able to consider some sort of attraction to the states with high energy. We will prove aging behaviour of these systems.

To illustrate better the connection between both models we construct here two trivial and rather artificial models, where it is easy to observe the transition between strong and weak disorder. Both these models are based on a well known object in probability theory, the sum of i.i.d. random variables. Let r_i, $i = 1, 2, \dots$, be a sequence of positive i.i.d. random variables. In the conductivity setting, r_i's can be simply regarded as the resistances of small conducting elements ranged linearly. Then the resistance of a system composed of n elements equals $S_n = \sum_{i=1}^{n} r_i$.

Consider now the "phase space" model, where the system follows simple transition rule: after leaving state i it jumps to state $i + 1$. Let r_i be, for the sake of simplicity, the time spent by the system in the i-th state. The time that the system needs for the first n jumps is then also S_n. Note that a very similar model was studied in [Bou92].

To illustrate the role of strong disorder, consider that r_i are in the domain of attraction of a totally asymmetric α-stable law. We use M_n to denote $\max\{r_i, i = 1, \dots, n\}$. If $\alpha < 1$, then it is known (see e.g. [Fel71] page 172) that the expectation of M_n/S_n converges to $1 - \alpha$ as $n \to \infty$. If $\alpha > 1$, then it converges to 0. Thus, the presence of strong disorder implies that the total resistivity is dominated by one particularly resistant element, or that during the first n jumps the system spends a non-negligible proportion of time in the site with the "energy" M_n.

The models we will treat later are obviously less trivial than a sum of i.i.d. random variables, but we will see that the effects of strong disorder are very similar to the above described trivial models.

Part I

CRITICAL PATH ANALYSIS

1. CRITICAL PATH ANALYSIS, TRANSPORT IN FRACTURED ROCKS

1.1 Introduction

This part of the thesis is devoted to the study of the conductivity of disordered systems. We focus on one method that has been widely used in physical and geological literature to estimate the global properties of complex disordered systems.

This method is called Critical Path Analysis (CPA). It was applied for the first time by Ambegaokar, Halperin, and Langer [AHL71] to justify the dependence of the conductivity of some semiconductors on temperature. The CPA method was further adapted by Katz and Thompson [KT86] to estimate the saturated hydraulic conductivity of porous rocks. It was then successfully used many times in different context. Charlaix, Guyon, and Roux [CGR87] used it to describe the properties of fractured rocks. Recently, this method was compared by Bernabe and Bruderer [BB98] with three other methods for the calculation of saturated hydraulic conductivity, and was found to be the most promising in many situations. It was also applied together with some fractal methods to estimate the unsaturated hydraulic conductivity of porous rocks by [HG02].

Our interest in this method originated from the project in collaboration with the group of Sanitary and Environmental Engineering at the Department of Rural Engineering at EPFL. The aim of this project was to estimate the transport properties of rocks. Needless to emphasise that the research in this domain is very important from the point of view of the applications. For many areas of human activity good estimations of the transport properties of rocks are crucial. These areas include construction and security of nuclear waste repositories, establishment of the productivity of petroleum reservoirs, pollution and cleaning of underground water, etc. The CPA method itself is, however, not bounded to the geological context. It can be applied in other areas of technological interest like porous electrochemical electrodes, filters and gels.

It was concluded already by Kirkpatick [Kir73] (who had done numerical simulations on transport in heterogeneous media), that percolation theories, and thus CPA, perform best of known approaches when disorder is (relatively) high. On the other hand, effective medium theories are more efficient when disorder is low. Porous and fractured rocks fall usually into the category of highly disordered systems.

It is known that the distribution of local hydraulic conductivity in rocks is usually very broad. This is mainly due to the very complex structure of void spaces in rocks. It follows from geological measurements that the ratio of the extreme values of size of these voids can spam several (up to 5) orders of magnitude in porous rocks. This ratio can be even larger in the case of fractured rocks. The void spaces are created during geological evolution of rocks. They include small pores between grains of sand that are created during sedimentation process, as well as huge fractures created for example by earthquakes, and whose length can exceed several kilometres.

The effect of voids of different sizes is further intensified by the fact that the dependence of hydraulic conductivity on the aperture of pores or fractures is very strong. For example, flat cracks of constant aperture δ have the hydraulic conductivity proportional to δ^3 (if we suppose very slow laminar flow, which is usually the case in rocks). For necks that connect pores between spherical grains, the power can be even larger.

For a mathematician there are other interesting aspects of the study of transport in rocks. A large variety of questions important for applications can be posed. It is necessary to estimate the average transport properties, because they are important e.g. for oil mining and extracting of underground water. On the other hand, the fastest paths, not the bulk transport, are fundamental for nuclear waste repositories.

Different physical and chemical processes in the rock can make the problems even more difficult and interesting. Among them we may mention the trapping of radioactive nucleotides by some kinds of argils, sedimentation of minerals dissolved in the water during transport, and others.

During our project we concentrated mainly on fractured rocks, since their study was of more interest for our geology partners. As we have already pointed out, the distribution of apertures in these rocks is usually very broad. This creates, under certain conditions, strongly conducting paths that contribute overwhelmingly to the permeability of the whole system. The study of these strongly conducting paths and mainly of the conditions that lead to their existence was the main subject of the project.

The idea about strongly conducting paths came up after geological observations that were done near Granada, Spain. It was apparent there that only a few fractures in the rock had been the preferential paths for hydraulic conductivity in the past. On the other hand, there were a lot of fractures in the same geological period that were relatively small with respect to the first set, and through which was almost no water flew. This second set of fractures became closed by sedimentation as time passed by.

The existence of strongly conducting paths can be important for simulation efforts since it allows to restrict the attention to a small subset of the system, and thus to decrease considerably the complexity of the computation. It is clear that such restriction cannot be done generally. Our goal was to clarify its domain of applicability. In a more general (not specifically geological) way, we wanted to justify that some global properties of disordered systems can be derived, if some conditions are satisfied, only from the properties of a small portion of the system.

Our research follow two main lines. The first one is to find a numerical justification of our working hypothesis. The second one is more theoretical and occupies the majority of the first part of this thesis. The theoretical part is concentrated on the construction of a model, where the justification of the Critical Path Analysis is possible. However, before we start to go into the details of the theoretical direction of our research, we make a short digression, and describe the numerical part of the project.

1.2 Numerical simulations

For the justification of the idea of Critical Path Analysis the software called CPA has been developed. The main part of the coding was done by the author of this thesis. The program has several cooperating modules that were developed separately.

Simulator of the rock. At the beginning of the project it was necessary to create a good simulator of the geometry of the fractured rock. This task is quite complicated, because the geometry of the rock tends to be considerably complex. All its properties depend on the geological history of the rock, which is never precisely known. Usually, the fractured rock contains several families of fractures (normally from two to five). All fractures in one family have approximately the same age. Their normals are distributed around some direction with the variance that depends also on the family. The density of fractures, their spacing, shape, size, aperture and surface roughness are other parameters that characterise every family. Moreover, due to geological history, there are various correlations between different families (e.g. the intersections of two fractures from different families can have shape of T more often than of X). The creation of a "realistic" rock simulator reveals thus to be quite complicated.

Another problem is connected to the simulation of the rock. It is practically impossible to obtain enough data to calibrate the simulator. It is caused by the fact that almost all geological measurements are done by using two-dimensional outcrops or one-dimensional boreholes, where only the traces of fractures are observed. The access to the interior of the rock is highly restricted. Moreover, such access usually changes the conditions inside of the rock. So, the measured characteristics (like e.g. appertures) do not correspond exactly to the reality.

It was therefore clear that the model we wanted to create should be a simplification, that should be, on one hand, sufficiently realistic for our geological partners and, on the other hand, simple to simulate with the data we had.

Some geometry simulators have already been described in the geological literature [BLE77, LRWW82, LB87]. Our simulator does not differ significantly from them. The fractures are represented by disks distributed in three-dimensional space. The program allows to tune the different parameters for the different families of fractures. The centres of fractures of each family are distributed according to a Poisson point process; the densities of different families can differ. For each family it is further possible to set the distributions of radii, of the normal direction and of the aperture.

Owing to the roughness of the fracture surface and to the sedimentation of the minerals inside fractures, a fracture cannot be considered as a two-dimensional object. Usually, one-dimensional channels are created in every fracture and the transport takes place in these channels. A different set of channels arises in the intersections of fractures from different families. For some types of rock the channel transport dominates. It is thus not completely unrealistic to replace the set of fractures by a network of channels. This is the approximation that we used; it allows to decrease the complexity of the program since the conducting objects become one-dimensional. At the start of the development channels that arise due to the surface roughness were ignored, and we concentrated mainly on the “intersection channels”. This omission was possible since the geological systems considered by our partners had such nature.

The program calculates the network of intersection channels from the fracture network. As the result of this calculation, a graph is obtained. Every vertex of this graph has assigned its position in space, and to every bond is associated its hydraulic conductivity.

Conductivity calculation. The second component of the program determines the conductivity of the graph prepared by the previous part. At the first stage of development, a very simple model was used for this computation. We used the analogy between water flow and electrical networks. This analogy is valid for slowly flowing liquid in one-dimensional channels (which is the case in most geological applications). The potential difference was applied on opposite sides of the rock and the current flow was calculated. The conductivity calculation was thus reduced to the solution of a possibly quite large system of linear equations, which was obtained from Kirchoff's and Ohm's laws. This system is usually very sparse and standard numerical methods were used to solve it.

Network reduction. Since the main aim of the simulation is to test the Critical Path hypothesis, we have to find a small subset of the original graph that can be a candidate for such a path and compute its conductivity. This conductivity (which is calculated again by the previous component) is then compared to the conductivity of the complete network. The critical path between the opposite sides of the rock is constructed by the following *pruning procedure*: First, all bonds from the graph are deleted. Then we start to re-add them step by step, in the order of decreasing conductivity. This means that first we add the most conducting one, etc. After each step we check for loops. If there is a loop in the newly created graph, we delete the bond we have just added and we continue with the next one. The procedure stops if a connection between the sides where potential difference was applied is produced. Then all dead-ends and isolated edges are removed. This leaves a one-dimensional path.

Let us finish this section with some comments on the results we obtain by numerical simulations and that are relevant for verification of Critical Path Analysis. Not surprisingly, we find that the hydraulic conductivity of the whole network and of the critical path differ if the local conductivity distribution is not sufficiently broad. If this distribution becomes "broader", the permeability of the critical path starts to be a good approximation of the permeability of the whole network.

There is one important observation. The "broadness" where the approximation starts to be good depends on the size of the sample. When the size of the sample increases, we have to take broader distributions of apertures to obtain a good agreement between both hydraulic conductivities. This is also easy to explain at the heuristic level. For any broad but bounded distribution there exists a length above which the system can be considered as homogeneous. If the size of the system exceeds this size, the CPA looses sense and the homogenisation techniques becomes more appropriate.

1.3 One simple model

We construct here one quite simple model where CPA can be proved. It is not very elaborate and serves only to illustrate some ideas that will be used later. However, already in this simple model it is possible to see how the required broadness of the distribution depends on the size of the box. Indeed, the level of disorder in the system should diverge as the size of the system increases.

The model that we use is motivated by [NS94], where a very similar model is used to discuss some problems connected with the ground state structure of Edwards-Anderson spin-glasses.

We consider the square lattice $\mathbb{Z}^2$, where every two points i, j satisfying $\mathrm{dist}(i,j) = 1$ are connected by the bond $\langle i,j\rangle$ with the conductivity c_{ij}. We will try to estimate the conductivity between the left and the right edge of the box $B_L = [0,L]^2 \cap \mathbb{Z}^d$.

The conductivities c_{ij} are i.i.d. random variables whose distribution will be specified later. We first describe the *key property* of these variables. We want the conductivity of any bond to exist at its own scale. More precisely, for a sufficiently large system, with overwhelming probability, the conductivity of each bond is at least twice the size of the next smaller conductivity. If this is satisfied, then for any bond $b = \langle i,j\rangle$, $i,j \in B_L$

$$c_b \geq \sum_{\substack{b'\in B_L\\ c_{b'}<c_b}} c_{b'} \qquad \text{and} \qquad c_b^{-1} \geq \sum_{\substack{b'\in B_L\\ c_{b'}>c_b}} c_{b'}^{-1}. \tag{1.1}$$

This property can be achieved by the following construction of c_{ij}. Let K_{ij} be i.i.d. positive random variables with continuous distribution (e.g. uniform on [0,1]). We set the conductivity c_{ij} as

$$c_{ij} = c_{ij}^{(L)} = \exp(\lambda_L K_{ij}). \tag{1.2}$$

The nonlinear scaling factor λ_L is chosen to diverge fast enough as $\Lambda \to \infty$ to ensure that (with probability 1) for all large L, each $c_{ij}^{(L)}$ in B_L is larger than at least twice the next one.

To see that such a choice is possible, note that for every distinct pair of bonds $\langle i,j\rangle$ and $\langle i',j'\rangle$, the function

$$\begin{aligned} g(\lambda_L) &= \mathbb{P}\big[1/2 \leq \exp(\lambda_L K_{ij})/\exp(\lambda_L K_{i'j'}) \leq 2\big] \\ &= \mathbb{P}\big[|K_{ij} - K_{i'j'}|\big] \leq \log 2/\lambda_L\big] \end{aligned} \tag{1.3}$$

decreases to 0 as $\lambda_L \to \infty$, because K_{ij} and $K_{i'j'}$ are independent random variables with continuous distribution. The probability that c_{ij}'s do not satisfy the key property is then bounded by

$$\sum_{\langle i,j\rangle\in B_L} \sum_{\langle i',j'\rangle\in B_L} g(\lambda_L) = O(L^4 g(\lambda_L)). \tag{1.4}$$

If we choose λ_L so that $g(\lambda_L) = O(L^{-4+1+\varepsilon})$ for some $\varepsilon > 0$, then the sum of (1.4) over L is finite. Hence, the Borel-Cantelli lemma gives that, with probability one, the key property is satisfied for L large enough. If K_{ij} are uniform on $[0,1]$, then $g(\lambda_L) = O(1/\lambda_L)$ and so $\lambda_L \geq L^{5+\varepsilon}$ is sufficiently fast divergence.

Let $p_c = 1/2$ denote the critical probability of the percolation on $\mathbb{Z}^2$, and let K_c be the solution of

$$\mathbb{P}[K_{ij} \leq K_c] = p_c. \tag{1.5}$$

We use $c_L^\star$ to denote the conductivity of the whole box B_L, and c_L^{CP} to denote the conductivity of the critical path constructed by the pruning procedure. Further, let c_L^{last} denote the conductivity of the last bond added during the pruning procedure. Then we can show the following proposition

Proposition 1.3.1. *For almost every realisation of the random variables K_{ij}, the conductivity $c_L^\star$ of the box satisfies*

$$\frac{1}{2} c_L^{\mathrm{last}} \leq c_L^{\mathrm{CP}} \leq c_L^\star \leq 2 c_L^{\mathrm{last}} \leq 4 c_L^{\mathrm{CP}} \tag{1.6}$$

and

$$\lim_{L\to\infty} \frac{1}{\lambda_L} \log c_L^{\mathrm{CP}} = \lim_{L\to\infty} \frac{1}{\lambda_L} \log c_L^\star = K_c \tag{1.7}$$

Proof. The proof is inspired by [GK84]. We start with the lower bound on $c_L^\star$. Clearly, $c_L^\star \geq c_L^{\mathrm{CP}}$. If L is large enough, then the conductivity of the critical path satisfies

$$(c_L^{\mathrm{CP}})^{-1} = \sum_{b \in \mathrm{CP}} c_b^{-1} \leq 2 (c_L^{\mathrm{last}})^{-1}, \tag{1.8}$$

where the sum runs over all bonds in the critical path and the last inequality follows from property (1.1) of c_{ij}. Using K_L^{last} for $\lambda_L^{-1} \log c_L^{\mathrm{last}}$ we get

$$\liminf_{L\to\infty} \frac{1}{\lambda_L} \log c_L^\star \geq \liminf_{L\to\infty} \frac{1}{\lambda_L} \log \frac{c_L^{\mathrm{last}}}{2} = \liminf_{L\to\infty} K_L^{\mathrm{last}}. \tag{1.9}$$

Using the exponential decay of crossing probabilities, it is easy to see that, with probability one for L large enough, $K_L^{\mathrm{last}} \geq K_c - \varepsilon$ for any $\varepsilon > 0$. This means that $\liminf K_L^{\mathrm{last}} \geq K_c$. This completes the proof of the lower bound.

The proof of the upper bound uses the self-duality of $\mathbb{Z}^2$. Let $\mathbb{Z}^2_\star$ denote the dual lattice. To every bond of the dual lattice we associate the same conductivity as has its dual bond. Since the adding of the bond with conductivity c_L^{last} during the pruning procedure produces a left-right crossing of B_L by bonds with large conductivity, there should exist a top-bottom crossing $\mathrm{CP}^\star$ of the rectangle $[\frac{1}{2}, L - \frac{1}{2}] \times [-\frac{1}{2}, L + \frac{1}{2}]$ in $\mathbb{Z}^2_\star$ using only bonds with conductivity smaller than c_L^{last}. Then (by setting the conductivity of all bonds that are not dual to some bond of this top-bottom crossing equal to infinity) it is easy to see

$$c_L^\star \le \sum_{b \in \mathrm{CP}^\star} c_b \le 2c_L^{\mathrm{last}} \le 4c_L^{\mathrm{CP}}. \tag{1.10}$$

Taking the limit and using the fact that for L large enough $K_L^{\mathrm{last}} \le K_c + \varepsilon$ with probability one we get easily the upper bound. $\square$

Let us make some comments on the obtained result. This simple model has a lot of features that we will see later in the more complicated situation. First, as we have already noted, the level of disorder should increase with the size of the box. Later we will use homogenisation techniques to estimate the conductivity. Using such techniques usually means that infinite volume limit should be considered. Hence, to be able to verify the validity of CPA, we will be forced to work in the limit of strong disorder.

From a technical point of view, observe also the role of dual crossing $CP^\star$. It is the traversal crossing of the box by bonds with small conductivities. This crossing creates a "barrier" that any flow should pass. We will use such barriers later to construct an upper bound on the conductivity.

The next chapter contains the main result of the first part of the thesis. We construct there a two-dimensional continuous model where we can verify the validity of the Critical Path Analysis.

2. CRITICAL PATH ANALYSIS FOR CONTINUUM PERCOLATION

Jiří Černý

Abstract. We prove the validity of the Critical path analysis for a continuum percolation model close to Golden-Kozlov one. This is obtained in the limit of strong disorder.

2.1 Introduction

One of the central issues of the theory of disordered materials is the determination of effective properties (like electrical conductivity or fluid permeability) from the knowledge of the micro-structural properties. In many areas of practical importance, the probability distribution of local physical characteristics is very broad. An interesting property of these so-called "highly disordered" systems is that the effective conductivity of the sample can often be approximated by the conductivity of a very small part of it. Such part is usually composed by a small number of paths that contribute overwhelmingly to the effective conductivity. It is thus important to find out the conditions that lead to this behaviour, since it is usually far less complex to compute the conductivity of a small number of paths than of the whole sample.

This idea was, for the first time, introduced by [AHL71] and is known in the physical literature as "Critical Path Analysis (CPA)". It was used successfully in many areas of physics [KT86, CGR87]. However, rigorous investigations are sparse up to now [NS96, GK99].

It should be obvious that the creation of strongly conducting paths (and thus the calculation of effective properties of the sample) is connected with the percolation of highly conducting areas. Let us explain this relation heuristically on a simple model. The procedure of reduction of the sample to a small set of "critical paths" follows [CMB94]. We will call this procedure a "pruning procedure".

Let Λ_N be the box of size N in $\mathbb{Z}^2$ and let $\mathbb{L}_N$ be the set of all bonds connecting nearest neighbours in Λ_N. Assign to each bond $b \in \mathbb{L}_N$ a random i.i.d. conductivity c_b. We want to compute the conductivity of the sample with the potential difference applied on the left and right edge of the box.

Now we start describing the "pruning procedure". First, we sort all the bonds in the graph $(\Lambda_N, \mathbb{L}_N)$ according to their conductivity. Then we delete all the bonds from the graph except the bonds that are contained in left of right edge of the box, and we start to re-add them bond by bond in the order of decreasing conductivity. After each step we check for loops. If there is a loop, we delete the bond just added and we continue with the next one. At the beginning of this procedure, there will be no connection between the left and right edge. After sufficiently many steps, adding the next bond produces a connection between the left and right edge. We stop the procedure at this moment. What we get at this point is a treelike structure containing one connection from left to right and many dead-ends that we can delete safely, because they do not contribute to the transport. The conductivity of this connection is easy to obtain. If the distribution of local characteristics is broad enough, then the CPA claims that the conductivity of this connection is close to the conductivity of the graph before the pruning.

One can go further in this type of reasoning. The conductivity of one-dimensional path of conducting elements with conductivities drawn from a very broad distribution is essentially determined by the element with the smallest conductivity. Applying this to the path constructed by pruning, one can conclude that the conductivity of the box is not far from the conductivity of the bond we have added as the last one. Denoting by $F(x) = \mathbb{P}(c_b \leq x)$ the distribution function of the local conductivity and by p_c the percolation threshold of the bond percolation, the conductivity of the box should be close to

$$\sup\{x : 1 - F(x) \geq p_c\}. \tag{2.1}$$

In this paper we construct a model where the above heuristic can be proved. The effective conductivity will be very close (at least in the limit of strong disorder) to the "critical local conductivity". This can be interpreted as a justification of the CPA for this model. The model we use is a continuous generalisation of the "chess-board" model used in [GK99].

2.2 Definitions and results

We consider the following two-dimensional medium. Let $X = X(\omega)$, $\omega \in \Omega$ be a homogeneous Poisson point process with density λ defined on some probability space Ω (see Section 2.3 for the definition). For every point $x \in \mathbb{R}^2$ let $S(x) = S(x, \omega)$ denote the minimal distance to some point of X,

$$S(x) = \inf\{d(x, y) : y \in X\}, \tag{2.2}$$

where $d(\cdot,\cdot)$ denotes the Euclidean distance of two points. We define the local conductivity of the medium by

$$\sigma(x,\mu) = \sigma(x,\mu,\omega) = \exp(\mu S(x)), \tag{2.3}$$

where μ is a positive parameter. That means that our medium can be considered as the set of insulating grains with the centres in the points of the point process. The parameter μ controls the amount of disorder of the system. We will be interested in the case where μ is very large.

The medium we have just defined is obviously statistically isotropic. Thus, its macroscopic properties can be described by one scalar effective conductivity $\sigma^\star(\mu,\omega)$ defined as follows. Let Λ_N be the box $[0,N]^2$ and let $u_N(x,\mu) = u_N(x,\mu,\omega)$ be the solution of the system

$$\begin{aligned} \operatorname{div}(\sigma(x,\mu)\nabla u_N(x,\mu)) &= 0 && x=(x^1,x^2)\in\Lambda_N \\ u_N(x,\mu) &= 0 && x^1 = 0 \\ u_N(x,\mu) &= N && x^1 = N \\ \frac{\partial u_N(x,\mu)}{\partial x^2} &= 0 && x^2 \in \{0,N\}. \end{aligned} \tag{2.4}$$

The function $u_N(x,\mu)$ is the electrical potential in the box Λ_N with the prescribed boundary conditions. Let $J_N(\mu) = J_N(\mu,\omega)$ denote the overall flow through the vertical line $x^2 = b$, $b \in (0,N)$,

$$J_N(\mu) = \int_0^N \sigma((b,x^2),\mu)\frac{\partial u_N((b,x^2),\mu)}{\partial x^1}\,dx^2, \tag{2.5}$$

which obviously does not depend on b. The effective conductivity is then defined by

$$\sigma^\star(\lambda,\mu,\omega) = \lim_{N\to\infty}\frac{1}{N}J_N(\mu,\omega). \tag{2.6}$$

Since our medium is evidently ergodic, it follows from the results of homogenisation theory that this limit exists almost surely and does not depend on ω (see [JKO94] Theorem 7.4).

To state our first theorem we need one quantity from the continuum percolation (for a good survey see [MR96]). It is well known that there exists a nontrivial value $S_c(\lambda)$, such that the set $\{x \in \mathbb{R}^2 : S(x) \le r\}$ percolates iff $r > S_c(\lambda)$, and its complement percolates iff $r < S_c(\lambda)$. We call $S_c(\lambda)$ the critical radius. As we have noted in the introduction, this value should be important for the estimation of the effective conductivity in the limit of the strong disorder. Actually, we have

Theorem 2.2.1. *For almost all realisations of the medium the value of the effective conductivity depends only on the parameters λ and μ and asymptotically satisfies*

$$\lim_{\mu\to\infty} \frac{1}{\mu} \log \sigma^\star(\lambda, \mu) = S_c(\lambda). \tag{2.7}$$

To clarify the relation of this result with equation (2.1) observe that Theorem 2.2.1 roughly says that $\sigma^\star(\lambda, \mu) \sim \exp(\mu S_c(\lambda))$. This value is the largest σ such that the domain where the conductivity is larger or equal to σ percolates.

The next theorem shows something that resembles the pruning that was described before, and also clarifies the meaning of Theorem 2.2.1. The pruning in this case cannot be defined in the same way as for the square lattice. However, it is possible to reduce our medium and to obtain a medium that essentially consists of points connected by tubes. These points will not be located on the square lattice, but this does not pose major problems for the pruning procedure.

As we have already noted, our medium can be regarded as an ensemble of insulating grains in the plain. Between every pair of neighbouring grains there is a domain where the conductivity is large. The structure of these grains can be identified with the Voronoi tessellation defined by the process $X(\omega)$. If μ is large, the conductivity decreases very rapidly with the distance from the borders of Voronoi cells. Hence, the contribution of a small neighbourhood of these borders to the effective conductivity should be very important. Thus, we should not make a large error if we consider the rest of the medium as totally insulating. We get a medium that consists only of the thin tubes around the borders of the Voronoi cells.

More precisely, let $\mathcal{V}(\omega) \subset \mathbb{R}^2$ denote the set of borders of Voronoi cells around the points of $X(\omega)$ and let $\rho > 0$ be a small positive constant. We define first the modified conductivity $\tilde\sigma(x)$

$$\tilde\sigma_\rho(x, \mu) = \begin{cases} \sigma(x, \mu) & \text{if } d(x, \mathcal{V}) < \rho \\ 0 & \text{if } d(x, \mathcal{V}) > 2\rho. \end{cases} \tag{2.8}$$

In the domain between ρ and 2ρ the function $\tilde\sigma_\rho(x)$ continuously and "monotonically" interpolates between the values on the boundary of this domain. The way how the interpolation is done is not important. We use it only to make the conductivity continuous and to avoid problems with the boundary conditions on the walls of the tubes.

The medium $\tilde\sigma_\rho(x)$ can be "pruned" further. It is obvious that at each bond b of $\mathcal{V}$ there is exactly one point s_b where the function $S(x)$ has the saddle point. Intuitively, the conductivity of the tube around the bond b should be proportional to the value of conductivity in the point s_b. Actually, it can be easily proved at least for μ large enough, but we will not need this claim later. Using this observation, one sees that the bonds with $\sigma(s_b)$ very small should not contribute too much to the overall conductivity. So we delete them. More formally, let $\mathcal{V}_\delta$ be the subset of $\mathcal{V}$ containing only the bonds with $S(s_b) > S_c(\lambda) - \delta$, i.e. the bonds that are far from the points of X. Let us define another modified medium $\hat{\sigma}_{\rho,\delta}(x, \mu)$ in the same way as we defined $\tilde{\sigma}_\rho(x, \mu)$ but using $\mathcal{V}_\delta$ instead of $\mathcal{V}$:

$$\tilde{\sigma}_{\rho,\delta}(x, \mu) = \begin{cases} \sigma(x, \mu) & \text{if } d(x, \mathcal{V}_\delta) < \rho \\ 0 & \text{if } d(x, \mathcal{V}_\delta) > 2\rho. \end{cases} \tag{2.9}$$

The medium $\hat{\sigma}_{\rho,\delta}$ consists of the tubes from $\tilde{\sigma}_\rho$ with large conductivity.

Note, that we do not define pruning in the inductive way that we have described before. The "pruned" medium $\hat{\sigma}_{\rho,\sigma}(x)$ does not consist of a single one-dimensional path crossing the box and it contains more tubes than it should. However, if the parameter δ is small (how small it should be, depends on the size of the box that we consider) the difference should not be substantial.

We use $\tilde{\sigma}^\star_\rho(\lambda, \mu)$ and $\hat{\sigma}^\star_{\rho,\delta}(\lambda, \mu)$ to denote the effective conductivities of the modified media. Then we have:

Theorem 2.2.2. *For every $\delta > 0$ and $\rho > 0$, the effective conductivities of the pruned media $\tilde{\sigma}^\star_\rho(\lambda, \mu)$ and $\hat{\sigma}^\star_{\rho,\delta}(\lambda, \mu)$ satisfy the same relation as the original medium, i.e.*

$$\lim_{\mu\to\infty} \frac{1}{\mu} \log \tilde{\sigma}^\star_\delta(\lambda, \mu) = \lim_{\mu\to\infty} \frac{1}{\mu} \log \hat{\sigma}^\star_{\rho,\delta}(\lambda, \mu) = S_c(\lambda). \tag{2.10}$$

At first sight, the results of our theorems can be found quite unsatisfactory, because they give us only the estimation in logarithmic scale and in the limit of the strong disorder. However, they can be useful to find out the dependence of the effective conductivity on other parameters. Indeed, let the local conductivity $\sigma(x, \alpha)$ be defined by $\exp(\mu f(S(x), \alpha))$, where f is a strictly increasing and differentiable in the first argument, and with the first derivative with respect to this argument in the point $S_c(\lambda)$ bounded away from zero and infinity. Then an easy modification of the arguments given in the proof of Theorem 2.2.1 gives

$$\lim_{\mu\to\infty} \frac{1}{\mu} \log \sigma^\star(\lambda, \mu, \alpha) = f(S_c(\lambda), \alpha). \tag{2.11}$$

This is essentially the way how the idea of CPA was used in the original article [AHL71].

Note also that there are two reasons for having results only in the logarithmic scale. The first one is the "non-gaussian" shape of the graph of the conductivity around the saddle points. This problem can be probably resolved by a more careful computation. However, there is still a second problem. We do not have enough control of the infinite cluster of continuum percolation near the critical point.

The proofs of Theorems 2.2.1 and 2.2.2 can be found in Section 2.4 and they use homogenisation techniques. In Section 2.3 we show some facts about continuum percolation in $\mathbb{R}^2$.

2.3 Percolation results

In this section we prove some facts that are known to be valid for discrete percolation. To our knowledge similar results do not exist in the case of continuum percolation. The proofs we present are rather standard modifications of the discrete versions. The reader familiar with the technical details can skip the rest of this section and read only Propositions 2.3.1 and 2.3.7 that will be used later.

Let N be a set of all finite counting measures assigning the weight at most one to singletons equipped with the usual σ-field $\mathcal{N}$ generated by sets of the form $\{n \in N : n(A) = k\}$, where $A \subset \mathbb{R}^2$ is a Borel set and $k \in \mathbb{N}$. Every $n \in N$ can be identified with a set of points in $\mathbb{R}^2$. This allows us to write $x \in n$, if n has an atom at $x \in \mathbb{R}$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be some probability space. The Poisson point process with density λ is an N-valued random variable which satisfies the following two conditions. $X(A)$ is a Poisson random variable with mean $\lambda|A|$, where $|A|$ denotes the Lebesgue measure of A. If $A_1, A_2 \subset \mathbb{R}^2$, $A_1 \cap A_2 = \emptyset$, then $X(A_1)$ and $X(A_2)$ are independent. We write $\mathbb{P}_\lambda$ for the law of X and $\mathbb{E}_\lambda$ for the corresponding expectation.

Let us now define set $\mathbb{X}(\omega)$, $\omega \in \Omega$, as the set $\{x \in \mathbb{R}^2 : S(x) \leq 1\}$. The set $\mathbb{X}$ is the union of unit disks with centres in $X(\omega)$. We will call it the occupied region. The complement of $\mathbb{X}(\omega)$ is called the vacant region. For any $A \subset \mathbb{R}^2$ we use $W(A)$ to denote the union of all components of $\mathbb{X}$ (occupied components) intersecting A. Similarly, we write $V(A)$ for the union of vacant components intersecting A. It is well known that in dimension two there exists a constant λ_c such that for every bounded set A the following holds

$$\begin{aligned}\lambda_c &= \sup\{\lambda : \mathbb{P}_\lambda[\operatorname{diam} V(A) = \infty] > 0\} = \inf\{\lambda : \mathbb{E}_\lambda[\operatorname{diam} V(A)] < \infty\} \\ &= \inf\{\lambda : \mathbb{P}_\lambda[\operatorname{diam} W(A) = \infty] > 0\} = \sup\{\lambda : \mathbb{E}_\lambda[\operatorname{diam} W(A)] < \infty\},\end{aligned} \tag{2.12}$$

i.e. occupied region percolates above λ_c and vacant region percolates below λ_c.

Let E be an event. We say that E is increasing event if from $\omega \in E$ follows $\omega' \in E$ for all ω' satisfying $X(\omega') \supset X(\omega)$. The event E is decreasing if E^c is increasing.

We now introduce some obvious geometrical notation. Let A_1, A_2, B be subsets of $\mathbb{R}^2$. We write $A_1 \underset{\text{in } B}{\overset{\text{occ}}{\longleftrightarrow}} A_2$ if A_1 is connected to A_2 in $B \cap \mathbb{X}$, i.e. there exists a continuous function $\phi : [0,1] \mapsto \mathbb{R}^2$ such that $\phi(0) \in A_1$, $\phi(1) \in A_2$, and $\phi(t) \in \mathbb{X} \cap B$ for every $t \in [0,1]$. If the set B is omitted, then it is understood $B = \mathbb{R}^2$. We use $A_1 \underset{\text{out } B}{\overset{\text{occ}}{\longleftrightarrow}} A_2$ for $A_1 \underset{\text{in } B^c}{\overset{\text{occ}}{\longleftrightarrow}} A_2$. Similarly, we write $A_1 \underset{\text{in } B}{\overset{\text{vac}}{\longleftrightarrow}} A_2$ if there exists a curve connecting A_1 and A_2 laying completely in $B \cap \mathbb{X}^c$.

Let $B_L(x)$ be the box $[x^1 - L, x^1 + L] \times [x^2 - L, x^2 + L]$. We say that the polygonal line $x_i, i = 0, \dots, n$, forms a left-right (LR) occupied crossing of $B_L(0)$ if all points x_i are in X, the disks around the successive points intersect (i.e. $d(x_{i-1}, x_i) \le 2$, $i = 1, \dots, n$), the points x_i, $i = 1, \dots n-1$, are in $B_L(0)$, and the first and the last disk intersect the left, resp. right, edge of $B_L(0)$ (i.e. $x_0^1 \in [-L-1, -L+1]$, $x_n^1 \in [L-1, L+1]$). Two LR occupied crossings are called disjoint if the corresponding polygonal lines do not intersect.

A smooth curve $\phi : [0,1] \mapsto \mathbb{R}^2$ is called LR vacant crossing of $B_L(0)$ if $\phi(0) \in \{-L\} \times [-L, L]$, $\phi(1) \in \{L\} \times [-L, L]$, and $\phi([0,1]) \in B_L(0) \cap \mathbb{X}^c$. Two LR vacant crossings ϕ and ϕ' are called disjoint if

$$\inf\big\{d(\phi(t), \phi'(t')) : t, t' \in [0,1]\big\} \ge 2. \tag{2.13}$$

The constant 2 has not any particular importance, any other positive constant can be chosen. Similarly, one defines the top-bottom (TB) crossings of $B_L(0)$. We will need the following proposition to prove Theorem 2.2.1.

Proposition 2.3.1. *(a) Let $\lambda > \lambda_c$, then there exist positive constants β, γ, L_0 depending only on λ such that*

$$\mathbb{P}_\lambda[\#\textit{ of disjoint occ. LR crossings of } B_L(0) \le \beta L] \le e^{-\gamma L} \tag{2.14}$$

for $L \ge L_0$.

(b) Let $\lambda < \lambda_c$, then there exist positive constants β', γ', L_0' depending only on λ such that

$$\mathbb{P}_\lambda[\#\textit{ of disjoint vac. LR crossings of } B_L(0) \le \beta' L] \le e^{-\gamma' L} \tag{2.15}$$

for $L \ge L_0'$.

We will prove part (a) of this proposition using the methods that are strongly inspired by discrete percolation (see [Gri99], Lemma 11.22). We start with the following lemma.

Lemma 2.3.2. *Let $\lambda > \lambda_c$, then there exists $\kappa > 0$, such that for L large enough*

$$\mathbb{P}_\lambda[\exists \text{ occ. LR crossing of } B_L(0)] \geq 1 - e^{-\kappa L}. \tag{2.16}$$

Proof. Using duality in $\mathbb{R}^2$ it is easy to see

$$\mathbb{P}_\lambda[\nexists \text{ occ. LR crossing of } B_L(0)] = \mathbb{P}_\lambda[\exists \text{ vac. TB crossing of } B_L(0)]. \tag{2.17}$$

If we place on the upper edge of $B_L(0)$ $2L+1$ boxes of size 2, then it is easy to see that the last expression can be bounded by

$$\begin{aligned} &\leq \sum_{i=-L}^{L} \mathbb{P}_\lambda[B_1((i,L)) \xleftrightarrow{\text{vac}} \text{ lower edge of } B_L(0)] \\ &\leq (2L+1)\mathbb{P}_\lambda[B_1(0) \xleftrightarrow{\text{vac}} \partial B_{2L}(0)]. \end{aligned} \tag{2.18}$$

We used the obvious notation $\partial B_L(0)$ for boundary of $B_L(0)$ and the translation invariance of the measure $\mathbb{P}_\lambda$.

Since $\lambda > \lambda_c$, it follows from (2.12) that $\mathbb{E}_\lambda[\text{diam}(V(B_1(0)))] < \infty$. Denoting by $\text{diam}'(A)$ the diameter of the set A in ∞-norm and using the obvious fact $\text{diam}'(A) \leq \text{diam}(A)$, we can write

$$\begin{aligned} \infty &> \mathbb{E}_\lambda[\,\text{diam}(V(B_1(0)))] \geq \mathbb{E}_\lambda[\,\text{diam}'(V(B_1(0)))] \\ &\geq \mathbb{E}_\lambda[\sup\{\|x\|_\infty : x \in V(B_1(0))\}] \\ &\geq \sum_{i=0}^{\infty} \mathbb{P}_\lambda[\sup\{\|x\|_\infty : x \in V(B_1(0))\} \geq i] \\ &= \sum_{i=0}^{\infty} \mathbb{P}_\lambda[B_1(0) \xleftrightarrow{\text{vac}} \partial B_i(0)]. \end{aligned} \tag{2.19}$$

From the last expression one can see that there exist k such that

$$4(k+2)\mathbb{P}(0 \xleftrightarrow{\text{vac}} \partial B_k(0)) \leq \eta < 1. \tag{2.20}$$

Indeed, suppose on the contrary that $P(0 \xleftrightarrow{\text{vac}} \partial B_k(0)) > \eta/4(k+2)$ for every k. Then the last sum in (2.19) is clearly infinite and we get the contradiction with the first inequality in (2.19).

Let $N \geq k+2$. By dividing the vacant connection from 0 to $\partial B_N(0)$ into two parts, first one from 0 to $\partial B_k(0)$ and second one from $\partial B_{k+2}(0)$ to $\partial B_N(0)$ we get

$$\begin{aligned} &\mathbb{P}_\lambda[B_1(0) \xleftrightarrow{\text{vac}} \partial B_N(0)] \\ &\qquad \leq \mathbb{P}_\lambda[(B_1(0) \xleftrightarrow{\text{vac}} \partial B_k(0)) \cap (\partial B_{k+2}(0) \underset{\text{out } B_{k+2}(0)}{\xleftrightarrow{\text{vac}}} \partial B_N(0))]. \end{aligned} \tag{2.21}$$

Further, let $\mathcal{Z}$ be the set of points laying on the segments composing the boundary of B_{k+2} that have the distance from the vertices of these segments divisible by 2. Around every point of $\mathcal{Z}$ we put a box whose edges have length 2. We get

$$\begin{aligned}\mathbb{P}_\lambda[B_1(0) \overset{\text{vac}}{\longleftrightarrow} \partial B_N(0)] \leq \\ \leq \mathbb{P}_\lambda\Big[B_1(0) \overset{\text{vac}}{\longleftrightarrow} \partial B_k(0) \cap \Big(\bigcup_{z\in\mathcal{Z}} B_1(z) \underset{\text{out } B_{k+2}(0)}{\overset{\text{vac}}{\longleftrightarrow}} \partial B_N(0)\Big)\Big].\end{aligned} \tag{2.22}$$

The events in the last equation are decreasing and are chosen to be disjoint (i.e. the disks, that can have influence on first event cannot change the second and vice versa). We can thus use BK inequality proved for continuum percolation by [GR99]. Hence,

$$\begin{aligned}&\mathbb{P}_\lambda[B_1(0) \overset{\text{vac}}{\longleftrightarrow} \partial B_N(0)] \\ &\qquad \leq \mathbb{P}_\lambda[B_1(0) \overset{\text{vac}}{\longleftrightarrow} \partial B_k(0)] \sum_z \mathbb{P}_\lambda[B_1(z) \underset{\text{out } B_{k+2}(0)}{\overset{\text{vac}}{\longleftrightarrow}} \partial B_N(0)] \\ &\qquad \leq 4(k+2)\mathbb{P}_\lambda[B_1(0) \overset{\text{vac}}{\longleftrightarrow} \partial B_k(0)]\mathbb{P}_\lambda[B_1(0) \overset{\text{vac}}{\longleftrightarrow} \partial B_{N-k-2}(0)] \\ &\qquad \leq \eta\mathbb{P}_\lambda[B_1(0) \overset{\text{vac}}{\longleftrightarrow} \partial B_{N-k-2}(0)].\end{aligned} \tag{2.23}$$

We used again the translation invariance of $\mathbb{P}_\lambda$ and (2.20). Iterating equation (2.23) until $N - j(k+2) \geq k+2$ we get

$$\mathbb{P}_\lambda[B_1(0) \overset{\text{vac}}{\longleftrightarrow} \partial B_N(0)] \leq \eta^{\lfloor N/(k+2)\rfloor}. \tag{2.24}$$

Substituting this into (2.18) we obtain

$$\mathbb{P}_\lambda[\nexists \text{ occ. LR crossing of } B_L(0)] \leq (2L+1)\eta^{\lfloor 2L/(k+2)\rfloor} \tag{2.25}$$

and the proof is finished taking L sufficiently large and κ slightly smaller than $-2\log\eta/(k+2) > 0$. □

To state the next lemma we need the following definition. Let E be an increasing event. We define the *r-kernel* $I_r(E)$ of this event as $I_r(E) = \{\omega \in E :$ every ω' such that $X(\omega) \supset X(\omega')$ and $|X(\omega) \setminus X(\omega')| \leq r$ is also in $E\}$. The event $I_r(E)$ is the set of configurations from which we can delete arbitrary r disks and E still occurs. The utility of this definition follows from the fact that the r-kernel of the event "there is a LR occupied crossing" is the event "there are $r+1$ LR occupied crossings". We have the following lemma (compare it with [Gri99], Theorem 2.45).

Lemma 2.3.3. *Let $\lambda_2 > \lambda_1$ and let E be an increasing event. Then*

$$1 - \mathbb{P}_{\lambda_2}[I_r(E)] \leq \left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right)^r (1 - \mathbb{P}_{\lambda_1}[E])\,. \tag{2.26}$$

Proof. Let X' be the λ_1/λ_2-thinning of X, i.e. the point process that we obtain from X by deleting each point independently with probability $1-\lambda_1/\lambda_2$. If X is the Poisson point process with density λ_2, then X' is again a Poisson point process, but this time with density λ_1. If $\omega \notin I_r(E)$, then there exists a set $B \subset X(\omega)$, such that $|B| \leq r$ and $\tilde{\omega}$ obtained from ω by deleting the points in B is not in E. If there are more such sets B, we choose one according to some predefined order. Conditionally on B, there is probability $(1-\lambda_1/\lambda_2)^{|B|}$ that we delete all points in B, i.e. we have

$$\begin{aligned} \mathbb{P}[X' \notin E | X \notin I_r(E)] &\geq \left(1 - \frac{\lambda_1}{\lambda_2}\right)^r \\ \mathbb{P}[X' \notin E] &\geq \left(\frac{\lambda_2 - \lambda_1}{\lambda_2}\right)^r \mathbb{P}[X \notin I_r(E)] \end{aligned} \tag{2.27}$$

and the claim follows easily. □

Proof of Proposition 2.3.1(a). Let A_L be the event that there exists an occupied LR crossing of $B_L(0)$. If $\lambda > \lambda_c$, then there exists λ', such that $\lambda > \lambda' > \lambda_c$ and $\kappa > 0$, such that

$$P_{\lambda'}[A_L] \geq 1 - e^{-\kappa L} \qquad \text{for } L \geq L_0. \tag{2.28}$$

Since $I_r(A_L) = \{\exists \text{ at least } r+1 \text{ disjoint LR occupied crossings}\}$ we choose $r = \beta L$. Using Lemma 2.3.3 we have

$$1 - \mathbb{P}_\lambda[\exists \text{ at least } \beta L \text{ occ. LR crossings}] \leq \left(\frac{\lambda}{\lambda - \lambda'}\right)^{\beta L} e^{-\kappa L}. \tag{2.29}$$

We now take β small enough to have $\gamma(\lambda, \lambda', \beta) = \kappa(\lambda') - \beta \log \frac{\lambda}{\lambda - \lambda'} > 0$. Using this choice we easily complete the proof. □

Proof of Proposition 2.3.1(b). The proof of this part is slightly more complicated since the vacant crossings do not have the discrete underlying structure. We will use a coarse graining to reduce this case to the discrete site percolation. We start with the following lemma.

Lemma 2.3.4. *Let $H(M, L)$ be the event that there is vacant crossing of the rectangle with sides M and $2L$ connecting the sides with length M. If $\lambda < \lambda_c$, then there exist positive constants C, ρ such that*

$$\mathbb{P}_\lambda[H(M, L)] \geq 1 - CLe^{-\rho M}. \tag{2.30}$$

Proof.

$$\begin{aligned}\mathbb{P}_\lambda[H(M,L)] &= 1-\mathbb{P}_\lambda[\exists \text{ occ. crossing in perpendicular direction}]\\ &\le 1-2L\mathbb{P}_\lambda[0 \overset{\text{occ}}{\longleftrightarrow} \partial B_M(0)] \le 1-2LCe^{-\rho M},\end{aligned} \tag{2.31}$$

In the last inequality we use the fact that if $\lambda < \lambda_c$, then (see page 38 of [MR96])

$$\mathbb{P}_\lambda(0 \overset{\text{occ}}{\longleftrightarrow} \partial B_M(0)) \le Ce^{-\rho M}. \tag{2.32}$$

This finishes the proof. □

Using this lemma we will prove a two dimensional version of coarse graining following closely the proof from [Gri99], page 191. We call the box $B_k(x)$ good if the next two conditions hold:

(i) there are both TB and LR vacant crossings of $B_k(x)$.

(ii) all other vacant clusters have diameter (in ∞-norm) smaller than k.

We want to prove the following lemma.

Lemma 2.3.5. *If $\lambda < \lambda_c$, then for every $\varepsilon > 0$ there exist k, such that*

$$\mathbb{P}[B_k(x) \text{ is good }] \ge 1-\varepsilon. \tag{2.33}$$

Proof. Without lost of generality we put $x = 0$. Let ρ be the constant from Lemma 2.3.4, $\nu > 1/\rho$, and k large enough such that $\nu \log k \le k$. We take four rectangles with sides $2k$ and $\nu \log k$ composing an "annulus" around the origin with the "outer radius" k and "inner radius" $k - \nu \log k$. More precisely, let R_1 be the rectangle $[-k, k] \times [-k, -k + \nu \log k]$ and let R_2, R_3 and R_4 be its images under rotations by $\pi/2$, π, and $3\pi/2$ around the origin.

Let B denote the event that there is a vacant crossing connecting the sides of length $\nu \log k$ inside of all these rectangles. The probability of this event can be bounded from below using the FKG inequality,

$$\mathbb{P}_\lambda[B] \ge \big(\mathbb{P}_\lambda[H(\nu \log k, k)]\big)^4. \tag{2.34}$$

Applying the previous lemma we have

$$\mathbb{P}_\lambda[B] \ge (1 - Ak^{1-\rho\nu})^4. \tag{2.35}$$

The last expression converges to 1 as k goes to infinity. Hence, we verified that condition (i) from the definition of the good block can be satisfied with arbitrarily large probability.

It remains to exclude the possibility that there is another cluster with diameter larger than k. This cluster has to cross the rectangle $[-k, k]\times[i, i+k]$

vertically or $[i, i+k] \times [-k, k]$ horizontally ($-k \le i \le 0$). However, the probability that there is horizontal or vertical vacant crossing of this rectangle tends exponentially to 1. Hence, this cluster is with overwhelming probability connected to the vacant crossing of one of the four rectangles $R_1, \dots, R_4$. This finishes the proof of Lemma 2.3.5. □

We now construct a block process Z_x, $x \in \mathbb{Z}^2$. Let $\varepsilon > 0$ and choose k large enough such that $\mathbb{P}_\lambda[B_k(0)$ is good $] \ge 1 - \varepsilon$. Let $Z_x = 1$ if $B_k(xk)$ is good, and $Z_x = 0$ otherwise. Obviously, Z_x is a dependent site percolation on $\mathbb{Z}^2$ with probability that $Z_x = 1$ larger than $1 - \varepsilon$. The definition of the good blocks implies the following property. For every nearest neighbours path $x_1, \dots, x_n$ in $\mathbb{Z}^2$ such that $Z_{x_i} = 1$, $i = 1, \dots, n$, there exist a vacant path of original continuum percolation passing through the blocks $B_k(x_i k)$. Hence, if we show that there is at least βL disjoint crossings of the square $B_{L/k}(0)$ for the process Z_x, the proof will be finished.

To prove this we use the standard method, namely stochastic domination. Let U_x and V_x be two families of random variables indexed by $x \in \mathbb{Z}^2$ and taking values in the set $\{0, 1\}$. We say that U stochastically dominates V if for all bounded, increasing, measurable functions $f : \{0, 1\}^{\mathbb{Z}^2} \to \mathbb{R}$ we have

$$\mathbb{E}(f(U)) \ge \mathbb{E}(f(V)). \tag{2.36}$$

We say that the family U_x is k-dependent if the random variables U_x and U_y are independent for all x, y such that $\|x - y\|_\infty > k$. The block process Z_x is clearly 2-dependent. Let Y_x^p denote the independent Bernoulli site percolation process on $\mathbb{Z}^2$ with the density p and let $\mathbb{P}_p^\star$ denote its measure. We use the following lemma from [LSS97].

Lemma 2.3.6. *Let V_x be a k-dependent family of random variables that satisfies $\mathbb{P}[V_x] \ge \delta$ for all $x \in \mathbb{Z}^2$. Then there exists a non-decreasing function $\pi(\delta) : [0, 1] \to [0, 1]$ satisfying $\pi(\delta) \to 1$ as $\delta \to 1$, such that V stochastically dominates $Y^{\pi(\delta)}$.*

We apply this lemma with $V = Z$. Let C be the event "there is at least βL disjoint LR crossings of $B_{L/k}(0)$." The event C is clearly increasing. Thus we have

$$\mathbb{P}_\lambda(C) \ge \mathbb{P}^\star_{\pi(1-\varepsilon)}(C). \tag{2.37}$$

We take ε such that $\pi(1 - \varepsilon)$ is larger than the percolation threshold p_c of independent site percolation. It is known that for independent site percolation above the threshold there exist constants $\tilde{\beta}$ and $\tilde{\gamma}$ such that

$$\mathbb{P}_p^\star[\text{there is at least } \tilde{\beta} L \text{ crossings of } B_L(0)] \ge 1 - e^{-\tilde{\gamma} L}. \tag{2.38}$$

Using this fact we easily complete the proof. □

For the proof of Theorem 2.2.2 we will need the following proposition. We recall that $\mathcal{V}$ denotes the set of borders of Voronoi cells around the points of the point process X. Let $\mathcal{W} \subset \mathcal{V}$. The LR crossing of $B_L(0)$ in $\mathcal{W}$ is the curve $\phi : [0,1] \to \mathbb{R}^2$ connecting the left and right side of $B_L(0)$ such that $\phi([0,1])$ is a subset of $\mathcal{W} \cap B_L(0)$. Two LR crossings are disjoint if they do not intersect.

Proposition 2.3.7. *Let $\mathcal{W}$ be the set of bonds b in $\mathcal{V}$ such that $d(b, X) \geq 1$ and let $0 < \lambda < \lambda_c$. Then there exist positive constants β'', γ'' and L_0'' depending only on λ such that*

$$\mathbb{P}_\lambda[\# \text{ of disjoint LR crossings of } B_L(0) \text{ in } \mathcal{W} \leq \beta'' L] \leq e^{-\gamma'' L} \tag{2.39}$$

for $L \geq L_0''$.

Proof. The proof of this proposition can be probably done by more elementary methods, but we prefer to use the previous result to prove it. We will use the fact that for every vacant crossing of $B_L(0)$ it is possible to find a path in $\mathcal{W}$ that is "not far" from this crossing.

To formalise the previous claim we first define the equivalence relation between LR vacant crossings of the strip $\mathcal{S}_L = [-L, L] \times \mathbb{R}$ (the LR vacant crossings of $\mathcal{S}_L$ are defined in the obvious way). We say that two crossings ϕ_1 and ϕ_2 are equivalent if there exists a continuous function $\Phi(t,s)$, such that $\Phi(t,0) = \phi_1(t)$, $\Phi(t,1) = \phi_2(t)$, for every fixed $s \in [0,1]$ $\Phi(t,s)$ is a LR crossing of $\mathcal{S}_L$, and $\Phi([0,1] \times [0,1]) \cap X = \emptyset$. Less formally, two crossings are not equivalent if there is a disk between them.

Observing now that every component W of the occupied region $\mathbb{X}$ is separated from $\mathbb{X} \setminus W$ by a loop in $\mathcal{W}$, it is easy to see that every vacant LR crossing of $B_L(0)$ is equivalent to a path in $\mathcal{W}$ that forms a crossing of $\mathcal{S}_L$ and, moreover, this path is almost uniquely determined (upto its starting and ending parts). There are two problems with this path. First, it can leave the box $B_L(0)$, secondly, two disjoint occupied crossings can be transformed to no disjoint paths in $\mathcal{W}$. Hence, we should construct a sufficient number of vacant crossings such that these two cases do not happen.

This can be achieved by a redefinition of the good blocks. We want to assure that the vacant crossing of the good block does not leave it after the transformation to a path in $\mathcal{W}$ and that the crossings of two neighbouring good blocks cannot be equivalent. The easiest way how to achieve it, is to force the good blocks to contain some disks that will force the paths in $\mathcal{W}$ to stay in the box. One way to do it is to consider the following definition of the good block.

We say that the block $B_{7k}(0)$ is good if every rectangle $[(2j-1)k, (2j+1)k] \times [-7k, 7k]$, $j \in \{-2, 0, 2\}$ contains a vertical vacant crossing and every rectangle $[-7k, 7k] \times [(2j-1)k, (2j+1)k]$ contains a horizontal vacant crossing. More over, every square

$$G_{jl} = [(2j-1)k+1, (2j+1)k-1] \times [(2l-1)k+1, (2l+1)k-1], \quad (2.40)$$

where $j, l \in \{-3, -1, 1, 3\}$, contains at least one disk. This construction is illustrated on Figure 2.1

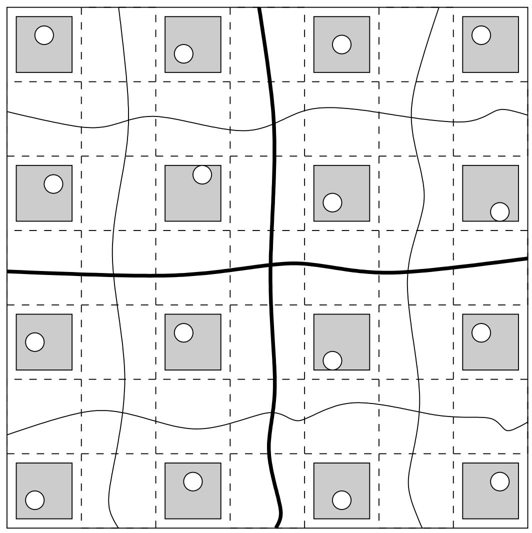

Fig. 2.1: Good block

The reader can verify that the disks in the squares G_{jl} do not permit the paths in $\mathcal{W}$ equivalent to the crossings of $[-k, k] \times [-7k, 7k]$ and $[-7k, 7k] \times [-k, k]$ to leave the box $B_{7k}(0)$. We define the box $B_{7k}(x)$ being good in the obvious way.

We should now show that the probability of the block being good can be made arbitrarily close to one. First, we observe that the crossings of the rectangles are independent of the configuration of X in the squares G_{jl}. The probability of having the long vacant crossings in all six rectangles can be bounded from below using the FKG inequality and Lemma 2.3.4 by $(1 - 7Ck\exp(-2k))^6$. The probability that there is at least one disk in any of G_{jl} is $1 - \exp(-\lambda(2k-2)^2)$. Hence

$$\mathbb{P}(B_{7k}(x) \text{ is good }) \geq (1 - 7Ck\exp(-2k))^6[1 - \exp(-\lambda(2k-2)^2)]^{16}. \quad (2.41)$$

Taking k large enough the right hand side of the previous expression can be made arbitrarily close to one.

We proceed in the obvious way. We define the process Z_x, $x \in \mathbb{Z}^2$. We set $Z_x = 1$ if the block $B_k(kx)$ is good. Otherwise we set $Z_x = 0$. As before, having path in Z assures us to have a crossing in $\mathcal{W}$ not leaving the boxes corresponding to the points of this path. Then we can continue exactly in the same manner as in the proof of Proposition 2.3.1. ☐

2.4 Proof of Theorems 2.2.1 and 2.2.2

Proof of Theorem 2.2.1. To prove Theorem 2.2.1 we apply the usual strategy. We express the effective conductivity $\sigma^\star(\lambda, \mu)$ in the form of a variational formula and we construct a test function that plugged into it will give us the required bound.

Upper bound: We use the following formula

$$\sigma^\star = \lim_{N\to\infty} \frac{1}{N^2} \inf_{u\in\mathcal{P}} \int_{\Lambda_N} \sigma(x)|\nabla u(x)|^2 \, dx, \tag{2.42}$$

where $\Lambda_N = [0, N]^2$ and

$$\mathcal{P} = \{u \in H^1(\Lambda_N) : u \text{ satisfies the boundary conditions in (2.4) } \}. \tag{2.43}$$

The infimum in (2.42) is attained by the solution of the system (2.4). That is why we are looking for a function that is not far from the solution and, moreover, the integral on the right-hand side of (2.42) is easy to compute.

Using the one dimensional analogy of our problem, it is not difficult to check that the potential u has large gradient in the places where there is a barrier to go through, i.e. where the conductivity is small. In the two-dimensional case such barriers should span all the width of the box. As we have already noted, our medium can be regarded as an ensemble of insulating grains around the points of the point process X. Hence, the easiest way how to construct a barrier is to have a chain of closely packed grains crossing the box from the top to the bottom. We need to specify what we mean by “closely packed”. According to the definition of $S_c(\lambda)$ we could not expect to find a crossing of the large box with the grains that have centres at a distance smaller than $2S_c(\lambda)$. Thus, we will choose the radius of grains slightly larger than $S_c(\lambda)$.

Let take $\varepsilon > 0$ and consider grains with the radius $S_c(\lambda) + \varepsilon$. We rescale temporarily the box Λ_N such that these grains become disks with radius 1. After the scaling we get a point process with density

$$\lambda' = \lambda(S_c(\lambda) + \varepsilon)^2. \tag{2.44}$$

From the definition (2.12) of λ_c it is easy to see that

$$S_c(\lambda_c) = 1. \tag{2.45}$$

Another application of scaling properties of the Poisson point process gives us

$$\lambda S_c(\lambda)^2 = \lambda_c. \tag{2.46}$$

If we put together the last three claims, we get $\lambda' > \lambda_c$. According to Proposition 2.3.1 (a), we know that there are with overwhelming probability at least

$$\beta(\lambda')N(2(S_c(\lambda)+\varepsilon))^{-1} \equiv \beta_\varepsilon N \tag{2.47}$$

top-bottom occupied crossings of rescaled box Λ_N with disks of radius one. If we now return to the original scale, we obtain $\beta_\varepsilon N$ chains of disks with radius $S_c(\lambda)+\varepsilon$ crossing Λ_N. Note that it will become clear in the next part of the proof why we need $O(N)$ crossings. One crossing would not be sufficient for our purposes.

We now define the test function that we will use. We use S_i to denote the crossings which we discussed in the previous paragraph. Let $i = 1, \ldots, R$, with R being the random number of crossings. We denote the crossings in the way that S_1 is the left most one, S_2 the second left one, etc. We recall that the occupied crossing was defined as a sequence of points from X with certain properties. We use $x_j^{(i)}$, $j = 1, \ldots, n_i$, to denote the points composing S_i in the way that $x_1^{(i)}$ is the point that is close to the lower edge and $x_{n_i}^{(i)}$ is close to the upper edge of Λ_N. We use $\bar{S}_i$ to denote the polygonal line connecting them. When $x_1^{(i)}$ is in the interior of Λ_N, we extend $\bar{S}_i$ by the vertical segment connecting $x_1^{(i)}$ with the lower edge of Λ_N. Similarly, if $x_{n_i}^{(i)}$ is in the interior of Λ_N, we connect it to the upper edge. Now, every line $\bar{S}_i$ divides the box into two disjoint parts.

We continue by smoothing off the lines $\bar{S}_i$. By smoothing we mean replacing the curves $\bar{S}_i$ by other set of curves that will be everywhere once differentiable and will have bounded curvature. The smoothing is necessary, it allows to construct a test-function that will have well defined gradient everywhere around these curves. The way how the smoothing is defined has no particular importance. For the sake of definiteness we chose the following one.

We will change the curves $\bar{S}_i$ only in the neighbourhoods $U(x_j^{(i)})$ of $x_j^{(i)}$ with the radius $S_c(\lambda)/10$. Choose one such point x. If there is no $y \in S_i$ such that $U(x) \cap U(y) \neq \emptyset$, we simply replace the two segments of $\bar{S}_i$ in $U(x)$ by a piece of circle. We do it in the way that the resulting curve is everywhere once differentiable. Since we can suppose that the minimal angle by any point $x_j^{(i)} \in S_i$ is $\pi/3$ (otherwise we can connect directly $x_{j-1}^{(i)}$ with $x_{j+1}^{(i)}$), we can bound the radius of the circle from below by some positive constant.

If, on the other hand, there is vertex $y \in S_i$ satisfying $U(x) \cap U(y) \neq \emptyset$, we argue in the following way. First, note that we can "optimise" the sets S_i in the way that for every point x there is at most one such y. Hence, we can consider only the pairs of "close" vertices. We should replace the polygonal line in the union of neighbourhoods $U(x)$, $U(y)$ by a smooth curve. We let the reader check that it is possible to make such replacement by two pieces of circle with the radii bounded from below.

Finally, we deform $\bar{S}_i$ slightly at its ends in the way that the smooth version is perpendicular to the boundary of Λ_N. We denote the smooth version of $\bar{S}_i$ by $\tilde{S}_i$. We use c_r to denote the lower bound on the radius of curvature of $\tilde{S}_i$.

Let us choose another constant $0 < d < c_r$. Denote by $\boldsymbol{S}_i$ the "tube" of radius d around $\tilde{S}_i$, i.e. the set $\{x \in \Lambda_N : d(x, \tilde{S}_i) \le d\}$. We use S_i^L, S_i^R to denote left and right boundary of $\boldsymbol{S}_i$. Let S_0^R, resp. S_{R+1}^L, be the left, resp. right, edge of Λ_N.

We construct the test function $u^\star(x)$ as follows. Let $u^\star(x)$ be constant between S_i^R and S_{i+1}^L, $i = 0, \dots, R$, and let $u^\star(x)$ grow linearly on the segments perpendicular to $\tilde{S}_i$ in the tubes $\boldsymbol{S}_i$. The condition $d < c_r$ ensures that for any point in $\boldsymbol{S}_i$ there is one and only one such segment. Let $u^\star(x)$ be continuous in Λ_N and let the difference of the values of $u^\star(x)$ on S_i^R and S_i^L be N/R. Such function is evidently in $\mathcal{P}$.

We plug the function $u^\star(x)$ into expression (2.42). Since $\nabla u^\star(x) = 0$ for all x outside the tubes $\boldsymbol{S}_i$ we have

$$\frac{1}{N^2} \int_{\Lambda_N} \sigma(x) |\nabla u^\star(x)|^2 \, dx = \frac{1}{N^2} \sum_{i=1}^{R} \int_{\boldsymbol{S}_i} \sigma(x) |\nabla u^\star(x)|^2 \, dx. \tag{2.48}$$

The value of $|\nabla u^\star(x)|^2$ we can bounded from above by

$$|\nabla u^\star(x)|^2 \le \frac{1}{4d^2} \cdot \frac{N^2}{R^2}. \tag{2.49}$$

Indeed, let x be an arbitrary point in $\boldsymbol{S}_i$ and let $s_x \ni x$ be the segment perpendicular to $\tilde{S}_i$ with the length $2d$ centred at $\tilde{S}_i$. The difference of the values of $u^\star$ on the ends of s_x is by definition N/R and function $u^\star$ is linear on s_x. Hence, the value of derivative of $u^\star$ in the direction of s_x is $N/2dR$. It remains to check that the derivative of $u^\star(x)$ in the direction perpendicular to s_x is zero. However, it is easy to verify using the fact that $\tilde{S}_i$ is composed by segments and pieces of circle, and that it is smooth.

We proceed by bounding the value of $\sigma(x)$. To achieve it, we divide every tube $\boldsymbol{S}_i$ into two disjoint regions. The good one

$$\boldsymbol{S}_i^g = \boldsymbol{S}_i \cap \{x \in \mathbb{R}^2 : S(x) \le S_c(\lambda) + \varepsilon\} \tag{2.50}$$

and the bad one $\boldsymbol{S}_i^b = \boldsymbol{S}_i \setminus \boldsymbol{S}_i^g$.

For $x \in \boldsymbol{S}_i^g$, the conductivity $\sigma(x)$ is smaller than $\exp(\mu(S_c(\lambda) + \varepsilon))$. To control the value of $\sigma(x)$ inside $\boldsymbol{S}_i^b$ we observe that $\boldsymbol{S}_i^b$ consists of parts similar

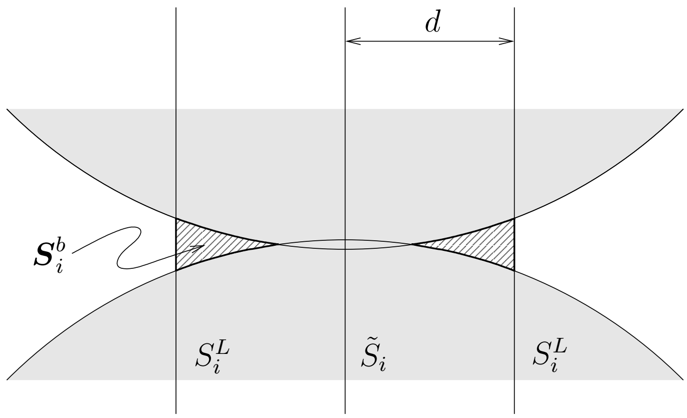

Fig. 2.2: Bad region of $\boldsymbol{S}_i$

to the striped regions on Figure 2.2. It is easy to check that there exists a constant $c_1 > 0$ such that for d small enough the area of one such piece is smaller than $c_1 d^3$. Similarly, we can find a constant $c_2 > 0$ such that the conductivity in the bad parts is bounded from above by $\exp(\mu(S_c(\lambda)+\varepsilon+c_2 d^2))$. Hence, we have

$$\int_{\boldsymbol{S}_i} \sigma(x)|\nabla u^\star(x)|^2\,dx = \int_{\boldsymbol{S}_i^g} \sigma(x)|\nabla u^\star(x)|^2\,dx + \int_{\boldsymbol{S}_i^b} \sigma(x)|\nabla u^\star(x)|^2\,dx \quad (2.51)$$

with

$$\int_{\boldsymbol{S}_i^g} \sigma(x)|\nabla u^\star(x)|^2\,dx \le \frac{1}{4d^2}\cdot\frac{N^2}{R^2}\exp(\mu(S_c(\lambda)+\varepsilon))|\boldsymbol{S}_i|, \quad (2.52)$$

and

$$\int_{\boldsymbol{S}_i^b} \sigma(x)|\nabla u^\star(x)|^2\,dx \le \frac{1}{4d^2}\cdot\frac{N^2}{R^2}\exp(\mu(S_c(\lambda)+\varepsilon+c_2 d^2))c_1 d^3 N_b, \quad (2.53)$$

where we use N_b to denote the number of bad pieces and $|A|$ to denote the Lebesgue measure of the set $A \subset \mathbb{R}^2$.

Since we try to find the result on the logarithmic scale only, we can use a rather crude bound, $|\bigcup_i \boldsymbol{S}_i| \le N^2$. We also claim that there exists a constant c_3 depending only on λ such that $N_b \le c_3 N^2$. The easiest way to see it, is to observe that bad pieces can come up only if there are two disks that almost touch in Λ_N. It is not possible to pack more than $O(N^2)$ disks that almost touch on R crossings of the box Λ_N. Putting all these estimates in expression

(2.42) we get

$$\begin{aligned}\sigma^\star(\lambda,\mu) \le \lim_{N\to\infty} \Bigg\{ & \frac{1}{N^2}\cdot\frac{1}{4d^2}\cdot\frac{N^2}{R^2}\exp[\mu(S_c(\lambda)+\varepsilon)]N^2 + \\ & \frac{1}{N^2}\cdot\frac{1}{4d^2}\cdot\frac{N^2}{R^2}\exp[\mu(S_c(\lambda)+\varepsilon+c_2d^2)]c_3N^2c_1d^3\Bigg\}. \end{aligned} \tag{2.54}$$

By Proposition 2.3.1(a) and Borel-Cantelli lemma for $\mathbb{P}$-a.e. realisation of the medium there is N_0 such that $R \ge \beta_\varepsilon N$ for all $N \ge N_0$. Hence, we have with probability one

$$\begin{aligned}\sigma^\star(\lambda,\mu) &\le Kd^{-2}\beta_\varepsilon^{-2}e^{\mu(S_c(\lambda)+\varepsilon)} + K'd\beta_\varepsilon^{-2}e^{\mu(S_c(\lambda)+\varepsilon+c_1d^2)} \\ &= e^{\mu(S_c(\lambda)+\varepsilon)}\beta_\varepsilon^{-2}\left(K\frac{1}{d^2}+K'de^{\mu d^2}\right),\end{aligned} \tag{2.55}$$

where K, K' are the constants that do not depend on μ, d and ε. From the last expression we easily get

$$\frac{1}{\mu}\log\sigma^\star(\lambda,\mu) \le S_c(\lambda)+\varepsilon+d^2+\frac{1}{\mu}\left[2\log\beta_\varepsilon-\log d+K''\right]. \tag{2.56}$$

We now set $d = d(\mu) = \exp(-\mu^{1/2})$ and compute the limit $\mu\to\infty$ of the last display. We obtain

$$\limsup_{\mu\to\infty}\frac{1}{\mu}\log\sigma^\star(\lambda,\mu) \le S_c(\lambda)+\varepsilon. \tag{2.57}$$

Since ε was arbitrary this gives the required upper bound.

Lower bound: For the lower bound we use the standard variational formula for the inverse of the homogenised matrix (see Chapters 1 and 8 of [JKO94] for its proofs for periodic, resp. random setting). The isotropic version of such formula can be written as

$$(\sigma^\star)^{-1} = \inf_{\boldsymbol{f}\in\mathcal{V}^2_{\rm sol}}\frac{1}{N^2}\int_{\Lambda_N}\sigma(x)^{-1}(\boldsymbol{e}^1+\boldsymbol{f}(x))^2\,dx, \tag{2.58}$$

where $\mathcal{V}^2_{\rm sol} = \{\boldsymbol{f} = (f_1,f_2) : f_1,f_2\in L^2(\Lambda_N), \operatorname{div}\boldsymbol{f} = 0, \int_{\Lambda_N}\boldsymbol{f}(x)\,dx = 0\}$, and $\boldsymbol{e}^1$ is the unit vector in x-direction.

The formula (2.58) can be rewritten using the fact that every function $\boldsymbol{f}\in\mathcal{V}^2_{\rm sol}$ can be written as $\boldsymbol{f} = (\frac{\partial v}{\partial x^2}, -\frac{\partial v}{\partial x^1})$ for some function $v\in H^1(\Lambda_N)$ that satisfies $v\equiv 0$ on $\partial\Lambda_N$. Setting $u(x^1,x^2) = v(-x^2,x^1)+x^1$, we have $\nabla u(x) = \boldsymbol{e}^1+\boldsymbol{f}(x)$. Thus (2.58) yields

$$\frac{1}{\sigma^\star} = \lim_{N\to\infty}\frac{1}{N^2}\inf_{u\in\mathcal{P}'}\int_{\Lambda_N}\sigma^{-1}(x)|\nabla u(x)|^2\,dx, \tag{2.59}$$

where $\mathcal{P}' = \{u \in H^1(\Lambda_N) : u(x^1, x^2) = x^1 \text{ on } \partial\Lambda_N\}$. This is the same variational formula as we used for the proof of the upper bound only with σ replaced by σ^{-1} and with $\mathcal{P}$ replaced by $\mathcal{P}'$. The second change corresponds to the change of boundary conditions. Since the boundary conditions do not influence the value of the effective conductivity we replace $\mathcal{P}'$ in (2.59) by $\mathcal{P}$. It allows us to use almost the same test-function as in the upper bound. The only difference is that the role of insulating grains and highly conducting domains between them will be reversed.

As in the proof of the upper bound we start by temporary rescaling of the box Λ_N. This time disks with radius $S_c(\lambda) - \varepsilon$ become disks with radius one. Using the same reasoning as in equations (2.44) and (2.46) we find that the density λ' of the rescaled point process is smaller than λ_c. According to Proposition 2.3.1 (b), there are at least $\beta'(\lambda')N(2(S_c(\lambda) - \varepsilon))^{-1} \equiv \beta'_\varepsilon N$ vacant crossings of rescaled box. Returning to the original scale we obtain the same number of paths traversing Λ_N in the complement of disks with radius $S_c(\lambda) - \varepsilon$.

We now use these crossings to construct the tubes similarly as in the upper bound. First note, that we can always deform them in the way that they will become once differentiable and will have the curvature bounded from above. We denote these smooth curves by $\tilde{S}_i$, $i = 1, \ldots, R$, and we construct the tubes $\boldsymbol{S}_i$ with the sufficiently small radius d and the function $u^\star(x)$ as before. The value of $|\nabla u^\star|^2$ in $\boldsymbol{S}_i$ is bounded from above by

$$|\nabla u^\star(x)|^2 \leq \frac{1}{4d^2} \cdot \frac{N^2}{R(\omega)^2} \tag{2.60}$$

and is zero in the rest of Λ_N. For $\sigma^{-1}(x)$ the following bound is valid in $\boldsymbol{S}_i$,

$$\sigma^{-1}(x) \leq \exp[-\mu(S_c(\lambda) - \varepsilon - d)]. \tag{2.61}$$

Plugging these two estimates into (2.59) we get

$$\frac{1}{\sigma^\star} \leq \lim_{N\to\infty} \frac{1}{N^2} \exp[-\mu(S_c(\lambda) - \varepsilon - d)] \frac{1}{4d^2} \cdot \frac{N^2}{R^2} \sum_{i=1}^{R} |\boldsymbol{S}_i|. \tag{2.62}$$

We bound the last sum by N^2 and use the fact that with overwhelming probability $R \geq \beta'_\varepsilon N$. Taking the logarithm we get

$$\frac{1}{\mu} \log \sigma^\star(\lambda, \mu) \geq S_c(\lambda) - \varepsilon - d - \frac{1}{\mu} \left[2|\log \beta'_\varepsilon| - \log d + K\right]. \tag{2.63}$$

Setting $d = d(\mu) = \exp(-\mu^{1/2})$ we obtain

$$\liminf_{\mu\to\infty} \frac{1}{\mu} \log \sigma^\star(\lambda, \mu) \geq S_c(\lambda) - \varepsilon. \tag{2.64}$$

Since ε was arbitrary it proves the lower bound. □

Proof of Theorem 2.2.2. From the fact

$$\hat{\sigma}_{\rho,\delta}(x,\mu) \le \tilde{\sigma}_{\rho}(x,\mu) \le \sigma(x,\mu). \tag{2.65}$$

and the variational formula (2.42) we easily get the upper bound,

$$\limsup_{\mu\to\infty} \frac{1}{\mu} \log \hat{\sigma}^{\star}_{\rho,\delta}(\lambda,\mu) \le \limsup_{\mu\to\infty} \frac{1}{\mu} \log \tilde{\sigma}^{\star}_{\delta}(\lambda,\mu) \le S_c(\lambda). \tag{2.66}$$

The dual variational formula (2.59) together with (2.65) imply that it is sufficient to prove the lower bound only for $\hat{\sigma}_{\rho,\delta}(x,\mu)$. We use the usual strategy to show it.

Let $\varepsilon > 0$ such that $\varepsilon \le \delta$. We rescale Λ_N in such a way that the disks with radius $S_c(\lambda) - \varepsilon$ become the disks with radius one. As in the proof of the lower bound for Theorem 2.2.1 we receive the process with sub-critical density λ'. The image of $\mathcal{V}_\varepsilon$ in this scaling is the set $\mathcal{W}$ defined in Proposition 2.3.7. As proved in that proposition there are at least $\beta''(\lambda')N(2(S_c(\lambda)-\varepsilon))^{-1} \equiv \beta''_\varepsilon N$ crossings of the rescaled box in $\mathcal{W}$. If we return back to the original scale, we conclude that there is $\beta''_\varepsilon N$ crossings of Λ_N in $\mathcal{V}_\varepsilon$. Moreover, it is not difficult to check that every crossing in $\mathcal{V}_\varepsilon$ can be smoothened in the way that the minimal radius of curvature is ρ and the tubes with radius ρ around the smooth version rest inside the tubes with the radius ρ around $\mathcal{V}_\varepsilon$. We use $\tilde{S}_i$ to denote the smooth crossings. We choose $d < \rho$ and we construct the test-function $u^\star(x)$ in the same way as before. Since $\boldsymbol{S}_i \subset \{x \in \mathbb{R}^2 : d(x,\mathcal{V}_\varepsilon) \le \rho\}$ and $\mathcal{V}_\varepsilon \subset \mathcal{V}_\delta$ we have

$$\hat{\sigma}_{\rho,\delta}(x,\mu) = \sigma(x,\mu) \quad \text{in} \quad \boldsymbol{S}_i. \tag{2.67}$$

After this observation the proof of the lower bound can be continued precisely in the same way as the proof of the lower bound for Theorem 2.2.1. □

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Part II

AGING IN BOUCHAUD'S TRAP MODEL

3. INTRODUCTION TO THE AGING PROBLEM

This part of the thesis is devoted to the study of aging in Bouchaud's trap model. Aging is one of the most interesting properties of the dynamics of complex disordered systems. It has been largely studied experimentally, numerically, and theoretically by physicists (see [BCKM98] for survey). The mathematical literature is relatively sparse, although some progress has been done in the last years [BBG02b, DGZ01, BG97].

In the physics literature, aging was studied particularly in the context of glassy dynamics. Bouchaud's trap model was proposed for the first time, in its simplest form, in [Bou92], and was further developed, among others, in [MB96] and [RMB01]. The model has a lot of dynamical properties that can be observed experimentally in glassy systems. Nevertheless, it can be studied rigorously.

In its most general form Bouchaud's trap model is defined in the following way. Let $G = (\mathcal{V}, \mathcal{E})$ be a connected graph. To every vertex $x \in \mathcal{V}$ is associated an energy E_x. The random variables E_x are usually chosen to be i.i.d. In the physics literature the distribution of E_x is mostly chosen to be exponential with mean one, but in this thesis we will consider a more general case. Bouchaud's trap model is a continuous time Markov chain $X(t)$ with state space $\mathcal{V}$. Its dynamics is determined by the transition rates between different vertices, $\mathbb{P}[X(t+dt) = y | X(t) = x] = w_{xy} dt$, where

$$w_{xy} = \begin{cases} \nu \exp\big\{ -\beta\big((1-a)E_x - aE_y\big)\big\} & \text{if } \langle x, y\rangle \in \mathcal{E}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

The constant β denotes the inverse temperature, the linear scaling factor ν changes only the time scale and is irrelevant for us. The parameter a characterises the symmetry of the dynamics. Its role will be precised later.

Originally, the model was studied in its "mean field form" [Bou92, MB96]. In this case the E_x's are updated after each jump. This simplifies the dynamics, since the time that is needed for n jumps becomes the sum of n i.i.d. random variables. It is evident that the properties of the graph G have in such a case very small relevance. In [MB96] the mean field case with $G = \mathbb{Z}^d$ was considered only to permit the study a kind of space correlations.

Bouchaud's trap model (already in its mean field version) undergoes dynamical phase transition at the temperature β_0 ($\beta_0 = 1$ if E_x are exponential with mean one). This transition can be observed for example considering the probability distribution $\mathcal{P}_E(t)$ of the energy of the site where the system is at time t. If $\beta < \beta_0$, then this distribution converges when $t \to \infty$ to an equilibrium distribution $\mathcal{P}_E$. On the other hand, if $\beta > \beta_0$, then $\mathcal{P}_E(t)$ does not converge.

Another way to observe this dynamical transition is suggested by the following argument put forward in the physics literature. The idea is to think in "the two-times plane", that is to consider the evolution of the system between two large times, generally denoted by t_w (like waiting time, t_w is the age of the system) and $t + t_w$ (t is then the duration of the observation of the system), and to let both t_w and t tend to infinity. The next step is to choose an appropriate two-point function, that is a function that depends on the evolution of the system during the time interval $(t_w, t + t_w)$, in order to measure how the system forgets its past during this interval. The simplest such function, considered frequently in the physics literature, is the probability that the system does not jump during the specified interval,

$$\Pi(t_w, t + t_w) = \mathbb{P}\big[X(t') = X(t_w)\,\forall t' \in [t_w, t + t_w]\big]. \tag{3.2}$$

How can the dynamical phase transition can be observed using this function? In the high temperature regime, $\beta < \beta_0$, since $\mathcal{P}_E(t)$ converges, there exists a nontrivial limit

$$\lim_{t_w \to \infty} \Pi(t_w, t + t_w) = f(t) \qquad \text{for all } t > 0 \text{ fixed.} \tag{3.3}$$

On the other hand, if $\beta > \beta_0$, then this limit equals one. This is true not only for constant times t, but also for all t satisfying $t = o(t_w^\gamma)$ for some $\gamma > 0$. Further, if $t \gg t_w^\gamma$, then the function $\Pi(t_w, t + t_w)$ tends to zero. Such a behaviour is referred to as *aging*.

Even more interesting is the behaviour of Π when t is taken in the critical regime, $t = \theta t_w^\gamma$. In this regime it is often possible to prove the existence of a nontrivial limit

$$\lim_{t_w \to \infty} \Pi(t_w, t_w + \theta t_w^\gamma) = f(\theta) \qquad \text{for all } \theta > 0. \tag{3.4}$$

Some authors reserve the term aging only for $\gamma = 1$. The cases $\gamma > 1$ and $\gamma < 1$ are then referred to as *superaging* and *subaging*.

The aging properties of Π are in fact closely related (by the Markov property) to the behaviour of the distribution $\mathcal{P}_E(t)$. As we will see later, in the aging regime the energy $E_{X(t)}$ of the trap where the system is at time t should be normalised in order to its distribution converges. To this end it is useful to introduce quantity

$$\tau_x = \exp(\beta E_x). \tag{3.5}$$

further referred to as the depth of the trap at site x. It is then possible to find a constant γ' such that the distribution of $\tau_{X(t)}/t^{\gamma'}$ converges. If $a = 0$ then $\gamma' = \gamma$.

This behaviour of the distribution of depth suggests which mechanism can be responsible for aging. At time t_w the system explores only a small part of its state space; the depth of the deepest trap is typically of the order $t_w^{\gamma'}$. In its future the system continues to explore its state space and can always find a much deeper trap, which slow it down more than all traps before.

There are other possible choices of two-point functions. Besides Π we will consider also the function

$$R(t_w, t + t_w) = \mathbb{P}[X(t_w) = X(t_w + t)]. \tag{3.6}$$

It is the probability that the system is in the same state at both times. Note, that both functions Π and R can be studied in both quenched and annealed regime, which means with and without taking the mean over all realisations of the random variables E_x.

Let us now explain how the dynamics of Bouchaud's trap model is related to the dynamics of real disordered systems. It is widely accepted in the physics literature that the energy landscape of a finite disordered system is extremely rough, with many local minima corresponding to metastable configurations. These minima are surrounded by rather high energy barriers. That is why such minima act like traps which get hold of the system during a certain time τ. The time τ clearly depends on the temperature of the system, but mainly on the depth of the valley around the particular local minimum.

The dynamics of Bouchaud's trap model is inspired by this picture. The configuration space of the original system is strongly reduced. To every state x of Bouchaud's trap model corresponds one local minimum of the energy landscape. The value E_x is then the depth of the valley around this minimum. The high energy configurations of the original system are neglected, because the time spent there is short. The deep traps in Bouchaud's trap model thus correspond to really very deep local minima of the energy landscape.

As we have already noted, the distribution of E_x is usually chosen to be exponential in physics literature. This choice is justified by the fact that only extremal configurations are considered, and that the tail of the exponential distribution is the same as the tail of the Gumble distribution of extreme value statistics. A more detailed discussion of the choice of E_x can be found in [BM97].

In the easiest case, when $a = 0$, the dynamics of Bouchaud's trap model is particularly simple, because the transition rates depend only on the state where X is, and not on its neighbours. Such dynamics is sometimes called the Random Hopping Time (RHT) dynamics. In this case the mean time that the system is trapped in state x is proportional to $\tau_x = \exp(\beta E_x)$. The probability distribution of τ_x becomes heavy tailed in the low temperature regime, $\beta > \beta_0$. Actually, τ_x satisfies

$$\mathbb{P}[\tau_x > u] \sim u^{-\alpha}, \tag{3.7}$$

with $\alpha = \beta^{-1}$. This implies that the mean time of stay in a trap has no expectation if the temperature is low. This induces very slow dynamics. The system needs an infinite time to explore its configuration space. Such behaviour is sometimes referred to in physics literature as *weak entropy breaking* [Bou92].

If $a \neq 0$, then the system is attracted by deep traps, and the dynamics becomes more difficult to handle. Note, however, that w_{xy} can be written as $w_{xy} = \nu \tau_x^{-(1-a)} \tau_y^a$. This means that τ_x still has some importance owing to the fact that the neighbouring sites of very deep traps are usually quite "normal" (this picture will be precised later). The importance of τ_x can be seen also from the fact that τ_x satisfies the detailed balance equation

$$\sum_{y \in \mathcal{V}} w_{xy} \tau_x = \sum_{y \in \mathcal{V}} w_{yx} \tau_y. \tag{3.8}$$

Hence, τ_x (if regarded as a measure on $\mathcal{V}$) is reversible for the Markov chain X, for all values of a.

The first mathematical treatment of Bouchaud's trap model can be found in [FIN02], where the behaviour of the function R in the case $\mathcal{V} = \mathbb{Z}$, $\mathcal{E} = \{\langle i, i+1 \rangle, i \in \mathbb{Z}\}$, and $a = 0$ is considered. In this thesis we generalise their results to $a \neq 0$, and we will also prove the subaging of the function Π (Chapter 4). Further (in Chapters 5 and 6), we will consider Bouchaud's trap model on d-dimensional cubic lattice, that is $\mathcal{V} = \mathbb{Z}^d$, $d \geq 2$, $\mathcal{E} = \{\langle i, j \rangle, \operatorname{dist}(i,j) = 1\}$, with the symmetry parameter a equal to 0. In all studied cases we will show that the aging occurs if the temperature is smaller than the critical temperature β_0. The critical temperature is characterised by the fact that for any $\beta > \beta_0$ the random variable $\tau_x = \exp(\beta E_x)$ is in the domain of the attraction of a α-stable law with index $\alpha < 1$. (To simplify the reasoning in $d \geq 2$ we will suppose that $\lim_{u \to \infty} u^\alpha \mathbb{P}[\tau_x > u] = 1$.)

We will prove that in all dimensions for $a = 0$ and in $d = 1$ for $a \in [0, 1]$, the two-point function R has the aging behaviour,

$$\lim_{t_w \to \infty} R(t_w, t_w + \theta t_w) = R(\theta), \tag{3.9}$$

where $R(\theta)$ does not depend on a in $d = 1$.

The behaviour of the two-point function Π is more interesting. The critical scaling between t_w and t depends on d and also on a if $d = 1$. We will show the (sub)aging behaviour of the two-point function Π,

$$\lim_{t_w \to \infty} \Pi(t_w, t_w + \theta f(t_w)) = \Pi(\theta), \tag{3.10}$$

where $f(t_w)$ satisfies

$$f(t_w) = \begin{cases} t_w^{\frac{1-a}{1+\alpha}} & \text{if } d = 1,\ a \in [0,1], \\ \frac{t_w}{\log t_w} & \text{if } d = 2,\ a = 0, \\ t_w & \text{if } d \geq 3,\ a = 0. \end{cases} \tag{3.11}$$

The different scaling follows from the fact that the Markov chain X in low dimensions is recurrent and visits any site an infinite number of times. In $d \geq 3$ any site is visited only a finite number of times. This implies that there is no difference between the scalings of Π and R.

The study of one-dimensional Bouchaud's trap model is contained in Chapter 4, which is taken over from [BČ02]. The two-dimensional model is studied in Chapter 5, which will be contained in [BČM]. The modifications of the proof given there that are needed if $d \geq 3$ can be found in Chapter 6.

4. BOUCHAUD'S MODEL EXHIBITS TWO DIFFERENT AGING REGIMES IN DIMENSION ONE

Gérard Ben Arous, Jiří Černý

Abstract. Let E_i be a collection of i.i.d. exponetnial random variables. Bouchaud's model on $\mathbb{Z}$ is a Markov chain $X(t)$ whose transition rates are given by $w_{ij} = \nu \exp(-\beta((1-a)E_i - aE_j))$ if i, j are neighbours in $\mathbb{Z}$. We study the behaviour of two correlation functions: $\mathbb{P}[X(t_w+t) = X(t_w)]$ and $\mathbb{P}\big[X(t') = X(t_w) \forall t' \in [t_w, t_w+t]\big]$. We prove the (sub)aging behaviour of these functions when $\beta > 1$ and $a \in [0,1]$.

4.1 *Introduction*

Aging is an out-of-equilibrium physical phenomenon that is gaining considerable interest in contemporary physics and mathematics. An extensive literature exists in physics (see [BCKM98] and their references). The mathematical literature is substantially smaller, although some progress was achieved in recent years ([BDG01, BBG02a, DGZ01, FIN02], see also [Ben02] for short summary).

The following model has been proposed by Bouchaud as a toy model for studying the aging phenomenon. Let $G = (\mathcal{V}, \mathcal{E})$ be a graph, $E = \{E_i\}_{i \in \mathcal{V}}$ be the collection of i.i.d. random variables indexed by vertices of this graph with the common exponential distribution with mean one. We consider the continuous time Markov chain $X(t)$ with state space $\mathcal{V}$, such that

$$\mathbb{P}(X(t+dt) = j | X(t) = i, E) = \begin{cases} w_{ij}dt & \text{if } i,\ j \text{ are connected in } G \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

The transition rates w_{ij} are defined by

$$w_{ij} = \nu \exp\big(-\beta((1-a)E_i - aE_j)\big). \tag{4.2}$$

The parameter β denotes, as usually, the inverse temperature and the parameter a, $0 \leq a \leq 1$, drives the "symmetry" of the model. The value of ν fixes the time scale and is irrelevant for our paper, we thus set $\nu = 1$.

This model has been studied when G is $\mathbb{Z}$ and $a = 0$ in [FIN99, FIN02]. It is an elementary model when G is the complete graph, which is a good ansatz for the dynamics of the REM (see [BBG02b]).

The time spent by the system at site i grows with the value of E_i. The value of E_i can thus be regarded as the depth of the trap at the site i. The model is sometimes referred to as "Bouchaud's trap model." It describes the motion of the physical system between the states with energies $-E_i$. It can be regarded as a useful rough approximation of spin-glass dynamics. The states of Bouchaud's trap model correspond to a subset of all possible states of the spin-glass system with exceptionally low energy. This justifies in a certain sense the exponential distribution of E_i since it is the distribution of extreme values. The idea behind this model is that the spin-glass dynamics spends most of the time in the deepest states and it passes through all others extremely quickly. Thus, only the extremal states are important for the long time behaviour of dynamics, which justifies formally the introduction of Bouchaud's model.

Usually, proving an aging result consists in finding a two-point function $F(t_w, t_w + t)$, a quantity that measures the behaviour of the system at time $t + t_w$ after it has aged for the time t_w, such that a nontrivial limit

$$\lim_{\substack{t\to\infty \\ t/t_w=\theta}} F(t_w, t_w + t) = F(\theta) \tag{4.3}$$

exists. The choice of the two-point function is crucial. For instance it has be shown in [RMB00] that a good choice is

$$R(t_w, t_w + t) = \mathbb{E}\mathbb{P}(X(t + t_w) = X(t_w)|E), \tag{4.4}$$

which is the probability that the system will be in the same state at the end of the observation period (i.e. at time $t + t_w$) as it was in the beginning (i.e. at time t_w). Another quantity exhibiting aging behaviour, which was studied in [FIN02] is

$$R^q(t_w, t_w + t) = \mathbb{E} \sum_{i\in\mathbb{Z}} [\mathbb{P}(X(t + t_w) = i|E, X(t_w))]^2, \tag{4.5}$$

which is the probability that two independent walkers will be at the same site after time $t + t_w$, if they were at the same site at time t_w. These authors have proved that, for these two two-point functions, aging occurs when $a = 0$. We extend this result to the case $a > 0$. The limiting object will be independent of a. Thus the parameter a could seem to be of no relevance for aging.

However, it is not the case for all two-point functions. For instance, for the function

$$\Pi(t_w, t_w + t) = \mathbb{E}\mathbb{P}(X(t') = X(t_w) \forall t' \in [t_w, t_w + t] | E), \tag{4.6}$$

giving the probability that the system does not change its state between t_w and $t_w + t$, it was predicted in [RMB00] that there exists a constant γ such that the limit $\lim_{t_w \to \infty} \Pi(t_w, t_w + \theta t_w^\gamma)$ exists and depends non-trivially on a. The name *subaging* was introduced for this type of behaviour, i.e. for the fact that there exists a constant $0 < \gamma < 1$ such that for some two-point function $F(t_w, t_w + t)$, there is a nontrivial limit

$$\lim_{\substack{t \to \infty \\ t/t_w^\gamma = \theta}} F(t_w, t_w + t) = F(\theta). \tag{4.7}$$

One of the main results of the present paper is the proof of the subaging behaviour of the function (4.6) for an arbitrary $a \in [0, 1]$.

Let us have a closer look at the role of the parameter a. If $a = 0$, the dynamics of the model is sometimes referred as "Random hopping time (RHT) dynamics" (cf. [Mat00]). In this case the rates w_{ij} do not depend on the value of E_j. Hence, the system jumps to all neighbouring sites with the same probability and the process $X(t)$ can be regarded as a time change of the simple random walk.

On the other hand, if $a > 0$, the system is attracted to the deepest traps and the underlying discrete time Markov chain is some kind of random walk in a random environment (RWRE). There are already some results about aging of RWRE in dimension one [DGZ01]. It that article Sinai's RWRE is considered. It is proved there that there is aging on the scale $\log t / \log t_w \to \text{const.}$

In our situation the energy landscape, far from being seen as a two-sided Brownian motion as in Sinai's RWRE, should be seen as essentially flat with few very narrow deep holes around the deep traps. The drifts on neighbouring sites are dependent and this dependency does not allow the existence of large domains with drift in one direction. This can be easily seen by looking at sites surrounding one particularly deep trap E_i. Here, the drift at site $i - 1$ pushes the system very strongly to the right and at site $i + 1$ to the left because the system is attracted to the site i. Moreover, these drifts have approximately the same size. A more precise description of this picture will be presented later (Section 4.5). However, these differences do not change notably the mechanism responsible for aging. Again, during the exploration of the random landscape, the process X finds deeper and deeper traps that slow down its dynamics.

It was observed numerically in [RMB00] that $X(t)$ ages only if the temperature is low enough, $\beta > 1$. (In the sequel we will consider only the low temperature regime.) This heuristically corresponds to the fact that if $a = 0$ and $\beta > 1$, the mean time $\mathbb{E}(\exp(\beta E_0))$ spent by $X(t)$ at arbitrary site becomes infinite. This implies that the distribution of the depth at which we find the system at time t does not converge as $t \to \infty$. The process $X(t)$ can find deeper and deeper traps where it stays longer.

If $a > 0$, the previous explanation is not precise. The time before the jump is shortened when a increases. On the other hand, the system is attracted to deep traps. This means that, instead of one long period spent in one deep trap, the process prefers to jump outside and then to return to it more often. For the two-point functions (4.4) and (4.5) these two effects cancel and the limiting behaviour is thus independent of a. For the two-point function (4.6), there cannot be cancellation, because the attraction to deeper traps has no influence on it.

Before stating the known results about the model we generalise it slightly. All statements in this paper do not actually require E_i to be an exponential random variable. The only property of E_i that we will need is that the random variable $\exp(\beta E_i)$ is in the domain of attraction of the totally asymmetric stable law with index $\beta^{-1} \equiv \alpha$. Clearly, the original exponential random variable satisfies this property.

Recently, this model was studied rigorously in [FIN99, FIN02] in connection with the random voter model and chaotic time dependence. In this paper only the RHT case, $a = 0$, was considered. If $d = 1$ and $\beta > 1$, they proved that the Markov chain $X(t)$ possess an interesting property called there localisation. Namely, it was shown there that

$$\limsup_{t\to\infty} \sup_{i\in\mathbb{Z}} \mathbb{P}(X(t) = i|E) > 0. \tag{4.8}$$

Also aging for the two-point functions (4.4) and (4.5) was proved there. In dimension $d \geq 2$, results of this paper imply that there is no localisation in the sense of (4.8). However, there is numerical evidence [RMB00] that the system ages. A rigorous proof of this claim will be presented in a forthcoming paper [BČM].

In this article we generalise the results of [FIN02] in dimension one to the general case, $a \neq 0$. As we have already noted, the main difficulty comes from the fact that the underlying discrete time Markov chain is not a simple random walk. We will prove aging for the quantities (4.4) and (4.5). We will then prove sub-aging for the two-point function (4.6).

As in [FIN02] we relate the asymptotic behaviour of quantities (4.4), (4.5), and (4.6) to the similar quantities computed using a singular diffusion $Z(t)$ in a random environment ρ — singular meaning here that the single time distributions of Z are discrete.

Definition 4.1.1 (Diffusion with random speed measure). *The random environment ρ is a random discrete measure, $\sum_i v_i \delta_{x_i}$, where the countable collection of (x_i, v_i)'s yields an inhomogeneous Poisson point process on $\mathbb{R} \times (0, \infty)$ with density measure $dx\, \alpha v^{-1-\alpha} dv$. Conditional on ρ, $Z(s)$ is a diffusion process (with $Z(0) = 0$) that can be expressed as a time change of a standard one-dimensional Brownian motion $W(t)$ with the speed measure ρ. Denoting $\ell(t, y)$ the local time of $W(t)$ at y, we define*

$$\phi^\rho(t) = \int \ell(t, y)\rho(dy) \tag{4.9}$$

and the stopping time $\psi^\rho(s)$ as the first time t when $\phi^\rho(t) = s$; then $Z(t) = W(\psi^\rho(t))$.

A more detailed description of time changes of Brownian motion can be found in Section 4.2.

Our main result about aging is the following

Theorem 4.1.2. *For any $\beta > 1$ and $a \in [0, 1]$ there exist nontrivial functions $R(\theta)$, $R^q(\theta)$ such that*

$$\begin{aligned}
\lim_{t\to\infty} R(t, t + \theta t) &= \lim_{t\to\infty} \mathbb{E}\mathbb{P}[X((1 + \theta)t) = X(t)|E] = R(\theta), \\
\lim_{t\to\infty} R^q(t, t + \theta t) &= \lim_{t\to\infty} \mathbb{E} \sum_{i\in\mathbb{Z}} [\mathbb{P}(X((1 + \theta)t) = i|E, X(t))]^2 = R^q(\theta).
\end{aligned} \tag{4.10}$$

Moreover, $R(\theta)$ and $R^q(\theta)$ can be expressed using the similar quantities defined using the singular diffusion Z:

$$\begin{aligned}
R(\theta) &= \mathbb{E}\mathbb{P}[Z(1 + \theta) = Z(1)|\rho], \\
R^q(\theta) &= \mathbb{E} \sum_{x\in\mathbb{R}} [\mathbb{P}(Z(1 + \theta) = x|\rho, Z(1))]^2.
\end{aligned} \tag{4.11}$$

For $a = 0$, this result is contained in [FIN02]. Since the diffusion $Z(t)$ does not depend on a, the functions $R(\theta)$ and $R^q(\theta)$ do not depend on it either. This is the result of the compensation of shorter visits of deep traps by the attraction to them.

We will also prove sub-aging for the quantity $\Pi(t_w, t_w + t)$. We use γ to denote the *subaging exponent*

$$\gamma = \frac{1}{1 + \alpha} = \frac{\beta}{1 + \beta}. \tag{4.12}$$

Theorem 4.1.3. *For any $\beta > 1$ and $a \in [0,1]$ there exist a nontrivial function $\Pi(\theta)$ such that*

$$\lim_{t\to\infty} \Pi(t, t + f_a(t,\theta)) \\ = \lim_{t\to\infty} \mathbb{E}\mathbb{P}\big[X(t') = X(t) \forall t' \in [t, t + f_a(t,\theta)] | E\big] = \Pi(\theta), \quad (4.13)$$

where the function f_a is given by

$$f_a(t,\theta) = \theta t^{\gamma(1-a)} L(t)^{1-a}, \quad (4.14)$$

and $L(t)$ is a slowly varying function that is determined only by the distribution of E_0. Its precise definition is given in Lemma 4.8.1. The function $\Pi(\theta)$ can be again written using the singular diffusion Z,

$$\Pi(\theta) = \int_0^\infty g_a^2(\theta u^{a-1}) dF(u), \quad (4.15)$$

where $F(u) = \mathbb{E}\mathbb{P}[\rho(Z(1)) \leq u | \rho]$, and where $g_a(\lambda)$ is the Laplace transform $\mathbb{E}(e^{-\lambda T_a})$ of the random variable

$$T_a = 2^{a-1} \exp(a\beta E_0)[\mathbb{E}(\exp(-2a\beta E_0))]^{1-a}. \quad (4.16)$$

If $a = 0$, (4.15) can be written as

$$\Pi(\theta) = \int_0^\infty e^{-\theta/u} dF(u). \quad (4.17)$$

Remark. Note that if E_i's are exponential random variables, the function $L(t)$ satisfies $L(t) \equiv 1$. The same is true if $\exp(\beta E_i)$ has a stable law.

As can be seen, in this case the function $\Pi(\theta)$ depends on a. This is not surprising since the compensation by attraction has no influence here and the jumps rates clearly depend on a.

This behaviour of the two-point functions $\Pi(t_w, t + t_w)$ and $R(t_w, t + t_w)$ is not difficult to understand, at least heuristically. One should first look at the behaviour of the distribution of the depth of the location of the process at time t_w. It can be proved that this depth grows like $t_w^{1/(1+\alpha)}$ (see Proposition 4.8.2). From this one can see that the main contribution to quantities (4.4) and (4.5) comes from trajectories of $X(t)$ that, between times t_w and $t_w + t$, leave $t_w^{(a+\alpha)/(1+\alpha)}$ times the original site and then return to it. Each visit of the original site lasts an amount of time of order $t_w^{(1-a)/(1+\alpha)}$.

In the case of the two-point function (4.6), we are interested only in the first visit and thus the time t should scale as $t_w^{1/(1+\alpha)}$. Proofs can be found in Sections 4.7, 4.8 and 4.9. In Section 4.2 we summarise some known results about time-scale changes of Brownian motion and about point-process convergence. In Section 4.3 we express the process X and its scaled versions as a time-scale change and in Section 4.4 we introduce a coupling between the different scales of X. In Section 4.5 we prove convergence of speed measures which is used for time-scale change and we apply this result to show the convergence of finite time distributions of rescaled versions of X to the finite time distributions of Z.

4.2 Definitions and known results

In this section we define some notations that we will use often later, and we summarise some known results.

4.2.1 Time-scale change of Brownian motion

The limiting quantities $R(\theta)$, $R^q(\theta)$, and $\Pi(\theta)$ are expressed using the singular diffusion defined by a time change of Brownian motion. So, it will be convenient to express also the chains with discrete state space as a time-scale change of Brownian motion. The scale change is necessary if $a \neq 0$, because the process $X(t)$ does not jump left or right with equal probabilities.

Consider a locally finite measure

$$\mu(dx) = \sum_i w_i \delta_{y_i}(dx) \tag{4.18}$$

which has atoms with weights w_i at positions y_i . The measure μ will be referred to as the speed measure. We denote positions of atoms y_i in the way that $y_i < y_j$ if $i < j$. Let S be a strictly increasing function defined on the set $\{y_i\}$. We call such S the scaling function. Let us introduce slightly nonstandard notation $S \circ \mu$ for the "scaled measure"

$$(S \circ \mu)(dx) = \sum_i w_i \delta_{S(y_i)}(dx). \tag{4.19}$$

We use $W(t)$ to denote the standard Brownian motion starting at 0. Let $\ell(t, y)$ be its local time. We define the function

$$\phi(\mu, S)(t) = \int_{\mathbb{R}} \ell(t, y)(S \circ \mu)(dy). \tag{4.20}$$

and the stopping time $\psi(\mu, S)(s)$ as the first time when $\phi(\mu, S)(t) = s$. The function $\phi(\mu, S)(t)$ is a nondecreasing, continuous function, and $\psi(\mu, S)(s)$ is its generalised right continuous inverse. It is an easy corollary of the results of [Sto63] that the process

$$X(\mu, S)(t) = S^{-1}(W(\psi(\mu, S)(t))) \tag{4.21}$$

is a nearest neighbours continuous time random walk on the set of atoms of μ. Moreover, every nearest neighbours random walk on a countable, nowhere dense subset of $\mathbb{R}$ satisfying some mild conditions on transition probabilities can be expressed in this way. We call the process $X(\mu, S)$ the time-scale change of Brownian motion. If S is the identity function, we speak only about time change.

The following proposition describes the properties of $X(\mu, S)$. It is the consequence of [Sto63], Section 3. The extra factor 2 comes from the fact that Stone uses the Brownian motion with generator $-\Delta$.

Proposition 4.2.1. *The process $X(\mu, S)(t)$ is a nearest neighbours random walk on the set $\{y_i\}$ of atoms of μ. The waiting time in the state y_i is exponentially distributed with mean*

$$2w_i \frac{(S(y_{i+1}) - S(y_i))(S(y_i) - S(y_{i-1}))}{S(y_{i+1}) - S(y_{i-1})}. \tag{4.22}$$

After leaving state y_i, $X(\mu, S)$ enters states y_{i-1} and y_{i+1} with respective probabilities

$$\frac{S(y_{i+1}) - S(y_i)}{S(y_{i+1}) - S(y_{i-1})} \quad \textit{and} \quad \frac{S(y_i) - S(y_{i-1})}{S(y_{i+1}) - S(y_{i-1})}. \tag{4.23}$$

It will be useful to introduce another process $Y(\mu, S)$ as

$$Y(\mu, S)(t) = X(S \circ \mu, \mathrm{Id})(t), \tag{4.24}$$

where Id is the identity function on $\mathbb{R}$. The process $Y(\mu, S)$ can be regarded as $X(\mu, S)$ before the final change of scale in (4.21). Actually,

$$Y(\mu, S)(t) = W(\psi(\mu, S)(t)). \tag{4.25}$$

We will also need processes that are not started at the origin but at some point $x \in \operatorname{supp} \mu$. They are defined in the obvious way using the Brownian motion started at $S(x)$. We use $X(\mu, S; x)$ and $Y(\mu, S; x)$ to denote them.

4.2.2 Point process convergence

To be able to work with quantities (4.4)–(4.6) that have a discrete nature (in the sense that they depend on the probability being exactly at some place) we recall the definition of the point process convergence of measures introduced in [FIN02]. Let $\mathcal{M}$ denote the set of locally finite measures on $\mathbb{R}$.

Definition 4.2.2 ([FIN02]). *Given a family ν, ν^ε, $\varepsilon > 0$, in $\mathcal{M}$, we say that ν^ε converges* in the point process sense *to ν, and write $\nu^\varepsilon \stackrel{pp}{\to} \nu$, as $\varepsilon \to 0$, provided the following holds: if the atoms of ν, ν^ε are, respectively, at the distinct locations y_i, $y^\varepsilon_{i'}$ with weights w_i, $w^\varepsilon_{i'}$, then the subsets of $V^\varepsilon \equiv \cup_{i'}\{(y^\varepsilon_{i'}, w^\varepsilon_{i'})\}$ of $\mathbb{R} \times (0,\infty)$ converge to $V \equiv \cup_i\{(y_i, w_i)\}$ as $\varepsilon \to 0$ in the sense that for any open U, whose closure $\bar{U}$ is a compact subset of $\mathbb{R} \times (0,\infty)$ such that its boundary contains no points of V, the number of points $|V^\varepsilon \cap U|$ in $V^\varepsilon \cap U$ is finite and equals $|V \cap U|$ for all ε small enough.*

Beside this type of convergence we will use the following two more common types of convergence

Definition 4.2.3. *For the same family as in the previous definition, we say that ν^ε converges* vaguely *to ν, and write $\nu^\varepsilon \stackrel{v}{\to} \nu$, as $\varepsilon \to 0$, if for all continuous real-valued functions f on $\mathbb{R}$ with bounded support $\int f(y)\nu^\varepsilon(dy) \to \int f(y)\nu(dy)$ as $\varepsilon \to 0$. We say that ν^ε converges* weakly*, and we write $\nu^\varepsilon \stackrel{w}{\to} \nu$, as $\varepsilon \to 0$, if the same is true for all bounded continuous functions on $\mathbb{R}$.*

To prove the point process convergence we will use the next lemma that is the copy of Proposition 2.1 of [FIN02].

Let ν, ν^ε be locally finite measures on $\mathbb{R}$ and let (y_i, w_i), $(y^\varepsilon_i, w^\varepsilon_i)$ be the sets of atoms of these measures (y_i is the position and w_i is the weight of the atom).

Condition 1. *For each l there exists a sequence $j_l(\varepsilon)$ such that*

$$(y^\varepsilon_{j_l(\varepsilon)}, w^\varepsilon_{j_l(\varepsilon)}) \to (y_l, w_l) \quad as \quad \varepsilon \to 0. \tag{4.26}$$

Lemma 4.2.4. *For any family ν, ν^ε, $\varepsilon > 0$, in $\mathcal{M}$, the following two assertions hold. If $\nu^\varepsilon \stackrel{pp}{\to} \nu$ as $\varepsilon \to 0$, then Condition 1 holds. If Condition 1 holds and $\nu^\varepsilon \stackrel{v}{\to} \nu$ as $\varepsilon \to 0$, then also $\nu^\varepsilon \stackrel{pp}{\to} \nu$ as $\varepsilon \to 0$.*

4.2.3 Convergence of the fixed time distributions

We want to formulate, for the future use, a series of results from [FIN02]. They will allow us to deduce the convergence of fixed time distributions from the convergence of speed measures.

Proposition 4.2.5. *Let μ^ε, μ be the collection of deterministic locally finite measures, and let Y^ε, Y be defined by*

$$Y^\varepsilon(t) = Y(\mu^\varepsilon, Id)(t) \quad and \quad Y(t) = Y(\mu, Id)(t). \tag{4.27}$$

For any deterministic $t_0 > 0$, let ν^ε denote the distribution of $Y^\varepsilon(t_0)$ and ν denote the distribution of $Y(t_0)$. Suppose

$$\mu^\varepsilon \xrightarrow{v} \mu \quad \text{and} \quad \mu^\varepsilon \xrightarrow{pp} \mu \quad \text{as} \quad \varepsilon \to 0. \tag{4.28}$$

(i) Then, as $\varepsilon \to 0$,

$$\nu^\varepsilon \xrightarrow{v} \nu \quad \text{and} \quad \nu^\varepsilon \xrightarrow{pp} \nu. \tag{4.29}$$

(ii) Let $(x_i^\varepsilon, v_i^\varepsilon)$ and (x_i, v_i) be the collections of atoms of μ^ε and μ. Similarly, let $(y_i^\varepsilon, w_i^\varepsilon)$ and (y_i, w_i) be the collections of atoms of ν^ε and ν. Then the sets of locations of the atoms are equal,

$$\{y_i^\varepsilon\} = \{x_i^\varepsilon\} \quad \text{and} \quad \{y_i\} = \{x_i\}. \tag{4.30}$$

(iii) Suppose that we have denoted x_i's and y_i's in such a way that $x_i = y_i$, $x_i^\varepsilon = y_i^\varepsilon$ (which is possible by (ii)). Let the sequence $j_l(\varepsilon)$ satisfy

$$(x_{j_l(\varepsilon)}^\varepsilon, v_{j_l(\varepsilon)}^\varepsilon) \to (x_l, v_l) \quad \text{as} \quad \varepsilon \to 0. \tag{4.31}$$

Then the sequence of corresponding atoms of ν^ε satisfies

$$(y_{j_l(\varepsilon)}^\varepsilon, w_{j_l(\varepsilon)}^\varepsilon) = (x_{j_l(\varepsilon)}^\varepsilon, w_{j_l(\varepsilon)}^\varepsilon) \to (y_l, w_l) \quad \text{as} \quad \varepsilon \to 0. \tag{4.32}$$

(iv) Parts (i)–(iii) stay valid if we replace the process $Y^\varepsilon(t)$ by the process started outside the origin $Y(\mu^\varepsilon, Id; z^\varepsilon)$, and similarly the process $Y(t)$ by $Y(\mu, Id; z)$ with $z^\varepsilon \to z$ as $\varepsilon \to 0$.

Part (i) of this proposition is stated as Theorem 2.1 in [FIN02]. Part (ii) is a consequence of Lemmas 2.1 and 2.3 of the same paper. Part (iii) follows from the proof of that theorem, but it is not stated there explicitly. Its proof is, however, the central part of the proof of (i). The remaining part is an easy consequence of (i)–(iii) and of the joint continuity of the local time $\ell(t, y)$.

4.3 Expression of $X(t)$ in terms of Brownian motion

To explore the asymptotic behaviour of the chain $X(t)$, we consider its scaling limit

$$X^\varepsilon(t) = \varepsilon X(t/\varepsilon c_\varepsilon). \tag{4.33}$$

The constant c_ε will be determined later. For the time being the reader can consider $c_\varepsilon \sim \varepsilon^{1/\alpha}$.

As we already noted in the previous section, it is convenient to express the walks $X(t)$ and $X^\varepsilon(t)$ as a time-scale change of the standard Brownian motion $W(t)$ started at 0. To achieve it we use Proposition 4.2.1. We define measures

$$\mu(dx) = \mu^1(dx) = \sum_{i\in\mathbb{Z}} \tau_i \delta_i(dx) \quad \text{and} \quad \mu^\varepsilon(dx) = c_\varepsilon \sum_{i\in\mathbb{Z}} \tau_i \delta_{\varepsilon i}(dx), \tag{4.34}$$

where

$$\tau_i = \frac{1}{2} \exp(\beta E_i)\mathbb{E}(\exp(-2a\beta E_0)). \tag{4.35}$$

We will consider the following scaling function. Let

$$r_i = \frac{\exp(-\beta a(E_i + E_{i+i}))}{\mathbb{E}(\exp(-2\beta a E_0))}, \tag{4.36}$$

and let

$$S(i) = \begin{cases} \sum_{j=0}^{i-1} r_j & \text{if } i \geq 0, \\ \sum_{j=i}^{-1} r_j & \text{otherwise.} \end{cases} \tag{4.37}$$

The constant factor $\mathbb{E}(\exp(-2\beta a E_0))$ that appears in (4.35) and (4.36) is not substantial, but it is convenient and it will simplify some expressions later.

We use $\tilde{X}^\varepsilon(t)$, $0 < \varepsilon \leq 1$, to denote the process

$$\tilde{X}^\varepsilon(t) = X(\mu^\varepsilon, \varepsilon S(\varepsilon^{-1}\cdot))(t), \tag{4.38}$$

which means that $\tilde{X}(t)$ is time-scale change of Brownian motion with speed measure μ^ε and scale function $\varepsilon S(\varepsilon^{-1}\cdot)$. If we write $\psi^\varepsilon(t)$ for $\psi(\mu^\varepsilon, \varepsilon S(\varepsilon^{-1}\cdot))(t)$, then we have

$$\tilde{X}^\varepsilon(t) = \varepsilon S^{-1}(\varepsilon^{-1} W^\varepsilon(\psi^\varepsilon(t))). \tag{4.39}$$

The process W^ε is the rescaled Brownian motion, $W^\varepsilon(t) = \varepsilon W(\varepsilon^{-2}t)$, which has the same distribution as $W(t)$. It is introduced only to simplify the proof of the next lemma. In the sequel we will omit the superscript if $\varepsilon = 1$, i.e. we will write $\tilde{X}(t)$ for $\tilde{X}^1(t)$, etc. Note that the function $S^{-1}(\cdot)$ is well defined for all values of its argument. Indeed, the set of atoms of $\varepsilon S(\varepsilon^{-1}\cdot) \circ \mu^\varepsilon$ is the set $\{\varepsilon S(i) : i \in \mathbb{Z}\}$, and thus $\varepsilon^{-1} W^\varepsilon(\psi^\varepsilon(t))$ takes values only in $\{S(i) : i \in \mathbb{Z}\}$.

Proposition 4.3.1. *The processes $\tilde{X}(t)$ and $\tilde{X}^\varepsilon(t)$ have the same distribution as $X(t)$ and $X^\varepsilon(t) = \varepsilon X(t/c_\varepsilon \varepsilon)$.*

Proof. We use the symbol $\sim$ to denote the equality in distribution. The time that $X(t)$ stays at site i is exponentially distributed with mean $(w_{i,i+1} + w_{i,i-1})^{-1}$. The probability that it jumps right or left is

$$\frac{w_{i,i+1}}{w_{i,i+1} + w_{i,i-1}} \quad \text{and} \quad \frac{w_{i,i-1}}{w_{i,i+1} + w_{i,i-1}}. \tag{4.40}$$

Plugging the definition (4.2) of w_{ij} into these expressions, it is easy to see that these values coincide with the same quantities for $\tilde{X}(t)$ which can be computed using Proposition 4.2.1. This implies that $X(t) \sim \tilde{X}(t)$.

To compare the distributions of $X^\varepsilon(t)$ and $\tilde{X}^\varepsilon(t)$, let us first look at the scaling of $\psi^\varepsilon(t)$. After an easy calculation, using the fact that the local time $\ell^\varepsilon(t,y)$ of W^ε satisfies $\ell^\varepsilon(t,y) = \varepsilon\ell(\varepsilon^{-2}t, \varepsilon^{-1}y)$, we obtain

$$\phi^\varepsilon(t) = \int \ell^\varepsilon(t,y)(\varepsilon S(\varepsilon^{-1}\cdot) \circ \mu^\varepsilon)(dy) = \varepsilon c_\varepsilon \phi(\varepsilon^{-2}t). \tag{4.41}$$

From it we get $\psi^\varepsilon(t) = \varepsilon^2\psi(t/\varepsilon c_\varepsilon)$. Hence,

$$\begin{aligned}\varepsilon\tilde{X}(t/\varepsilon c_\varepsilon) &= \varepsilon S^{-1}(W(\psi(t/\varepsilon c_\varepsilon))) = \varepsilon S^{-1}(W(\varepsilon^{-2}\psi^\varepsilon(t))) \\ &= \varepsilon S^{-1}(\varepsilon^{-1}W^\varepsilon(\psi^\varepsilon(t))) = \tilde{X}^\varepsilon(t),\end{aligned} \tag{4.42}$$

where we used the scaling of $W(t)$ and (4.39). Since $\tilde{X}(t)$ has the same distribution as $X(t)$, the same is valid for $\tilde{X}^\varepsilon(t)$ and $X^\varepsilon(t)$. □

4.4 A coupling for walks on different scales

It is convenient to introduce the processes $Y(t)$ and $Y^\varepsilon(t)$ that are only a time change of Brownian motion with speed measures $S \circ \mu$ and $\varepsilon S(\varepsilon^{-1}\cdot) \circ \mu^\varepsilon$. Namely,

$$Y^\varepsilon(t) = Y(\mu^\varepsilon, \varepsilon S(\varepsilon^{-1}\cdot))(t) \quad \text{and} \quad Y(t) = Y(\mu, S)(t). \tag{4.43}$$

Using (4.25) we have

$$Y(t) = W(\psi(t)) \quad \text{and} \quad Y^\varepsilon(t) = W(\psi^\varepsilon(t)). \tag{4.44}$$

The original processes X and X^ε are related to them by

$$X(t) = S^{-1}(Y(t)) \quad \text{and} \quad X^\varepsilon(t) = \varepsilon S^{-1}(\varepsilon^{-1}Y^\varepsilon(t)). \tag{4.45}$$

In the sequel we want to use Proposition 4.2.5 to prove the convergence of the finite time distributions of Y^ε. Thus, we want to apply this proposition to the sequence of random speed measures μ^ε. It is easy to see that convergence in distribution of this sequence is not sufficient for its application. That is why we will construct a coupling between measures μ^ε on different scales ε on a larger probability space. Using this coupling we obtain the a.s. convergence on this space. It is not surprising that the same coupling as in [FIN02] does the job.

Consider the Lévy process $V(x)$, $x \in \mathbb{R}$, $V(0) = 0$, with stationary and independent increments and cadlag paths defined on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ given by

$$\bar{\mathbb{E}}\big[e^{ir(V(x+x_0)-V(x_0))}\big] = \exp\Big[x\alpha \int_0^\infty (e^{irw} - 1)w^{-1-\alpha}dw\Big]. \tag{4.46}$$

Let $\bar{\rho}$ be the random Lebesgue-Stieltjes measure on $\mathbb{R}$ associated to V, i.e. $\bar{\rho}(a, b] = V(b) - V(a)$. It is a known fact that $\bar{\rho}(dx) = \sum_j v_j \delta_{x_j}(dx)$, where (x_j, v_j) is an inhomogeneous Poison point process with density $dx\, \alpha v^{-1-\alpha} dv$. Note that $\bar{\rho}$ has the same distribution as ρ which we used as speed measure in the definition of the singular diffusion Z.

For each fixed $\varepsilon > 0$, we will now define the sequence of i.i.d. random variables E_i^ε such that E_i^ε's are defined on the same space as V and $\bar{\rho}$ and they have the same distribution as E_0.

Define a function $G : [0, \infty) \mapsto [0, \infty)$ such that

$$\bar{\mathbb{P}}(V(1) > G(x)) = \mathbb{P}(\tau_0 > x). \tag{4.47}$$

The function G is well-defined since $V(1)$ has continuous distribution, it is nondecreasing and right continuous, and hence has nondecreasing right-continuous generalised inverse G^{-1}. Let $g_\varepsilon : [0, \infty) \mapsto [0, \infty)$ be defined as

$$g_\varepsilon(x) = c_\varepsilon G^{-1}(\varepsilon^{-1/\alpha} x) \quad \text{for all} \quad x \geq 0, \tag{4.48}$$

where

$$c_\varepsilon = \Big(\inf[t \geq 0 : \mathbb{P}(\tau_0 > t) \leq \varepsilon]\Big)^{-1}. \tag{4.49}$$

Note that if τ_0 is the α stable random variable with characteristic function

$$\mathbb{E}(e^{ir\tau_0}) = \exp\Big[\alpha \int_0^\infty (e^{irw} - 1)w^{-1-\alpha}dw\Big], \tag{4.50}$$

the choice of c_ε and g_ε can be simplified (although it does not correspond to the previous definition)

$$c_\varepsilon = \varepsilon^{1/\alpha} \quad \text{and} \quad g_\varepsilon(y) \equiv y. \tag{4.51}$$

The reader who is not interested in the technical details should keep this choice in mind.

Lemma 4.4.1. *Let*

$$\tau_i^\varepsilon = \frac{1}{c_\varepsilon} g_\varepsilon(V(\varepsilon(i+1)) - V(\varepsilon i)), \tag{4.52}$$

and

$$E_i^\varepsilon = \frac{1}{\beta} \log\left(\frac{2\tau_i^\varepsilon}{\mathbb{E}(\exp(-2a\beta E_0))}\right). \tag{4.53}$$

Then for any $\varepsilon > 0$, the τ_i^ε are i.i.d. with the same law as τ_0, and $\{E_i^\varepsilon\}_{i\in\mathbb{Z}}$ have the same distribution as $\{E_i\}_{i\in\mathbb{Z}}$.

Proof. By stationarity and independence of increments of V it is sufficient to show $\bar{\mathbb{P}}(\tau_0^\varepsilon > t) = \mathbb{P}(\tau_0 > t)$. However,

$$\bar{\mathbb{P}}(\tau_0^\varepsilon > t) = \bar{\mathbb{P}}(V(\varepsilon) > \varepsilon^{1/\alpha} G(t)) \tag{4.54}$$

by the definitions of τ_0^ε and G. The result then follows from (4.47) and the scaling invariance of V: $V(\varepsilon) \sim \varepsilon^{1/\alpha} V(1)$. The second claim follows easily using (4.35). □

Let us now define the random speed measures $\bar{\mu}^\varepsilon$ using the collections $\{E_i^\varepsilon\}$ from the previous lemma,

$$\bar{\mu}^\varepsilon(dx) = \sum_{i \in \mathbb{Z}} c_\varepsilon \tau_i^\varepsilon \delta_{\varepsilon i}(dx). \tag{4.55}$$

We also define the scaling functions S_ε similarly as in (4.37). Let

$$r_i^\varepsilon = \frac{\exp(-\beta a(E_i^\varepsilon + E_{i+1}^\varepsilon))}{\mathbb{E}(-2a\beta E_0)} \tag{4.56}$$

and

$$S_\varepsilon(i) = \begin{cases} \sum_{j=0}^{i-1} r_j^\varepsilon & \text{if } i \geq 0, \\ \sum_{j=i}^{-1} r_j^\varepsilon & \text{otherwise.} \end{cases} \tag{4.57}$$

It is an easy consequence of Lemma 4.4.1 that $\bar{\mu}^\varepsilon \sim \mu^\varepsilon$ and $S_\varepsilon \sim S$ for any $\varepsilon \in (0, 1]$.

4.5 Convergence of speed measures

The following proposition proves the convergence of the scaled speed measures. If S is the identity, i.e. $a = 0$, it corresponds to Proposition 3.1 of [FIN02].

Proposition 4.5.1. *Let $\bar{\mu}^\varepsilon$ and $\bar{\rho}$ be defined as above. Then*

$$\varepsilon S_\varepsilon(\varepsilon^{-1} \cdot) \circ \bar{\mu}^\varepsilon \xrightarrow{v} \bar{\rho} \quad \textit{and} \quad \varepsilon S_\varepsilon(\varepsilon^{-1} \cdot) \circ \bar{\mu}^\varepsilon \xrightarrow{pp} \bar{\rho} \quad \textit{as } \varepsilon \to 0 \quad \bar{\mathbb{P}}\textit{-a.s.} \tag{4.58}$$

The proof requires three technical lemmas.

Lemma 4.5.2. *As $\varepsilon \to 0$ we have*

$$\varepsilon S_\varepsilon(\lfloor \varepsilon^{-1} y \rfloor) \to y \quad \textit{as } \varepsilon \to 0 \quad \bar{\mathbb{P}}\textit{-a.s.} \tag{4.59}$$

uniformly on compact intervals.

Notice that this lemma sheds more light on the difference between the discrete time embedded walk of the process X and the Sinai's RWRE. In the case of Sinai's RWRE the scale function S corresponds, loosely speaking, to the function

$$S'(n) = \sum_{i=1}^{n} \rho_1 \dots \rho_n, \tag{4.60}$$

where $\rho_i = (1-p_i)/p_i$, p_i is the probability going right at i, and p_i's are i.i.d. In our case $\rho_i = r_i/r_{i-1}$. An easy computation gives that the product $\rho_1 \dots \rho_n$ depends only on E_0 and E_{n+1}. Thus, $S'(n)$ is in our situation essentially a sum of i.i.d. random variables which is definitively not the case for the Sinai's RWRE.

Proof of Lemma 4.5.2. We consider only $y > 0$. The proof for $y < 0$ is very similar. By definition of S_ε we have $\varepsilon S_\varepsilon(\lfloor \varepsilon^{-1} y \rfloor) = \varepsilon \sum_{j=0}^{\lfloor \varepsilon^{-1} y \rfloor - 1} r_j^\varepsilon$, where for fixed ε the sequence r_i^ε is an ergodic sequence of bounded positive random variables. Moreover, r_i^ε is independent of all r_j^ε with $j \notin \{i-1, i, i+1\}$. The $\bar{\mathbb{P}}$-a.s. convergence for fixed y is then a consequence of the strong law of large numbers for triangular arrays. Note that this law of large numbers can be easily proved in our context using the standard methods, because the variables r_i^ε are bounded and thus their moments of arbitrary large degree are finite. The uniform convergence on compact intervals is easy to prove using the fact that $S_\varepsilon(i)$ is increasing and the identity function is continuous. $\square$

The next two lemmas correspond to Lemmas 3.1 and 3.2 of [FIN02]. We state them without proofs.

Lemma 4.5.3. *For any fixed $y > 0$, $g^\varepsilon(y) \to y$ as $\varepsilon \to 0$.*

Lemma 4.5.4. *For any $\delta' > 0$, there exist constants C' and C'' in $(0, \infty)$ such that*

$$g_\varepsilon(x) \le C' x^{1-\delta'} \quad \textit{for} \quad \varepsilon^{1/\alpha} \le x \le 1 \quad \textit{and} \quad \varepsilon \le C''. \tag{4.61}$$

Proof of Proposition 4.5.1. We first prove the vague convergence. Let f be a bounded continuous function with compact support $I \subset \mathbb{R}$. Then,

$$\int f(x)(\varepsilon S_\varepsilon(\varepsilon^{-1}x) \circ \bar{\mu}^\varepsilon)(dx) = \sum_{i \in J_0^\varepsilon} f(\varepsilon S_\varepsilon(i)) g_\varepsilon(V(\varepsilon(i+1)) - V(\varepsilon i)), \tag{4.62}$$

where we used the notation

$$J_y^\varepsilon = \{i \in \mathbb{Z} : \varepsilon S_\varepsilon(i) \in I, V(\varepsilon(i+1)) - V(\varepsilon i) \ge y\}. \tag{4.63}$$

Choose now $\delta > 0$. To estimate the last sum, we treat separately the sums over J_δ^ε, $J_{\varepsilon^{1/\alpha}}^\varepsilon \setminus J_\delta^\varepsilon$ and $J_0^\varepsilon \setminus J_{\varepsilon^{1/\alpha}}^\varepsilon$.

Due to the convergence of $\varepsilon S_\varepsilon(\varepsilon^{-1}\cdot)$ to the identity, we know that for ε small enough there is a small neighbourhood I' of I such that $J_0^\varepsilon \subset \varepsilon^{-1}I'$. The process V has $\bar{\mathbb{P}}$-a.s. only finitely many jumps larger than δ in I', so the first sum has only a finite number of terms. Using the continuity of f and applying Lemmas 4.5.2 and 4.5.3 we have

$$\sum_{i\in J_\delta^\varepsilon} f(\varepsilon S_\varepsilon(i))g_\varepsilon(V(\varepsilon(i+1))-V(\varepsilon i)) \to \sum_{j:v_j\geq\delta} f(x_j)v_j, \tag{4.64}$$

with (x_i,v_i) being the set of atoms of $\bar{\rho}$. In the previous expression we also use the fact that $i\varepsilon\to x_i$ for the corresponding terms in the sums.

By Lemma 4.5.4 we have for some δ' such that $\delta'+\alpha\leq 1$

$$\begin{aligned}\sum_{i\in J^\varepsilon_{\varepsilon^{1/\alpha}}\setminus J^\varepsilon_\delta} & f(\varepsilon S_\varepsilon(i))g_\varepsilon(V(\varepsilon(i+1))-V(\varepsilon i))\\ &\leq C\sum_{i\in J^\varepsilon_{\varepsilon^{1/\alpha}}\setminus J^\varepsilon_\delta}\big(V(\varepsilon(i+1))-V(\varepsilon i)\big)^{1-\delta'}\leq C\sum_{\substack{j:v_j\leq\delta\\ x_j\in I'}} v_j^{1-\delta'}=H_\delta.\end{aligned}\tag{4.65}$$

From the definition of the point process (x_i,v_i) we have

$$\bar{\mathbb{E}}(H_\delta)\leq\alpha|I'|\int_0^\delta w^{1-\delta'}w^{-1-\alpha}dw\to 0 \quad\text{as}\quad \delta\to 0. \tag{4.66}$$

Since H_δ is decreasing and positive, the limit $\lim_{\delta\to 0}H_\delta$ exists $\bar{\mathbb{P}}$-a.s. The dominated convergence theorem then gives $\bar{\mathbb{E}}\lim_{\delta\to 0}H_\delta=0$, and thus $\lim_{\delta\to 0}H_\delta=0$ $\bar{\mathbb{P}}$-a.s.

The third part of the sum is also negligible for ε small enough. Indeed, by monotonicity of g_ε, we have $g_\varepsilon(x)\leq g_\varepsilon(\varepsilon^{1/\alpha})\leq Cc_\varepsilon$ for all $x\leq\varepsilon^{1/\alpha}$. Hence,

$$\begin{aligned}\sum_{i\in J_0^\varepsilon\setminus J^\varepsilon_{\varepsilon^{1/\alpha}}} & f(\varepsilon S_\varepsilon(i))g_\varepsilon(V(\varepsilon(i+1))-V(\varepsilon i))\\ &\leq C'c_\varepsilon\sum_{i\in\varepsilon^{-1}I'\cap\mathbb{Z}}1\leq C''c_\varepsilon\varepsilon^{-1}\to 0\quad\text{as}\quad\varepsilon\to 0.\end{aligned}\tag{4.67}$$

In the last equation we use the fact that if τ_0 is in the domain of attraction of the stable law with index α, there exists $\kappa>0$ such that the function c_ε can be bounded from above by $C\varepsilon^{-\kappa+1/\alpha}$ with $-\kappa+1/\alpha>1$.

Putting now all three parts together, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}&\sum_{i\in J_0^\varepsilon} f(\varepsilon S_\varepsilon(i))g_\varepsilon(V(\varepsilon(i+1))-V(\varepsilon i))\\ &=\lim_{\delta\to 0}\sum_{j:v_j\geq\delta} f(x_j)v_j=\int f\,d\bar{\rho}.\end{aligned}\tag{4.68}$$

This proves the vague convergence.

To prove the point process convergence we use Lemma 4.2.4. Since we have already proved the vague convergence, we must only verify Condition 1 for the measures $\varepsilon S_\varepsilon(\varepsilon^{-1}\cdot) \circ \bar{\mu}^\varepsilon$ and $\bar{\rho}$. Thus, for any atom (x_l, v_l) of $\bar{\rho}$ we want to find a sequence $j_l(\varepsilon)$ such that

$$\varepsilon S_\varepsilon(j_l(\varepsilon)) \to x_l \quad \text{and} \quad g_\varepsilon\big(V(\varepsilon(j_l(\varepsilon)+1)) - V(\varepsilon j_l(\varepsilon))\big) \to v_l. \tag{4.69}$$

Choose $j_l(\varepsilon)$ such that $x_l \in \big(\varepsilon j_l(\varepsilon), \varepsilon(j_l(\varepsilon)+1)\big]$. Then by Lemma 4.5.2 we have the first statement of (4.69), and by Lemma 4.5.3 we have the second. This finishes the proof of Proposition 4.5.1. □

4.6 Change of scale for fixed time distributions

Write $\bar{X}^\varepsilon$ and $\bar{X}$ for the processes defined as in (4.38), but using the speed measures $\bar{\mu}^\varepsilon$ and the scaling functions S_ε. Since $\bar{\mu}^\varepsilon \sim \mu^\varepsilon$ and $S_\varepsilon \sim S$, we have $\bar{X}^\varepsilon \sim X^\varepsilon$. Similarly, we define the processes $\bar{Y}^\varepsilon$, $\bar{Y}$ as in (4.44), and $\bar{Z}$ as in Definition 4.1.1 using the measures with bars. Evidently, $\bar{Y}^\varepsilon \sim Y^\varepsilon$, $\bar{Y} \sim Y$ and $\bar{Z} \sim Z$. The following proposition is a consequence of Propositions 4.2.5 and 4.5.1.

Proposition 4.6.1. *Fix $t_0 > 0$. Write $\bar{\nu}^\varepsilon_{Y,V}$ for the distribution of $\bar{Y}^\varepsilon(t_0)$ and $\bar{\nu}_V$ for the distribution of $\bar{Z}(t_0)$ conditionally on V. Then, $\bar{\mathbb{P}}$-a.s we have*

$$\bar{\nu}^\varepsilon_{Y,V} \xrightarrow{v} \bar{\nu}_V \quad \textit{and} \quad \bar{\nu}^\varepsilon_{Y,V} \xrightarrow{pp} \bar{\nu}_V \quad \textit{as} \quad \varepsilon \to 0. \tag{4.70}$$

The proof of the convergence of the fixed time distribution of $\bar{X}^\varepsilon$ will be finished if we can compare the limits of $\bar{X}^\varepsilon$ and $\bar{Y}^\varepsilon$.

Proposition 4.6.2. *Fix t_0 as in Proposition 4.6.1. Let $\bar{\nu}^\varepsilon_{X,V}$ denote the distribution of $\bar{X}^\varepsilon(t_0)$ conditionally on V. Then, $\bar{\mathbb{P}}$-a.s. we have*

$$\lim_{\varepsilon\to 0} \bar{\nu}^\varepsilon_{X,V} = \lim_{\varepsilon\to 0} \bar{\nu}^\varepsilon_{Y,V} = \bar{\nu}_V, \tag{4.71}$$

where the limits are taken in both the vague and the point process sense.

Proof. As an easy consequence of Lemma 4.5.2 we have

$$\varepsilon S_\varepsilon^{-1}(\varepsilon^{-1} y) \to y \qquad \bar{\mathbb{P}}\text{-a.s.} \tag{4.72}$$

We will again apply Lemma 4.2.4 to prove the convergence. Let f be a continuous function with bounded support $I \subset \mathbb{R}$. By continuity of f and (4.72), choosing the fixed realisation of Brownian motion W, we have $\bar{\mathbb{P}}$-a.s.

$$\lim_{\varepsilon\to 0} f(\bar{X}^\varepsilon(t_0)) = \lim_{\varepsilon\to 0} f(\bar{Y}^\varepsilon(t_0)). \tag{4.73}$$

A standard application of the dominated convergence theorem yields

$$\lim_{\varepsilon\to 0}\int f\, d\bar\nu^\varepsilon_{X,V} = \lim_{\varepsilon\to 0}\int f\, d\bar\nu^\varepsilon_{Y,V} = \int f\, d\bar\nu_V. \tag{4.74}$$

We finally verify Condition 1. Write $(x_i^\varepsilon, v_i^\varepsilon)$, $(y_i^\varepsilon, w_i^\varepsilon)$ for the collections of atoms of $\bar\nu^\varepsilon_{X,V}$ and $\bar\nu^\varepsilon_{Y,V}$. By Proposition 4.2.5(ii) we can choose $x_i^\varepsilon = \varepsilon i$ and $y_i^\varepsilon = \varepsilon S_\varepsilon(i)$, setting eventually v_i^ε, resp. w_i^ε, equal to zero if there is no atom at x_i^ε, resp. y_i^ε. Using this choice of x_i^ε and y_i^ε and the relation (4.45) we have $v_i^\varepsilon = w_i^\varepsilon$. Let (z_l, u_l) be the collection of atoms of $\bar\nu_V$ and $j_l(\varepsilon)$ be the sequence of indexes such that $(y_{j_l(\varepsilon)}, w_{j_l(\varepsilon)}) \to (z_l, u_l)$. Then by (4.72) we have $(x_{j_l(\varepsilon)}, v_{j_l(\varepsilon)}) \to (z_l, u_l)$ which completes the proof. □

4.7 Proof of Theorem 4.1.2

We first express the quantities that we are interested in using the processes $\bar X^\varepsilon$. From the definition of $\tilde X^\varepsilon$, Proposition 4.3.1, and the fact that $\bar X^\varepsilon \sim \tilde X^\varepsilon$ we get

$$\begin{aligned}\lim_{t_w\to\infty} \mathbb{E}\mathbb{P}\big[X((1+\theta)t_w) = X(t_w)|E\big]& \\ = \lim_{\varepsilon\to 0} \bar{\mathbb{E}}\bar{\mathbb{P}}\big[\bar X^\varepsilon(1+\theta) = \bar X^\varepsilon(1)|V\big] &\equiv \lim_{\varepsilon\to 0} R_\varepsilon(\theta)\end{aligned} \tag{4.75}$$

and similarly

$$\begin{aligned}\lim_{t_w\to\infty} \mathbb{E}\sum_{i\in\mathbb{Z}} \big[\mathbb{P}(X((1+\theta)t_w) = i|E, X(t_w))\big]^2& \\ = \lim_{\varepsilon\to 0} \bar{\mathbb{E}}\sum_{i\in\mathbb{Z}} \big[\bar{\mathbb{P}}(\bar X^\varepsilon(1+\theta) = i\varepsilon|V, \bar X^\varepsilon(1))\big]^2 &\equiv \lim_{\varepsilon\to 0} R^q_\varepsilon(\theta).\end{aligned} \tag{4.76}$$

We introduce some notation for the sets of atoms of the measures we will consider. In the following everything depends on the realisation of the Lévy process V and we will not denote this dependence explicitly. We write

$$\bar\mu^\varepsilon = \sum_i v_i^\varepsilon \delta_{x_i^\varepsilon} \quad \text{and} \quad \bar\rho = \sum_i v_i \delta_{x_i}. \tag{4.77}$$

The atoms of the distribution ν_1^ε of $\bar X^\varepsilon(1)$ will be denoted by $(x_i^\varepsilon, w_i^\varepsilon)$. Similarly, (x_i, w_i) denotes the atoms of the distribution ν_1 of $\bar Z(1)$. The weights of the joint distribution of $\bar X^\varepsilon(1)$ and $\bar X^\varepsilon(1+\theta)$ will be denoted by w_{ij}^ε,

$$\begin{aligned} w_{ij}^\varepsilon &= \bar{\mathbb{P}}\big[(\bar X^\varepsilon(1) = x_i^\varepsilon) \cap (\bar X^\varepsilon(1+\theta) = x_j^\varepsilon)|V\big],\\ w_{ij} &= \bar{\mathbb{P}}\big[(\bar Z(1) = x_i) \cap (\bar Z(1+\theta) = x_j)|V\big].\end{aligned} \tag{4.78}$$

The last measure we will introduce is the distribution $\nu^\varepsilon_{1+\theta}(\cdot|x^\varepsilon_i)$ of $\bar{X}^\varepsilon(1+\theta)$ conditioned on $\bar{X}^\varepsilon(1)=x^\varepsilon_i$. We denote its atoms by $(x^\varepsilon_j,u^\varepsilon_{ij})$. Thus,

$$\begin{aligned} u^\varepsilon_{ij} &= \bar{\mathbb{P}}\big[\bar{X}^\varepsilon(1+\theta)=x^\varepsilon_j|\bar{X}^\varepsilon(1)=x^\varepsilon_i,V\big],\\ u_{ij} &= \bar{\mathbb{P}}\big[\bar{Z}(1+\theta)=x_j|\bar{Z}(1)=x_i,V\big]. \end{aligned} \tag{4.79}$$

Observe that $w^\varepsilon_{ij}=w^\varepsilon_i u^\varepsilon_{ij}$ and $w_{ij}=w_iu_{ij}$.

Using this notation we can rewrite (4.75) and (4.76),

$$R_\varepsilon(\theta)=\bar{\mathbb{E}}\Big[\sum_i w^\varepsilon_i u^\varepsilon_{ii}\Big] \quad\text{and}\quad R^q_\varepsilon(\theta)=\bar{\mathbb{E}}\Big[\sum_i w^\varepsilon_i \sum_j (u^\varepsilon_{ij})^2\Big], \tag{4.80}$$

where the expectations are taken over all realisations of V. Obviously we have

$$R(\theta)=\bar{\mathbb{E}}\Big[\sum_i w_iu_{ii}\Big] \quad\text{and}\quad R^q(\theta)=\bar{\mathbb{E}}\Big[\sum_{i,j} w_i(u_{ij})^2\Big]. \tag{4.81}$$

If we prove the $\bar{\mathbb{P}}$-a.s. convergence of the expressions inside the expectations in (4.80) to the corresponding expressions in (4.81), the proof will follow easily using the dominated convergence theorem. We want to use the results of Proposition 4.6.2, namely the point process convergence of ν^ε_1 to ν_1 and $\nu^\varepsilon_{1+\theta}(\cdot|x^\varepsilon_{j_i(\varepsilon)})$ to $\nu_{1+\theta}(\cdot|x_i)$. Here, as usually, $j_i(\varepsilon)$ satisfies $(x_{j_i(\varepsilon)},v_{j_i(\varepsilon)})\to(x_i,v_i)$ as $\varepsilon\to0$. Note that the point process convergence of $\nu^\varepsilon_{1+\theta}(\cdot|x^\varepsilon_{j_i(\varepsilon)})$ follows from Propositions 4.6.2 and 4.2.5(iv).

In the proof we will need one property of the atoms of different measures that is connected with Condition 1. From the point process convergence of $\bar{\mu}^\varepsilon$ we know that for every atom (x_l,v_l) of $\bar{\rho}$ there is a function $j_l(\varepsilon)$ such that $(x^\varepsilon_{j_l(\varepsilon)},v^\varepsilon_{j_l(\varepsilon)})$ converges to (x_l,v_l). From Proposition 4.2.5(iii) we can see that for the same function $w^\varepsilon_{j_l(\varepsilon)}\to w_l$, $u^\varepsilon_{j_l(\varepsilon),j_k(\varepsilon)}\to u_{lk}$, and thus $w^\varepsilon_{j_l(\varepsilon),j_k(\varepsilon)}\to w_{lk}$ as $\varepsilon\to0$. This observation is essential, because only the point process convergence of all measures is not sufficient to imply our results.

We prove the convergence only for the quantity $R(\theta)$. The proof for $R^q(\theta)$ is entirely similar. Point process convergence, Condition 1, and the observation of the previous paragraph give

$$\sum_i w_iu_{ii}=\lim_{\varepsilon\to0}\sum_i w^\varepsilon_{j_i(\varepsilon)}u^\varepsilon_{j_i(\varepsilon),j_i(\varepsilon)}\le\liminf_{\varepsilon\to0}\sum_i w^\varepsilon_i u^\varepsilon_{ii}. \tag{4.82}$$

To show the opposite bound we choose $\delta>0$, and divide the sum in (4.80) into sums over three disjoint sets

$$\begin{aligned} A_\varepsilon(\delta)&=\{i:w^\varepsilon_i>\delta,u^\varepsilon_{ii}>\delta\}\\ B_\varepsilon(\delta)&=\{i:u^\varepsilon_{ii}\le\delta\}\\ C_\varepsilon(\delta)&=\{i:w^\varepsilon_i\le\delta,u^\varepsilon_{ii}>\delta\}. \end{aligned} \tag{4.83}$$

The sum over $A_\varepsilon(\delta)$ has necessarily finite number of terms. From point process convergence we have

$$\limsup_{\varepsilon\to 0} \sum_{i\in A_\varepsilon(\delta)} w_i^\varepsilon u_{ii}^\varepsilon = \sum_{i\in A(\delta)} w_i u_{ii}, \tag{4.84}$$

where $A(\delta)$ has the obvious meaning. For the second part we have

$$\limsup_{\varepsilon\to 0} \sum_{i\in B_\varepsilon(\delta)} w_i^\varepsilon u_{ii}^\varepsilon = \delta \limsup_{\varepsilon\to 0} \sum_{i\in B_\varepsilon(\delta)} w_i^\varepsilon \le \delta, \tag{4.85}$$

since ν_1^ε is the probability measure. The last part satisfies

$$\limsup_{\varepsilon\to 0} \sum_{i\in C_\varepsilon(\delta)} w_i^\varepsilon u_{ii}^\varepsilon \le \limsup_{\varepsilon\to 0} \sum_{i\in C_\varepsilon(\delta)} w_i^\varepsilon \le 1 - \liminf_{\varepsilon\to 0} \sum_{i:w_i^\varepsilon>\delta} w_i^\varepsilon. \tag{4.86}$$

The sum in the last expression has a finite number of terms. Hence

$$\limsup_{\varepsilon\to 0} \sum_{i\in C_\varepsilon(\delta)} w_i^\varepsilon u_{ii}^\varepsilon \le 1 - \sum_{i:w_i>\delta} w_i, \tag{4.87}$$

and the last sum goes to 1 as $\delta \to 0$, because ν_1 is a purely discrete measure. From (4.84)–(4.87) it is easy to see that

$$\limsup_{\varepsilon\to 0} \sum_{i} w_i^\varepsilon u_{ii}^\varepsilon \le \sum_{i\in A(\delta)} w_i u_{ii} + \delta + (1 - \sum_{i:w_i>\delta} w_i) \tag{4.88}$$

and the proof is finished by taking the limit $\delta \to 0$.

4.8 Proof of sub-aging in the symmetric case

We start the proof by a technical lemma that will provide the connection between the rescaled processes at time $t = 1$ and the process X at some large time t. Let $\varepsilon(t)$ be defined by

$$\varepsilon(t) c_{\varepsilon(t)} t = 1. \tag{4.89}$$

Solution to this equation always exists, at least for t large enough, because c_ε is continuous nondecreasing function of ε as can be easily seen from (4.49). Until the end of the proof $\varepsilon = \varepsilon(t)$ will be connected with t and we will not denote the dependence explicitly. We will also sometimes write c_t for $c_{\varepsilon(t)}$.

The next lemma defines the slowly varying function $L(t)$ that is used in Theorem 4.1.3. Note that all slowly varying function that we use are slowly varying at infinity.

Lemma 4.8.1. *There exists a slowly varying function $L(t)$ such that*

$$c_t t^\gamma L(t) = 1. \tag{4.90}$$

The proof of this lemma is postponed to the end of the section.

The main step in proving Theorem 4.1.3 is the following proposition that describes the scaling of the distribution of the depth of the site where X stays at time t. We recall that

$$\gamma = \frac{\beta}{1+\beta} = \frac{1}{1+\alpha}. \tag{4.91}$$

Proposition 4.8.2. *Let $F_t(u) = \mathbb{E}\mathbb{P}(\tau(X(t))/t^\gamma L(t) \le u|E)$. Then*

$$\lim_{t\to\infty} F_t(u) = \mathbb{E}\mathbb{P}(\rho(Z(1)) \le u|\rho) \equiv F(u) \tag{4.92}$$

for all points of continuity of $F(u)$.

We use this proposition to prove subaging for $a = 0$.

Proof of Theorem 4.1.3 in the symetric case. The process X stays at the site i for an exponentially long time with mean τ_i. Using the Markov property we can write

$$\begin{aligned}\mathbb{P}\big[X(t') = X(t)\forall t' \in [t, t+\theta t^\gamma L(t)]\big]& \\ = \int_0^\infty e^{-\theta t^\gamma L(t)/u}\, dF_t\big(u/(t^\gamma L(t))\big) &= \int_0^\infty e^{-\theta/u}\, dF_t(u).\end{aligned} \tag{4.93}$$

By the weak convergence stated in Proposition 4.8.2, the last expression converges to $\int e^{-\theta/u} dF(u) = \Pi(\theta)$. ☐

The proof of Theorem 4.1.3 for the asymmetric case is postponed to the next section because it is relatively complicated and relies on some notation introduced later in this section.

Proof of Proposition 4.8.2. We follow the similar strategy as in the proof of aging. Again we start with some notations. We write

$$\bar\mu^\varepsilon(dx) = \sum_{i\in\mathbb{Z}} c_\varepsilon \tau_i^\varepsilon \delta_{i\varepsilon}(dx) \quad \text{and} \quad \bar\rho(dx) = \sum_{i\in\mathbb{Z}} v_i \delta_{x_i}(dx). \tag{4.94}$$

Similarly, the distributions of $\bar X^\varepsilon(1)$ and $\bar Z(1)$ satisfy

$$\bar\nu_1^\varepsilon(dx) = \sum_{i\in\mathbb{Z}} w_i^\varepsilon \delta_{i\varepsilon}(dx) \quad \text{and} \quad \bar\nu_1(dx) = \sum_{i\in\mathbb{Z}} w_i \delta_{x_i}(dx). \tag{4.95}$$

Here again we used the fact that the sets of positions of atoms of $\bar{\rho}$ and $\bar{\nu}_1$ are equal. We also introduce the distributions of the depth at the time one

$$\pi_1^\varepsilon(dx) = \sum_{i\in\mathbb{Z}} w_i^\varepsilon \delta_{c_\varepsilon \tau_i^\varepsilon}(dx) \tag{4.96}$$

and

$$\pi_1(dx) = \sum_{i\in\mathbb{Z}} w_i \delta_{\bar{\rho}(x_i)}(dx) = \sum_{i\in\mathbb{Z}} w_i \delta_{v_i}(dx). \tag{4.97}$$

We claim that

Lemma 4.8.3.

$$\pi_1^\varepsilon \xrightarrow{v} \pi_1 \quad \textit{and} \quad \pi_1^\varepsilon \xrightarrow{pp} \pi_1 \quad \textit{as} \quad \varepsilon \to 0 \quad \bar{\mathbb{P}}\textit{-a.s.} \tag{4.98}$$

Proof. As usually we prove the vague convergence and Condition 1. To verify the second property, let us first observe that for any atom (v_l, w_l) of π_1 there exists x_l such that (x_l, v_l) is an atom of $\bar{\rho}$, and (x_l, w_l) is an atom of $\bar{\nu}_1$. From the point process convergences $\mu^\varepsilon \xrightarrow{pp} \bar{\rho}$, $\bar{\nu}_1^\varepsilon \xrightarrow{pp} \bar{\nu}_1$, and from the direct part of the Lemma 4.2.4 we have that for any l there exist sequences $j_l(\varepsilon)$ and $k_l(\varepsilon)$, such that $(\varepsilon j_l(\varepsilon), c_\varepsilon \tau_{j_l(\varepsilon)}^\varepsilon) \to (x_l, v_l)$ and $(\varepsilon k_l(\varepsilon), w_{k_l(\varepsilon)}^\varepsilon) \to (x_l, w_l)$ as $\varepsilon \to 0$. Moreover, it can be seen from Proposition 4.2.5(iii) that $j_l(\varepsilon) = k_l(\varepsilon)$. Putting together the last three claims we easily show that $(c_\varepsilon \tau_{j_l(\varepsilon)}^\varepsilon, w_{j_l(\varepsilon)}^\varepsilon) \to (v_l, w_l)$ as $\varepsilon \to 0$.

We should now verify the vague convergence. Let f be a nonnegative, continuous function with compact support. We use I_δ to denote the open rectangle $(-\delta^{-1}, \delta^{-1}) \times (\delta, 2)$. By (4.96) we have

$$\begin{aligned} \int f(x)\pi_1^\varepsilon(dx) &= \sum_{i\in\mathbb{Z}} w_i^\varepsilon f(c_\varepsilon \tau_{i\varepsilon}^\varepsilon) \\ &= \sum_{i:(i\varepsilon, w_i^\varepsilon)\in I_\delta} w_i^\varepsilon f(c_\varepsilon \tau_{i\varepsilon}^\varepsilon) + \sum_{i:(i\varepsilon, w_i^\varepsilon)\notin I_\delta} w_i^\varepsilon f(c_\varepsilon \tau_{i\varepsilon}^\varepsilon). \end{aligned} \tag{4.99}$$

From the point process convergence of $\bar{\nu}_1^\varepsilon$ we know that for all but countably many $\delta > 0$ and for ε large enough the number of atoms of $\bar{\nu}_1^\varepsilon$ in I_δ is finite and is equal to the number of atoms of $\bar{\nu}_1$ in I_δ. Moreover, by the first part of Lemma 4.2.4 we have for any such atom (x_l, w_l) the sequence of atoms $(\varepsilon j_l(\varepsilon), w_{j_l(\varepsilon)}^\varepsilon)$ converging to (x_l, w_l). By the same reasoning as in the previous paragraph the sequence $c_\varepsilon \tau_{j_l(\varepsilon)}^\varepsilon$ converges as $\varepsilon \to 0$ to $\bar{\rho}(x_l) = v_l$. Thus, by continuity of f we have

$$\lim_{\varepsilon\to 0} \sum_{i:(i\varepsilon, w_i^\varepsilon)\in I_\delta} w_i^\varepsilon f(c_\varepsilon \tau_{i\varepsilon}^\varepsilon) = \sum_{i:(x_i, w_i)\in I_\delta} w_i f(v_i). \tag{4.100}$$

The right hand side of the last equation is bounded by $\|f\|_\infty$ and increases as δ decreases. Thus, its limit as $\delta \to 0$ exist and is equal to $\int f(x)\pi_1(dx)$.

The second sum in (4.99) is bounded by

$$C \sum_{i:(i\varepsilon, w_i^\varepsilon)\notin I_\delta} w_i^\varepsilon = C\Big(1 - \sum_{i:(i\varepsilon, w_i^\varepsilon)\in I_\delta} w_i^\varepsilon\Big). \tag{4.101}$$

Using the same argument as in (4.87) we have

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\Big(1 - \sum_{i:(i\varepsilon, w_i^\varepsilon)\in I_\delta} w_i^\varepsilon\Big) = \lim_{\delta\to 0}\Big(1 - \sum_{i:(x_i, w_i)\in I_\delta} w_i\Big) = 0, \tag{4.102}$$

since the finite time distribution of $\bar{Z}$ is discrete. □

We can now finish the proof of Proposition 4.8.2. By definition of $X^\varepsilon(t)$ we have

$$F_t(u) = \mathbb{P}[\tau(X(t))/t^\gamma L(t) \le u] = \mathbb{P}[\tau(\varepsilon^{-1}X^\varepsilon(t\varepsilon c_\varepsilon))/t^\gamma L(t) \le u]. \tag{4.103}$$

Inserting the definition of μ^ε into the last claim yields

$$F_t(u) = \mathbb{P}[c_\varepsilon^{-1}\mu^\varepsilon(X^\varepsilon(t\varepsilon c_\varepsilon))/t^\gamma L(t) \le u]. \tag{4.104}$$

Using $t\varepsilon c_\varepsilon = 1$, the equality of the distributions $\bar{X}^\varepsilon \sim X^\varepsilon$, $\bar{\mu}^\varepsilon \sim \mu^\varepsilon$, and Lemma 4.8.1, we get

$$F_t(u) = \bar{\mathbb{P}}[\bar{\mu}^\varepsilon(\bar{X}^\varepsilon(1)) \le u]. \tag{4.105}$$

By definition of π_1^ε we have

$$1 - F_t(u) = \bar{\mathbb{E}}\bar{\mathbb{P}}[\bar{\mu}^\varepsilon(\bar{X}^\varepsilon(1)) > u|V] = \bar{\mathbb{E}}\Big[\sum_{i:c_\varepsilon\tau_i^\varepsilon > u} w_i^\varepsilon\Big]. \tag{4.106}$$

The point process convergence proved in Lemma 4.8.3 implies that the sum in the last expectation converges $\bar{\mathbb{P}}$-a.s. for all u such that $u \neq v_i$ for all i.

$$\lim_{\varepsilon\to 0}\sum_{i:c_\varepsilon\tau_i^\varepsilon > u} w_i^\varepsilon = \sum_{i:v_i>u} w_i = \bar{\mathbb{P}}[\bar{\rho}(\bar{Z}(1)) > u|V] \tag{4.107}$$

Using the fact that (ρ, Z) has the same distribution as $(\bar{\rho}, \bar{Z})$ and applying dominated convergence theorem it is easy to finish the proof. □

Proof of Lemma 4.8.1. One should only prove that $L(t)$ is slowly varying. Since τ_0 is in the domain of attraction of the stable variable with index α, there exists a slowly varying function $L_1(t)$ such that

$$\mathbb{P}[\tau_0 > t] = t^{-\alpha}L_1(t). \tag{4.108}$$

From definition (4.49) of c_ε we get

$$\varepsilon^{-1}\mathbb{P}[\tau_0 > c_\varepsilon^{-1}] \to 1 \qquad \text{as} \quad \varepsilon \to 0. \tag{4.109}$$

Indeed, it is easy to see that $\varepsilon^{-1}\mathbb{P}[\tau_0 > c_\varepsilon^{-1}] \leq 1$. Take $\eta > 0$, the lower bound follows from

$$(1+2\eta)^{-\alpha} = \lim_{\varepsilon\to 0} \frac{\mathbb{P}[\tau_0 > \frac{1+2\eta}{1+\eta}c_\varepsilon^{-1}]}{\mathbb{P}[\tau_0 > \frac{1}{1+\eta}c_\varepsilon^{-1}]} \leq \liminf_{\varepsilon\to 0} \varepsilon^{-1}\mathbb{P}[\tau_0 > c_\varepsilon^{-1}] \tag{4.110}$$

since η is arbitrary. From (4.109) and (4.108) we get

$$\varepsilon^{-1}c_\varepsilon^\alpha L_1(c_\varepsilon^{-1}) \to 1 \qquad \text{as} \quad \varepsilon \to 0. \tag{4.111}$$

Applying (4.89) we have

$$c_t t^\gamma L_1^\gamma(c_t^{-1}) \to 1 \qquad \text{as} \quad t \to \infty. \tag{4.112}$$

We want to show that $c_t = t^{-\gamma}L(t)^{-1}$ where $L(t)$ is slowly varying. Choose $k > 0$ and define $d_t = L(t)/L(kt)$. Take $\eta > 0$ small and assume that $\liminf_{t\to\infty} d_t < 1 - 2\eta$. We choose $\delta > 0$ and we consider t large enough such that $c_t t^\gamma L_1^\gamma(c_t^{-1}) \in (1-\delta, 1+\delta)$. This can be done by (4.112). We have

$$d_t = \frac{L(t)}{L(kt)} = \frac{c_{kt}}{c_t}k^\gamma \geq \frac{1-\delta}{1+\delta} \cdot \frac{L_1^\gamma(c_t^{-1})}{L_1^\gamma(c_{kt}^{-1})} = \frac{1-\delta}{1+\delta} \cdot \frac{L_1^\gamma(c_t^{-1})}{L_1^\gamma(d_t^{-1}c_t^{-1}k^\gamma)}. \tag{4.113}$$

Our assumption implies that there exists a sequence t_n such that $d_{t_n}^{-1} > 1 + \eta$ for all n. Since L_1 is slowly varying, we know that for arbitrary $\theta > 0$ there exists x_0 such that for all $l > 1 + \eta$ and $x > x_0$ we have $L_1(lx) \leq l^\theta L_1(x)$. This implies that for n large enough we have

$$d_{t_n} \geq \frac{1-\delta}{1+\delta} \cdot \frac{L_1^\gamma(c_{t_n}^{-1})}{d_{t_n}^{-\gamma\theta} L_1^\gamma(c_{t_n}^{-1}k^\gamma)}. \tag{4.114}$$

Taking the limit $n \to \infty$, using that $c_{t_n} \to \infty$ and that L_1 is slowly varying we get

$$\liminf_{n\to\infty} d_{t_n}^{1+\gamma\theta} \geq \frac{1-\delta}{1+\delta}. \tag{4.115}$$

For every η we can take δ and θ such that the last equation is in contradiction with $\liminf_{t\to\infty} d_t < 1 - 2\eta$. Thus $\liminf_{t\to\infty} d_t \geq 1$. The proof of the upper bound follows from

$$d_{t_n} \leq \frac{1+\delta}{1-\delta} \cdot \frac{d_{t_n}^{\gamma\theta} L_1^\gamma(k^{-\gamma}c_{kt_n}^{-1})}{L_1^\gamma(c_{kt_n}^{-1})}. \tag{4.116}$$

This can be proved if one assumes that $\limsup_{t\to\infty} d_t \geq 1 + 2\eta$ and it leads to a contradiction similarly as in (4.115). $\square$

4.9 Proof of sub-aging in the non-symmetric case

If $a > 0$, the jump rates depend also on the depths of the neighbouring sites. As is easy to see from definition of τ_i^ε, the depth of the neighbouring sites of some very deep trap does not converge $\bar{\mathbb{P}}$-a.s. (By very deep trap we mean here a trap where X has a large chance to stay at time t.) On the other hand, we expect (see [RMB00]) that the depth of these sites is, at least if t_w is large, almost independent of the diffusion and has the same distribution as E_0.

The idea of the proof is to enlarge the probability space $\bar{\Omega}$ and insert in the neighbourhood of very deep traps additional sites with depths not depending on V. On this larger probability space we almost recover the a.s. convergence.

We first define the set of sites whose neighbours we will modify. Choose $m > 0$ large and $\eta > 0$ small. We use $J_m^\eta = J_m^\eta(V)$ to denote the set of deep traps not far from the origin

$$J_m^\eta = \{x \in [-m, m] : V(x) - V(x-) \geq \eta\}. \tag{4.117}$$

To simplify the following definitions we will suppose that ε is small enough such that the minimal distance of two points in $J_m^\eta \cup 0$ is larger then 2ε. The set $T_m^\eta(\varepsilon)$ will be the set of sites corresponding to J_m^η at the scale ε

$$T_m^\eta(\varepsilon) = \{i \in \mathbb{Z} : (i\varepsilon, (i+1)\varepsilon] \cap J_m^\eta \neq \emptyset\}. \tag{4.118}$$

Note that J_m^η and $T_m^\eta(\varepsilon)$ are $\bar{\mathbb{P}}$-a.s. finite sets.

Let τ_i^+ and τ_i^- be two independent sequences of i.i.d. random variables defined on $\bar{\Omega}$ with the same distribution as τ_0 that are also independent of V. We now define the new environments $\hat{\tau}_i^\varepsilon$. They are essentially the same as τ_i^ε only in the neighbourhood of the sites from $T_m^\eta(\varepsilon)$ we insert the new variables τ_i^+ and τ_i^-. The precise definition of $\hat{\tau}_i^\varepsilon$ follows.

Let $J_m^\eta = \{y_1, \dots, y_n\}$ with $y_i < y_{i+1}$ and $y_{r-1} < 0 < y_r$. Here we ignore the zero probability event $0 \in J_m^\eta$. Let $i_k^\varepsilon \in T_m^\eta(\varepsilon)$ be such that $y_k \in (i_k^\varepsilon \varepsilon, (i_k^\varepsilon + 1)\varepsilon]$ and let $\bar{i}_k^\varepsilon = 1 + i_k^\varepsilon + 2(k - r)$. We write $\bar{T}_m^\eta(\varepsilon)$ for $\{\bar{i}_1^\varepsilon, \dots, \bar{i}_n^\varepsilon\}$. Then

$$\hat{\tau}_i^\varepsilon = \begin{cases} \tau_{i_k^\varepsilon}^\varepsilon & \text{if } i = \bar{i}_k^\varepsilon \\ \tau_k^+ & \text{if } i = \bar{i}_k^\varepsilon + 1 \\ \tau_k^- & \text{if } i = \bar{i}_k^\varepsilon - 1 \\ \tau_{i-2(k-r)}^\varepsilon & \text{if } i \in \{\bar{i}_{k-1}^\varepsilon + 2, \dots, \bar{i}_k^\varepsilon - 2\}. \end{cases} \tag{4.119}$$

Lemma 4.9.1. *The sequence $\hat{\tau}_i^\varepsilon$ has the same distribution as τ_i.*

Proof. The independence of the $\hat{\tau}_i^\varepsilon$'s is a consequence of independence of V and $\tau^\pm$. It is thus sufficient to show that the distribution of $\hat{\tau}_j^\varepsilon$ is the same as that of τ_0. Let F be the event that $|\bar{i}_k^\varepsilon - j| = 1$ for some k. Then

$$\bar{\mathbb{P}}(\hat{\tau}_j^\varepsilon \leq u) = \bar{\mathbb{P}}(\hat{\tau}_j^\varepsilon \leq u | F)\bar{\mathbb{P}}(F) + \bar{\mathbb{P}}(\hat{\tau}_j^\varepsilon \leq u | F^c)\bar{\mathbb{P}}(F^c) \tag{4.120}$$

However, $\bar{\mathbb{P}}(\hat{\tau}_j^\varepsilon \le u|F) = \bar{\mathbb{P}}(\tau^\pm \le u) = \mathbb{P}(\tau_0 \le u)$ and similarly $\bar{\mathbb{P}}(\hat{\tau}_j^\varepsilon \le u|F^c) = \mathbb{P}(\tau_0 \le u)$. We have thus

$$\bar{\mathbb{P}}(\hat{\tau}_j^\varepsilon \le u) = \mathbb{P}(\tau_0 \le u)(\mathbb{P}(F) + \mathbb{P}(F^c)) = \mathbb{P}(\tau_0 \le u). \tag{4.121}$$

□

We define the measures $\hat{\mu}^\varepsilon$ and the scaling functions $\hat{S}_\varepsilon$ similarly as in (4.55) and (4.57) but using $\hat{\tau}_i^\varepsilon$ instead of τ_i^ε. Similarly as in Proposition 4.5.1 we have

Lemma 4.9.2. *For every fixed realisation of τ_i^+ and τ_i^-, $\bar{\mathbb{P}}$-a.s.*

$$\varepsilon \hat{S}_\varepsilon(\varepsilon^{-1}\cdot) \circ \hat{\mu}^\varepsilon \xrightarrow{v} \bar{\rho} \quad \textit{and} \quad \varepsilon \hat{S}_\varepsilon(\varepsilon^{-1}\cdot) \circ \hat{\mu}^\varepsilon \xrightarrow{pp} \bar{\rho} \quad \textit{as } \varepsilon \to 0. \tag{4.122}$$

Proof. The proof is the same as that of Lemma 4.5.2 and Proposition 4.5.1, the finite number of additional random variables looses its influence as $\varepsilon \to 0$. To demonstrate it, we will show here the differences that appear in the proof of Lemma 4.5.2.

We want to show that $\varepsilon \hat{S}_\varepsilon(\lfloor \varepsilon^{-1} y \rfloor) = \varepsilon \sum_{j=0}^{\lfloor \varepsilon^{-1} y \rfloor} \hat{r}_j^\varepsilon$ converges to y. Since J_m^η is finite only the finite number of $\hat{r}_j^\varepsilon$'s are influenced by changing the sequence of τ's. The contribution of this part of the sum goes to zero as $\varepsilon \to 0$. The rest of the sum can be treated in the same way as in the proof of Lemma 4.5.2. □

Further, we define the processes $\hat{X}^\varepsilon(t)$ as

$$\hat{X}^\varepsilon(t) = X(\hat{\mu}^\varepsilon, \varepsilon \hat{S}_\varepsilon(\varepsilon^{-1}\cdot))(t). \tag{4.123}$$

As follows from Lemma 4.9.1 these processes have the same distribution as $X^\varepsilon(t)$ and from Lemma 4.9.2 and Proposition 4.2.5 we know that their fixed time distributions converge to the distribution of $\bar{Z}$ at the same time.

The following proposition can be regarded as a stronger version of the localisation effect (4.8). It claims that we can find a finite set $A_\varepsilon \subset \mathbb{Z}$ such that $\varepsilon^{-1} X^\varepsilon(1) \in A_\varepsilon$ with arbitrarily large probability. The size of A_ε is independent of ε.

Proposition 4.9.3. *For every $\delta > 0$ there exist m, η, and ε_0 such that for $\varepsilon < \varepsilon_0$*

$$\bar{\mathbb{P}}\big[\bar{\mathbb{P}}(\varepsilon^{-1} \hat{X}^\varepsilon(1) \in \bar{T}_m^\eta(\varepsilon) | V, \tau_i^+, \tau_i^-) > 1 - \delta\big] > 1 - \delta. \tag{4.124}$$

We postpone the proof of this proposition and we first finish the proof of sub-aging. We consider the function $\Pi(t, t + f_a(t,\theta))$. By its definition we have

$$\Pi(t, t + f_a(t,\theta)) = \mathbb{E}\Big[\sum_{i \in \mathbb{Z}} \mathbb{P}(X(t) = i | E) \exp(-(w_{i,i+1} + w_{i,i-1}) f_a(t,\theta))\Big]. \tag{4.125}$$

The rates $w_{i,i+1}$ and $w_{i,i-1}$ can be expressed using the variables τ_i

$$w_{i,i+1} + w_{i,i-1} = \frac{\tau_{i-1}^a + \tau_{i+1}^a}{\tau_i^{1-a}} \left[\frac{\mathbb{E}(\exp(-2a\beta E_0))}{2}\right]^{1-2a}. \tag{4.126}$$

We use K to denote the constant in the brackets in the last expression.

Let ε be such that $\varepsilon c_\varepsilon t = 1$ similarly as in the proof of Proposition 4.8.2. From Lemma 4.9.1, it follows that (4.126) can be rewritten using the measures with hats,

$$\begin{aligned}\Pi(t, t+f_a(t,\theta))& \\ &= \bar{\mathbb{E}}_{\tau^\pm}\bar{\mathbb{E}}_V\Big[\sum_{i\in\mathbb{Z}} \hat{w}_i^\varepsilon \exp\Big(-f_a(t,\theta)K\frac{(\hat{\tau}_{i+1}^\varepsilon)^a + (\hat{\tau}_{i-1}^\varepsilon)^a}{(\hat{\tau}_i^\varepsilon)^{1-a}}\Big)\Big],\end{aligned} \tag{4.127}$$

where $(\hat{x}_i^\varepsilon, \hat{w}_i^\varepsilon)$ is defined similarly as in (4.95) and $\bar{\mathbb{E}}_V$ and $\bar{\mathbb{E}}_{\tau^\pm}$ are the expectations over all realisations of V, resp. $\tau^\pm$. Let $\delta > 0$ and take m, η and ε_0 as in Proposition 4.9.3. We consider only $\varepsilon < \varepsilon_0$.

We divide the sum in the last expression into two parts. The first one over $i \in \bar{T}_m^\eta(\varepsilon)$ and the second one over the rest. The second part is not important. Indeed,

$$\begin{aligned}\bar{\mathbb{E}}_{\tau^\pm}\bar{\mathbb{E}}_V\Big[\sum_{i\in\mathbb{Z}\setminus\bar{T}_m^\eta(\varepsilon)} \hat{w}_i^\varepsilon \exp\Big(-f_a(t,\theta)K\frac{(\hat{\tau}_{i+1}^\varepsilon)^a + (\hat{\tau}_{i-1}^\varepsilon)^a}{(\hat{\tau}_i^\varepsilon)^{1-a}}\Big)\Big]& \\ \le \bar{\mathbb{E}}_{\tau^\pm}\bar{\mathbb{E}}_V\Big[\sum_{i\in\mathbb{Z}\setminus\bar{T}_m^\eta(\varepsilon)} \hat{w}_i^\varepsilon\Big] \le 2\delta&\end{aligned} \tag{4.128}$$

as follows from Proposition 4.9.3. Let us look at the limit of the first part. We have

$$\begin{aligned}\lim_{\varepsilon\to 0}\sum_{i\in\bar{T}_m^\eta(\varepsilon)} \hat{w}_i^\varepsilon \exp\Big(-f_a(t,\theta)K\frac{(\tau_{i+1}^\varepsilon)^a + (\tau_{i-1}^\varepsilon)^a}{(\tau_i^\varepsilon)^{1-a}}\Big)& \\ = \sum_{j=1}^n \hat{w}(y_j)\exp\Big(-K\theta t^{\gamma(1-a)}L(t)^{1-a}\frac{(\tau_j^+)^a + (\tau_j^-)^a}{(c_\varepsilon^{-1}\bar{\rho}(y_j))^{1-a}}\Big),&\end{aligned} \tag{4.129}$$

where $\hat{w}(y_j)$ is the weight of the atom of distribution of $\bar{Z}(1)$ at y_j. Here we used the fact that the values of $\hat{\tau}$ for the neighbours of $\bar{T}_m^\eta(\varepsilon)$ do not depend on ε. Applying Lemma 4.8.1, we get from the last two claims

$$\begin{aligned}\limsup_{t_w\to\infty} &\Pi(t,t+f_a(t,\theta))\\ &\le \bar{\mathbb{E}}_V\bar{\mathbb{E}}_{\tau^\pm}\Big[\sum_{j=1}^n \hat{w}(y_j)\exp\Big(-K\theta\frac{(\tau_j^+)^a+(\tau_j^-)^a}{\bar{\rho}(y_j)^{1-a}}\Big)\Big]+2\delta,\\ \liminf_{t_w\to\infty} &\Pi(t,t+f_a(t,\theta))\\ &\ge \bar{\mathbb{E}}_V\bar{\mathbb{E}}_{\tau^\pm}\Big[\sum_{j=1}^n \hat{w}(y_j)\exp\Big(-K\theta\frac{(\tau_j^+)^a+(\tau_j^-)^a}{\bar{\rho}(y_j)^{1-a}}\Big)\Big].\end{aligned} \tag{4.130}$$

The expectation over $\tau^\pm$ is easy to calculate since the distribution of $\tau_i^\pm$ is same as the distribution of $\exp(\beta E_0)\mathbb{E}(\exp(-2a\beta E_0))/2$. Thus $K(\tau_i^+)^a$ has the same distribution as

$$2^{a-1}\exp(a\beta E_0)\big(\mathbb{E}(\exp(-2a\beta E_0))\big)^{1-a}\equiv T_a. \tag{4.131}$$

If we use $g_a(\lambda)=\mathbb{E}(e^{-\lambda T_a})$ to denote the Laplace transform of T_a and add inside the sum the remaining atoms (making again an error of order at most 2δ), we get

$$\lim_{t_w\to\infty}\Pi(t,t+f_a(t,\theta))=\bar{\mathbb{E}}_V\big[\sum_j \hat{w}_j g_a^2(\theta\bar{\rho}(\hat{x}_j)^{a-1})\big]\pm 4\delta. \tag{4.132}$$

Since δ was arbitrary we have

$$\Pi(\theta)=\int_0^\infty g_a^2(\theta u^{a-1})dF(u), \tag{4.133}$$

which finishes the proof of sub-aging in the asymmetric situation. We still have to show Proposition 4.9.3

Proof of Proposition 4.9.3. The claim follows from the existence of η and m such that

$$\bar{\mathbb{P}}\big[\bar{\mathbb{P}}(\bar{Z}(1)\in J_m^\eta|V)\ge 1-\delta/2\big]\ge 1-\delta/2, \tag{4.134}$$

and from the $\bar{\mathbb{P}}$-a.s. point process convergence of the distribution of $\hat{X}^\varepsilon(1)$ to that of $\bar{Z}(1)$. Namely, for $\bar{\mathbb{P}}$-a.e. realisation of V it follows from Proposition 4.2.5(iii) that there is $\varepsilon(V)>0$ such that for $\varepsilon<\varepsilon(V)$

$$\big|\bar{\mathbb{P}}(\bar{Z}(1)\in J_m^\eta|V)-\bar{\mathbb{P}}(\hat{X}^\varepsilon(1)\in\bar{T}_m^\eta(\varepsilon)|V)\big|\le\delta/2. \tag{4.135}$$

We then take ε_0 such that $\bar{\mathbb{P}}(\varepsilon(V)>\varepsilon_0)>1-\delta/2$.

We should still verify (4.134). It is equivalent to

$$\bar{\mathbb{P}}[\bar{\mathbb{P}}(\bar{Z}(1)\notin J_m^\eta|V)\le\delta/2]\ge 1-\delta/2. \tag{4.136}$$

The last claim can be easily verified if we show

$$\bar{\mathbb{P}}[\bar{Z}(1) \notin J_m^\eta] = \bar{\mathbb{E}}[\bar{\mathbb{P}}(\bar{Z}(1) \notin J_m^\eta | V)] \le \delta^2/4. \tag{4.137}$$

Indeed, assume that (4.136) is not true, i.e.

$$\bar{\mathbb{P}}[\bar{\mathbb{P}}(\bar{Z}(1) \notin J_m^\eta | V) > \delta/2] > \delta/2. \tag{4.138}$$

Then clearly

$$\bar{\mathbb{E}}[\bar{\mathbb{P}}(\bar{Z}(1) \notin J_m^\eta | V)] > \delta^2/4, \tag{4.139}$$

in contradiction with (4.137).

We establish claim (4.137) using two lemmas.

Lemma 4.9.4. *Let $\eta(t) = t^{1/(1+\alpha)}$ and $m(t) = t^{\alpha/(1+\alpha)}$. Then*

$$\bar{\mathbb{P}}(\bar{Z}(1) \in J_m^\eta) = \bar{\mathbb{P}}(\bar{Z}(t) \in J_{m(t)}^{\eta(t)}). \tag{4.140}$$

Lemma 4.9.5. *For every δ' there exist m' and η' such that*

$$\int_0^1 \bar{\mathbb{P}}(\bar{Z}(t) \in J_{m'}^{\eta'})\, dt \ge 1 - \delta'. \tag{4.141}$$

We first finish the proof of Proposition 4.9.3. The Lemma 4.9.5 ensures the existence of $t \in (0,1)$ such that $\bar{\mathbb{P}}(Z(t) \in J_{m'}^{\eta'}) \ge 1 - \delta'$. The claim (4.137) then follows from Lemma 4.9.4, choosing $\delta' = \delta^2/4$, $m = t^{-\alpha/(1+\alpha)} m'$, and $\eta = t^{-1/(1+\alpha)} \eta'$. $\square$

Proof of Lemma 4.9.4. The pair

$$(W_\lambda(t), V_\lambda(x)) \equiv \big(\lambda W(\lambda^{-2} t), \lambda^{1/\alpha} V(\lambda^{-1} x)\big) \tag{4.142}$$

has the same distribution as $(W(t), V(x))$. The measure $\bar{\rho}_\lambda$ associated to V_λ can be written as

$$\bar{\rho}_\lambda = \sum_{x_i} (V_\lambda(x_i) - V_\lambda(x_i-)) \delta_{x_i} = \lambda^{1/\alpha} \sum_{y_i} (V(y_i) - V(y_i-)) \delta_{\lambda y_i}. \tag{4.143}$$

We thus have

$$\begin{aligned} \phi_\lambda(t) &\equiv \int \ell_\lambda(t,y) \bar{\rho}_\lambda(dy) = \int \lambda \ell(\lambda^{-2} t, \lambda^{-1} y) \bar{\rho}_\lambda(dy) \\ &= \sum_{y_i} \lambda \ell(\lambda^{-2} t, y_i) \lambda^{1/\alpha} (V(y_i) - V(y_i-)) = \lambda^{(\alpha+1)/\alpha} \phi(\lambda^{-2} t) \end{aligned} \tag{4.144}$$

and therefore its generalised inverse satisfies $\psi_\lambda(t) = \lambda^2\psi(\lambda^{-(\alpha+1)/\alpha}t)$. The rescaled singular diffusion defined by $\bar{Z}_\lambda = W_\lambda(\psi_\lambda(t))$ that has the same distribution as $\bar{Z}$ thus satisfies

$$\bar{Z}_\lambda(t) = W_\lambda(\psi_\lambda(t)) = \lambda\bar{Z}(\lambda^{-(\alpha+1)/\alpha}t). \tag{4.145}$$

Clearly, the triplet $(W_\lambda, V_\lambda, \bar{Z}_\lambda)$ has the same distribution as $(W, V, \bar{Z})$ too. We thus have

$$\bar{\mathbb{P}}(\bar{Z}(1) \in J_m^\eta(V)) = \bar{\mathbb{P}}(\bar{Z}_\lambda(1) \in J_m^\eta(V_\lambda)). \tag{4.146}$$

The set $J_m^\eta(V_\lambda)$ satisfies $J_m^\eta(V_\lambda) = \lambda J_{m\lambda^{-1}}^{\eta\lambda^{-1/\alpha}}(V)$ as can be easily verified from the scaling of V or from (4.143) and thus

$$\begin{aligned}\bar{\mathbb{P}}(\bar{Z}(1) \in J_m^\eta(V)) &= \bar{\mathbb{P}}(\lambda\bar{Z}(\lambda^{-(\alpha+1)/\alpha}) \in \lambda J_{m\lambda^{-1}}^{\eta\lambda^{-1/\alpha}}(V)) \\ &= \bar{\mathbb{P}}(\bar{Z}(\lambda^{-(\alpha+1)/\alpha}) \in J_{m\lambda^{-1}}^{\eta\lambda^{-1/\alpha}}(V)).\end{aligned} \tag{4.147}$$

The proof is finished taking λ satisfying $\lambda^{-(\alpha+1)/\alpha} = t$. □

Proof of Lemma 4.9.5. The claim of the lemma is equivalent with

$$\int_0^1 \bar{\mathbb{P}}(\bar{Z}(t) \notin J_{m'}^{\eta'})\,dt \le \delta'. \tag{4.148}$$

Let m be large enough such that

$$\bar{\mathbb{P}}(\bar{Z} \text{ leaves } [-m, m] \text{ before time } 1) < \delta'/2. \tag{4.149}$$

We use σ to denote the first time $\bar{Z}$ leaves $[-m, m]$. Then we have

$$\begin{aligned}\int_0^1 \bar{\mathbb{P}}(\bar{Z}(t) \notin J_{m'}^{\eta'})\,dt &\le \int_0^\sigma \bar{\mathbb{P}}(\bar{Z}(t) \notin J_{m'}^{\eta'})\,dt + \bar{\mathbb{P}}(\sigma < 1) \\ &\le \int_0^\sigma \bar{\mathbb{P}}(\bar{Z}(t) \notin J_{m'}^{\eta'})\,dt + \delta'/2.\end{aligned} \tag{4.150}$$

We should bound the integral in the last expression by $\delta'/2$. We establish this bound by proving

$$\bar{\mathbb{P}}\left[\int_0^\sigma \bar{\mathbb{P}}(\bar{Z}(t) \notin J_{m'}^{\eta'}|V)\,dt \ge \delta'/4\right] \le \delta'/4. \tag{4.151}$$

The integral inside the brackets can be written as

$$\int_0^\sigma \bar{\mathbb{P}}(\bar{Z}(t) \notin J_{m'}^{\eta'}|V)\,dt = \sum_{\substack{x_i \in [-m,m] \\ v_i < \eta}} G_m(0, x_i)v_i, \tag{4.152}$$

where as usually (x_i, v_i) is the collection of atoms of $\bar{\rho}$ and $G_m(x, y)$ is the Green's function of the standard Brownian motion killed on exit from $[-m, m]$. There exists a constant k depending only on m such that $G(0, x) \leq k$ for all $x \in [-m, m]$. We thus have

$$\bar{\mathbb{P}}\left[\int_0^\sigma \bar{\mathbb{P}}(\bar{Z}(t) \notin J_{m'}^{\eta'}|V)\,dt \geq \delta'/4\right] \leq \bar{\mathbb{P}}\left[k \sum_{\substack{x_i \in [-m,m] \\ v_i < \eta}} v_i \geq \delta'/4\right]. \tag{4.153}$$

The sum in the last equation has the same distribution as the Lévy process V without jumps larger then η at the time $2m$. One can thus easily choose η small enough, such that the last probability is smaller then $\delta'/4$. $\square$

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5. AGING IN TWO-DIMENSIONAL BOUCHAUD'S MODEL

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5.1 Introduction

Bouchaud's trap model [Bou92] is a model that was proposed in the physics literature to study dynamical properties of complex physical systems like for example the spin glasses (see [BCKM98] for survey, [RMB01] for recent numeric results). It is well known that the relaxation to the equilibrium of these systems is very slow below a certain temperature, but their dynamics has also other interesting features. They can be observed choosing a convenient correlation function $F(t_w, t_w + t)$ that depends on the behaviour of the system during the time interval $[t_w, t_w + t]$. The value t_w represents the time that passed between the preparation of the experiment and the start of measurements, the value t is the duration of the measurements. It was observed experimentally that for some such functions a nontrivial limit

$$\lim_{t_w \to \infty} F(t_w, t_w + f(t_w)) \tag{5.1}$$

exists with f being an increasing function. Such behaviour is referred as aging. More precisely, one speaks usually about aging if $f(t_w) = \theta t_w$ with $\theta > 0$. If $f(t_w) = o(t_w)$ one speaks about subaging. The classical example of such a function is $f(t_w) = t_w^\gamma$ with $0 < \gamma < 1$, but other possibilities can be relevant for different models as we will see later in this paper. The choice of a correlation function F is crucial. The function f strongly depends on this choice.

Bouchaud proposed the following model. Let $G = (\mathcal{V}, \mathcal{E})$ be a connected graph. To every vertex x of this graph is associated a random variable E_x. These variables are usually chosen i.i.d. with the exponential distribution. Bouchaud's model is a continuous time, nearest neighbours random walk on G. The jump rates are given by

$$w_{xy} = \nu \exp\big[-\beta((1-a)E_x - aE_y)\big] \quad \text{if } x \sim y, \tag{5.2}$$

and zero otherwise. The parameter β denotes as usually the inverse temperature, ν fixes the time scale and is irrelevant in our situation. The value of a

tunes the influence of the energy of the neighbouring sites on the dynamics. The value $-E_x$ can be regarded as the energy of state x.

Let us describe roughly the physical meaning of this model. The vertices of the graph represent a subset of all states of some complex physical system. Usually, the states with exceptionally low energy are chosen. This justifies the choice of E_x being exponential, because it is a distribution of extremes. The states with very low energy are important for the dynamics of the system because the time spent inside of them is large. States with high energy have no particular representation in the model since the time spent there is negligible. The set of edges of the graph represents the pairs of low energy states that are in some sense close from the dynamical point of view, for example for the spin glass models one can obtain one from the other by flipping a limited number of spins. Clearly the choice of the graph G influences strongly the properties and also the physical relevance of the model. For example, the complete graph is a good ansatz for the Random Energy Model. In this case the rigorous analysis of the model is rather straightforward. Bouchaud's trap model with $G = \mathbb{Z}$ was studied in [BČ02], where the reader can find also more detailed description of the role of parameter a.

We consider here a special case of Bouchaud's model on $\mathbb{Z}^2$. We first define our simplified model and then we comment its relation with the original one. Let $\boldsymbol{\tau} = \{\tau_x\}_{x\in\mathbb{Z}^2}$ be a collection of i.i.d. positive random variables with a law in the domain of the attraction of an α-stable law with $0 < \alpha < 1$. For simplicity we assume

$$\lim_{u\to\infty} u^\alpha \mathbb{P}(\tau_0 \geq u) = 1. \tag{5.3}$$

We study the process $X(t)$ that stays at the site $x \in \mathbb{Z}^2$ an exponentially distributed time with mean τ_x and then it jumps with the equal probability to one of the four neighbouring sites. Formally, let $X_d(i)$, $i = 0, 1, \dots,$ denote the discrete time simple random walk on $\mathbb{Z}^2$ started at origin, and let e_i be a collection of i.i.d. exponential random variables with mean one. We use $S(n)$ to denote the "time change" of the simple random walk

$$S(n) = \sum_{i=0}^{n-1} e_i \tau_{X_d(i)}. \tag{5.4}$$

Then $X(t) = X_d(j)$ if $S(j) \leq t < S(j+1)$.

The process is trapped at the site x a random time that is proportional to the value of τ_x, that is why we call this value the depth of the trap at site i.

The process $X(t)$, as we have defined, corresponds to Bouchaud's model with $a = 0$, $\beta = 1/\alpha$, E_x being e.g. exponential with mean one, $\tau_x = \exp(\beta E_x)$, and $\nu = 1/4$.

As in [BČ02], we consider two two-point functions to study the aging properties of Bouchaud's model:

$$R(t_w, t_w + t) = \mathbb{P}[X(t_w + t) = X(t_w)|\boldsymbol{\tau}], \tag{5.5}$$

which is the probability that the process is at to the same site at time $t_w + t$ as it was at time t_w, and

$$\Pi(t_w, t_w + t) = \mathbb{P}[X(t') = X(t_w) \forall t' \in [t_w, t_w + t]|\boldsymbol{\tau}], \tag{5.6}$$

which is the probability that the process does not jump between the times t_w and $t+t_w$. Unlike as in [BČ02] we study here the so called quenched two-point functions. As usually, this means that we do not take the average over all realisations of the environment. The functions R and Π depends thus on $\boldsymbol{\tau}$, but we will not denote this dependence explicitly.

We prove the aging behaviour for the function R.

Theorem 5.1.1. *There exists a function $R(\theta)$ independent of $\boldsymbol{\tau}$ such that for $\mathbb{P}$-a.e. realisation of the environment $\boldsymbol{\tau}$*

$$\lim_{t_w \to \infty} R(t_w, t_w + \theta t_w) = R(\theta). \tag{5.7}$$

Moreover, the function $R(\theta)$ can be explicitly calculated (see Proposition 5.7.1) and it satisfies

$$\lim_{\theta \to 0} R(\theta) = 1 \quad \textit{and} \quad \lim_{\theta \to \infty} R(\theta) = 0. \tag{5.8}$$

The subaging result is contained in the following theorem.

Theorem 5.1.2. *There exists a function $\Pi(\theta)$ independent of $\boldsymbol{\tau}$ such that for $\mathbb{P}$-a.e. realisation of the environment $\boldsymbol{\tau}$*

$$\lim_{t_w \to \infty} \Pi\left(t_w, t_w + \theta \frac{t_w}{\log t_w}\right) = \Pi(\theta). \tag{5.9}$$

The function $\Pi(\theta)$ can be again made explicit (see Proposition 5.8.2) and it satisfies the same relations (5.8) *as $R(\theta)$.*

The results of both theorems can be described heuristically in the following way. After time t_w the system is typically in a trap whose depth is of order $t_w/\log t_w$ (as can be seen from Theorem 5.1.2). After passing a time of that order in the trap the process X makes excursions from it and returns there of order $\log t_w$ times. Then X leaves the neighbourhood of this trap and continues to explore the lattice.

We describe here the strategy that will be used to prove both the theorems. Let $n \in \mathbb{N}$. We consider the process $X(t)$ only before the exit from the disk $\mathbb{D}(n)$ with the area $m2^n n^{1-\alpha}$ around the origin. The constant m will be chosen later in order that the walk can stay a sufficiently long time inside $\mathbb{D}(n)$. We are interested mainly in the time that the walk spends in traps of the depth larger than $\varepsilon 2^{n/\alpha}/n$ (such traps will be referred to as deep traps). In the disk $\mathbb{D}(n)$ there are approximately mn of such traps. Since the probability of hitting a particular point in $\mathbb{D}(n)$, that is sufficiently far from the walk's initial point, before the exit from $\mathbb{D}(n)$ is of order n^{-1}, the walk has a reasonable chance to hit at least one deep trap. The constant ε will be chosen small enough to ensure that the walk spends negligible proportion of time in shallower traps.

We cut the trajectory of the process X into short parts. Every part is finished when X exits for the first time the disk of area $2^n n^\beta$ around the initial point of the part. At this moment a new part is started. Clearly, we should take $\beta < 1-\alpha$. For every such part we look at the time that the walk spends in the traps which we have specified in the previous paragraph. It will be proved that, with overwhelming probability, the walk hits at most one such trap in every part. Moreover, the same trap is almost never hit again in the next parts before the exit from $\mathbb{D}(n)$. To the i-th part of the trajectory we associate a random variable s_i that we call score of that part, and that is roughly the time spent by X in the deep trap that was hit during this part (the score will be defined in Section 5.2). It will be proved that for n sufficiently large the random variables s_i are essentially independent and the well rescaled trajectory of the sum $\sum s_i$ converges to a pure jump, increasing Lévy process. It will be also shown that this sum is a good approximation for the well rescaled time change $S(n)$.

The proof of both theorems relies on the fact that the events that we are interested in, that is the probabilities of staying a long time at the same place, mainly occur if well rescaled values of times t_w and $t + t_w$ falls into one jump of the Lévy process, or more precisely if the intersection of the range of the Lévy process with the rescaled interval $[t_w, t + t_w]$ is empty.

The theorems are proved in Sections 5.7 and 5.8 where the reader can also find the explicit expressions for functions $R(\theta)$ and $\Pi(\theta)$. The proof of the convergence of well rescaled sums of scores occupies Sections 5.2–5.6.

5.2 The coarse-graining of $X(t)$

We introduce some notations needed later. We use $D_x(m)$, and $B_x(m)$ to denote the disk, resp. the box, with area m around the site x. If x is omitted the disk (box) is centred around the origin. Both these objects are understood as subsets of $\mathbb{Z}^2$. In the following we will very often use the claim that the disk

$D(m)$ contains m sites from $\mathbb{Z}^2$, although it is not precisely true. An error we introduce by this consideration is negligible for m large enough.

Let $n \in \mathbb{N}$ large. We consider the process $X(t)$ before the first exit from the disk $\mathbb{D}(n) \equiv D(m2^n n^{1-\alpha})$. We write

$$\begin{aligned} \Lambda_d(n) &= \inf\{i \in \mathbb{N} : X_d(i) \notin \mathbb{D}(n)\}, \\ \Lambda(n) &= \inf\{t \in \mathbb{R} : X(t) \notin \mathbb{D}(n)\} \end{aligned} \tag{5.10}$$

for the exit times of discrete, resp. continuous, time process from $\mathbb{D}(n)$. We will often skip the dependence on n in our notations.

We use $T_\varepsilon^M(n)$ to denote the set

$$T_\varepsilon^M(n) = \Big\{x \in \mathbb{D}(n) : \frac{\varepsilon 2^{n/\alpha}}{n} \le \tau_x < \frac{M 2^{n/\alpha}}{n}\Big\}. \tag{5.11}$$

If M or ε are omitted, it is understood $M = \infty$, resp. $\varepsilon = 0$. The constants ε and M will be chosen later. However, we always suppose that $\varepsilon \ll 1 \ll M$. We call the traps from T^ε shallow traps, T_ε^M is the set of deep traps, and T_M is the set of very deep traps.

We write $\mathcal{E}(n)$ for the set of sites that are sufficiently far from the set $T_\varepsilon^M(n)$,

$$\mathcal{E}(n) = \mathbb{D}(n) \setminus \bigcup_{y \in T_\varepsilon^M(n)} D_y(2^n n^{-\kappa}). \tag{5.12}$$

The constant $\kappa = \kappa(\alpha)$ can be taken arbitrarily large, but will be fixed while $n \to \infty$. The value $\kappa = 5/(1-\alpha)$ is sufficient for our purposes. The role of the set $\mathcal{E}(n)$ will be clarified later.

Further, we introduce a function $L(a)$ satisfying

$$\mathbb{P}[\tau_0 \ge u] = u^{-\alpha} L(u). \tag{5.13}$$

From (5.3) we know that $\lim_{a\to\infty} L(a) = 1$. It is also not difficult to see that $L(a)$ is bounded.

We write $\chi(A)$ for the indicator function of the set A. We use the letters C, c to denote positive constants that have no particular importance. The value of these constants can change during computations. On the other hand, the letter K is reserved for constants with particular meaning.

We define now the coarse-graining of the trajectory of the process X. Let $\beta < 1-\alpha$. We set $j_0^n = 0$, and then we define recursively

$$j_i^n = \min\{k > j_{i-1}^n : X_d(k) \notin D_{X_d(j_{i-1}^n)}(2^n n^\beta)\}, \tag{5.14}$$

with the convention that the minimum of an empty set is equal to infinity. We use x_i^n to denote the starting points of the parts of trajectory, $x_i^n = X_d(j_i^n)$.

The part of the trajectory of X_d between the times j,k is denoted by $X_d[j,k)$, i.e. $X_d[j,k) = \{X_d(l) : j \le l < k\}$. We will now define the score s_i^n of the part $X_d[j_i^n, j_{i+1}^n)$. Let λ_1 be the first time when X_d hits a deep trap after the start of this part,

$$\lambda_1 = \min\{k \ge j_i^n : X_d(k) \in T_\varepsilon^M\}. \tag{5.15}$$

Let $y = X_d(\lambda_1)$ be the first visited deep trap. Further, let λ_2 be the exit time from the disk $D_y(2^n n^{-\kappa})$,

$$\lambda_2 = \min\{k > \lambda_1 : X_d(k) \notin D_y(2^n n^{-\kappa})\}. \tag{5.16}$$

The last time that we need is

$$\lambda_3 = \min\left(\left\{k > \lambda_1 : X_d(k) \in T_\varepsilon^M \setminus y\right\} \cup \left\{k \ge \lambda_2 : X_d(k) \in T_\varepsilon^M\right\}\right). \tag{5.17}$$

It is the first time after λ_1 when X_d hits a deep trap, but we do not consider the successive hits of the trap y before the time λ_2.

If $\lambda_1 < \lambda_2 \le j_{i+1}^n \le \lambda_3$, $j_{i+1}^n \le \Lambda_d$, and y is farther then $\sqrt{\pi^{-1}2^n n^{-\kappa}}$ from the border of $D_{x_i^n}(2^n n^\beta)$, we define the score by

$$s_i^n = \sum_{k=\lambda_1}^{\lambda_2} e_k \tau_y \chi(X_d(k) = y). \tag{5.18}$$

The last condition assures that the movement of X inside $D_y(2^n n^{-\kappa})$ is not influenced by the border of $D_{x_i^n}(2^n n^\beta)$. If $\lambda_1 \ge j_{i+1}^n$ and $j_{i+1}^n \le \Lambda_d$, we set $s_i^n = 0$. In both previous cases the score is simply the time spent in the first visited deep trap. In all other cases we set $s_i^n = \infty$. This value has no particular meaning, it only marks the parts of trajectory where something "unusual" happens. By unusual we mean here that

(a) $X_d[j_i^n, j_{i+1}^n)$ contains two deep traps, and so $\lambda_3 < j_{i+1}^n$

(b) X_d exits $\mathbb{D}(n)$ before j_{i+1}^n, and so $\Lambda_d < j_{i+1}^n$.

(c) X_d returns to the first deep trap after exiting a disk of area $2^n n^{-\kappa}$ around it, i.e. again $\lambda_3 < j_{i+1}^n$

(d) Disk $D_y(2^n n^{-\kappa})$ intersects the complement of $D_{x_i^n}(2^n n^\beta)$, i.e. X hits a deep trap that is too close to the border of $D_{x_i^n}(2^n n^\beta)$.

We will study the behaviour of the trajectory of the process

$$Y^n(t) = \frac{1}{2^{n/\alpha}} \sum_{i=0}^{\lfloor tn^{1-\alpha-\beta}\rfloor} s_i^n. \tag{5.19}$$

The value of this process becomes infinite if any of the possibilities from the previous paragraph happen. Therefore, we will redefine Y^n. Let $J_1(n)$ be the index of the first part of trajectory where s_i^n is infinite, $J_1(n) = \min\{i : s_i^n = \infty\}$. For technical reasons we introduce another three bad events. Let

$$J_2(n) = \min\{i : x_{i+1}^n \notin \mathcal{E}(n)\}, \tag{5.20}$$

that means that the end of the J_2-th part of the trajectory is too close to some deep trap. The reason why we introduce this time is that when X starts a part of the trajectory too close to some deep trap, it has a big chance of hitting this trap, and thus the value of the score is strongly influenced by the depth of this trap.

For similar reasons we introduce

$$J_3(n) = \min\left\{i : \operatorname{dist}(x_i^n, \mathbb{D}(n)^c) \le \sqrt{\pi^{-1} 2^n n^\beta}\right\}. \tag{5.21}$$

That means that the part J_3 is the first part that starts too close to the border of $\mathbb{D}(n)$ and X can therefore exit from the large disk during it.

Further, let

$$J_4(n) = \min\{i : X_d[0, j_i^n) \cap T_\varepsilon^M \cap X_d[j_i^n, j_{i+1}^n) \neq \emptyset\}, \tag{5.22}$$

which means that X_d returns during part J_4 to some deep trap visited in previous parts of the trajectory. Let $J(n) = \min\{J_1(n), \dots, J_4(n)\}$. The value of J is the index of the fist part of the trajectory where at least one of the following “bad” possibilities happens

(i) X_d visits two different deep traps

(ii) X_d can exit $\mathbb{D}(n)$

(iii) X_d returns to some deep trap y (possibly visited in previous parts) after exiting $D_y(2^n n^{-\kappa})$

(iv) the end of this part of trajectory is too close to some deep trap.

(v) X_d hits a deep trap that is too close to the border of $D_{x_i^n}(2^n n^\beta)$.

Note that (iii) includes (c) from the previous list, (ii) contains (d), and (i), (v) is same as (a), (d).

Let now $\tilde{s}_i^n$ be a suitably chosen collection of i.i.d. random variables whose distribution will be defined later (see proof of Proposition 5.7.1). We set

$$\bar{s}_i^n = \begin{cases} s_i^n & \text{if } i < J(n), \\ \tilde{s}_i^n & \text{otherwise.} \end{cases} \tag{5.23}$$

We redefine the process Y^n by

$$Y^n(t) = \frac{1}{2^{n/\alpha}} \sum_{i=0}^{\lfloor tn^{1-\alpha-\beta} \rfloor} \bar{s}_i^n. \tag{5.24}$$

We want to compare this process with the well rescaled time change $S(n)$, namely with

$$\bar{S}^n(t) = \frac{1}{2^{n/\alpha}} S(j^n_{\lfloor tn^{1-\alpha-\beta} \rfloor}). \tag{5.25}$$

To this end we should control several quantities. First, we should estimate the time spent in the shallow traps, that is in T^ε (Section 5.3). Second, we need to control the probability that X_d hits T_M before Λ, because we did not include the very deep traps into the definition of the score (Section 5.4). Finally, we need to be sure that the value of J is large enough, otherwise the process Y^n has no relevance for our model (Section 5.5).

If all these condition are satisfied, that means that Y^n is a good approximation of $\bar{S}_n$ at least at the start of the trajectory, we should study the behaviour of the sequence Y^n. We will show that it converges to certain Lévy process (Section 5.6).

5.3 The shallow traps

As we already noted in the previous section, we want to show that the proportion of time that X spends in the shallow traps is negligible. It will be shown later that the time that X needs to leave disk $\mathbb{D}(n)$ is of order $2^{n/\alpha}$. We thus need to prove that the time spent in T^ε can be made arbitrarily small with respect to $2^{n/\alpha}$. This is the result of the following lemma, whose prove occupies the rest of this section.

Lemma 5.3.1. *There exists K_1 independent of ε such that for $\mathbb{P}$-a.e. random environment $\boldsymbol{\tau}$ and for n large enough*

$$\mathbb{E}\Big[\sum_{i=0}^{\Lambda_d - 1} e_i \tau_{X_d(i)} \chi\{X_d(i) \in T^\varepsilon\} \Big| \boldsymbol{\tau} \Big] \le K_1 \varepsilon^{1-\alpha} 2^{n/\alpha}. \tag{5.26}$$

To prove this lemma we first describe the distribution of the shallow traps in the disk $\mathbb{D}(n)$. We divide the shallow traps into several groups. Let $i_0(n)$ be an integer satisfying $1 \le \varepsilon 2^{-i_0(n)} \frac{2^{n/\alpha}}{n} < 2$. For any $i \in \{1, \dots, i_0(n)\}$ we define, similarly as in (5.11),

$$T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}} = \Big\{ x \in \mathbb{D}(n) : \varepsilon 2^{-i} \frac{2^{n/\alpha}}{n} \le \tau_x < \varepsilon 2^{-i+1} \frac{2^{n/\alpha}}{n} \Big\}. \tag{5.27}$$

Let C be a large positive constant. We use $H_1 = H_1(n, C, \varepsilon)$ to denote the event

$$H_1(n, C, \varepsilon) = \big\{\boldsymbol{\tau} : \big|T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\big| \leq C n \varepsilon^{-\alpha} 2^{i\alpha}, \forall i \in \{1, \dots, i_0(n)\}\big\}. \tag{5.28}$$

We show that H_1 occurs with an overwhelming probability.

Lemma 5.3.2. *There exists K_2 independent of ε such that for n large enough and for some positive constants C and c.*

$$\mathbb{P}[H_1(n, K_2, \varepsilon)] \geq 1 - Cn \exp(-cn). \tag{5.29}$$

The proof is postponed.

Convention. At this place it is convenient to introduce one convention. During the following parts of this paper we will need different properties of the environment that we will denote H_i, $i = 1, 2, \dots$ For all these properties we will prove a result that allows an application of Borel-Cantelli lemma. When we prove such result we will suppose that these properties are verified. We thus ignore a set of "unusual" environments whose probability is zero .

Proof of Lemma 5.3.1. The proof is divided into two parts. We first bound the time spent in very shallow traps. Let ξ be large enough such that

$$(1 - \xi)(1 - \alpha) + 1 < 0 \tag{5.30}$$

We define the set $\mathcal{S}$ of very shallow traps by

$$\mathcal{S} = \big\{x \in \mathbb{D}(n) : \tau_x \leq 2^{n/\alpha} n^{-\xi} \ll \varepsilon 2^{n/\alpha}/n\big\}, \tag{5.31}$$

Let $G_{\mathbb{D}(n)}(\cdot, \cdot)$ denote the Green's function of the simple random walk in the disk $\mathbb{D}(n)$. Then we have

$$\mathbb{E}\Big[\sum_{j=0}^{\Lambda_d - 1} e_j \tau_{X_d(j)} \chi\{X_d(j) \in \mathcal{S}\}\Big|\boldsymbol{\tau}\Big] = \sum_{x \in \mathbb{D}(n)} G_{\mathbb{D}(n)}(0, x) \tau(x) \chi\{x \in \mathcal{S}\}, \tag{5.32}$$

The Green's function can be bounded by (see (5.235) in Appendix 5.A)

$$G_{\mathbb{D}(n)}(0, x) \leq cn \quad \text{for all } x \in \mathbb{D}(n). \tag{5.33}$$

We thus have

$$\mathbb{E}\Big[\sum_{j=0}^{\Lambda_d - 1} e_j \tau_{X_d(j)} \chi\{X_d(j) \in \mathcal{S}\}\Big|\boldsymbol{\tau}\Big] \leq cn \sum_{x \in \mathbb{D}(n)} \tau(x) \chi\{x \in \mathcal{S}\}. \tag{5.34}$$

Let $i_1(n)$ be an integer satisfying

$$2^{-1+n/\alpha}n^{-\xi} \le 2^{-i_1(n)}\varepsilon\frac{2^{n/\alpha}}{n} \le 2^{n/\alpha}n^{-\xi}. \tag{5.35}$$

It is easy to verify that $i_1(n) \sim (\xi-1)\log_2 n$. The expression (5.34) is bounded from above by

$$cn\sum_{x\in\mathbb{D}(n)}\tau(x)\chi\{\tau(x)\le 2\} + cn\sum_{i=i_1(n)}^{i_0(n)}\sum_{x\in\mathbb{D}(n)}\tau(x)\chi\{x\in T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\}. \tag{5.36}$$

By Lemma 5.3.2 and (5.30) this can be bounded by

$$\begin{aligned}
&\le 2cnm2^n n^{1-\alpha} + Cn\sum_{i=i_1(n)}^{i_0(n)}\varepsilon 2^{-i+1}\frac{2^{n/\alpha}}{n}\cdot n\varepsilon^{-\alpha}2^{i\alpha}\\
&\le Cn\varepsilon^{1-\alpha}2^{n/\alpha}\sum_{i=i_1(n)}^{i_0(n)}2^{i(\alpha-1)} + o(2^{n/\alpha})\\
&\le C\varepsilon^{1-\alpha}2^{n/\alpha}n^{1+(1-\xi)(1-\alpha)} + o(2^{n/\alpha}) = o(2^{n/\alpha}).
\end{aligned} \tag{5.37}$$

This finishes the first part.

In the second part we bound the time spent in $T^\varepsilon\setminus\mathcal{S}$. We treat separately the time spent in $T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}$ for $i\in 1,\dots,i_1(n)$, where $i_1(n)$ is defined as above. Let K' be a large positive constant and let $A(n,i)$ be the event

$$A(n,i) = \Big\{\sum_{x\in T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}} G_{\mathbb{D}(n)}(0,x)\tau(x) \ge K'2^{n/\alpha}\varepsilon^{1-\alpha}2^{-i(1-\alpha)}\Big\}. \tag{5.38}$$

From the definition of $T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}$ we have

$$\mathbb{P}[A(n,i)] \le \mathbb{P}\Big[2\sum_{x\in T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}} G_{\mathbb{D}(n)}(0,x) \ge K'n\varepsilon^{-\alpha}2^{\alpha i}\Big]. \tag{5.39}$$

By Lemma 5.3.2, there are at most $K_2 n\varepsilon^{-\alpha}2^{i\alpha}$ sites in $T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}$ $\mathbb{P}$-a.s. for large n. For $i=i_1(n)$ this number is of order $n^{1+\alpha(\xi-1)}$, for all others i's it is smaller.

Let y_i, $i=1,\dots,R$, be a collection of uniformly, independently chosen points in $\mathbb{D}(n)$. By an easy combinatorial argument it is possible to prove that if R is $o(2^{n/2}n^{(1-\alpha)/2})$, then the probability that two of them are at the same place tends to zero. Since this is evidently satisfied for the number of sites in any of $T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}$, we can bound the sum in (5.39) by the sum over the random collection y_i, $i = 1, \dots, K_2 n\varepsilon^{-\alpha}2^{i\alpha}$. For some small c and for n large enough we thus have

$$\mathbb{P}[A(n,i)] \le (1+c)\mathbb{P}\Big[2 \sum_{i=0}^{K_2 n\varepsilon^{-\alpha}2^{i\alpha}} G_{\mathbb{D}(n)}(0,y_i) \ge K' n\varepsilon^{-\alpha}2^{\alpha i}\Big]. \tag{5.40}$$

It is known that there exist constants λ and C not depending on n such that (see Lemma 5.A.2 for proof of this claim)

$$\mathbb{E}\big[\exp\big(\lambda G_{\mathbb{D}(n)}(0,y_1)\big)\big] \le C. \tag{5.41}$$

By standard argument we can thus choose K' not depending on i such that

$$\mathbb{P}[A(n,i)] \le c\exp(-c' n\varepsilon^{-\alpha}2^{i\alpha}). \tag{5.42}$$

Since $i_1(n) \le n/\alpha$, we get by summation

$$\mathbb{P}\Big[\bigcup_{i=1}^{i_1(n)} A(n,i)\Big] \le cn\exp(-c' n\varepsilon^{-\alpha}), \tag{5.43}$$

and thus for n large enough none of $A(n,i)$ occurs $\mathbb{P}$-a.s. However, if it is the case, we have (using also the result of the first part of the proof)

$$\begin{aligned}\mathbb{E}\Big[\sum_{j=0}^{\Lambda_d-1} e_j \tau_{X_d(j)}\chi(X_d(j)\in T^\varepsilon)\Big|\boldsymbol{\tau}\Big]&\\ \le \sum_{i=0}^{i_1(n)} K' 2^{n/\alpha}\varepsilon^{1-\alpha}2^{-i(1-\alpha)} + o(2^{n/\alpha}) &\le K_1 2^{n/\alpha}\varepsilon^{1-\alpha}.\end{aligned} \tag{5.44}$$

This finishes the proof. □

It remains to show Lemma 5.3.2.

Proof of Lemma 5.3.2. We first study the size of $T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}$ for some fixed index i. The probability $p_{n,i}$ that a site in $\mathbb{D}$ is in $T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}$ is

$$p_{n,i} = \varepsilon^{-\alpha}\frac{n^\alpha}{2^n}2^{i\alpha}\Big[L\Big(\varepsilon 2^{-i}\frac{2^{n/\alpha}}{n}\Big) - \Big(\frac{1}{2}\Big)^\alpha L\Big(\varepsilon 2^{-i+1}\frac{2^{n/\alpha}}{n}\Big)\Big]. \tag{5.45}$$

Since L is bounded, the expression in the brackets can be bounded from above uniformly in i by some constant depending only on the function L. Hence,

$$p_{n,i} \le c\varepsilon^{-\alpha}\frac{n^\alpha}{2^n}2^{i\alpha}. \tag{5.46}$$

Applying exponential Markov bound we get for $\lambda > 0$

$$\begin{aligned}\mathbb{P}\big[\big|T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\big| \geq K_2 n\varepsilon^{-\alpha}2^{i\alpha}\big] &\leq \exp(-\lambda K_2 n\varepsilon^{-\alpha}2^{i\alpha})\mathbb{E}\big[\exp\big(\lambda\big|T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\big|\big)\big] \\ &= \exp(-\lambda K_2 n\varepsilon^{-\alpha}2^{i\alpha})\big[(1-p_{n,i})+p_{n,i}e^{\lambda}\big]^{m2^n n^{1-\alpha}} \\ &\leq \exp\big[n\varepsilon^{-\alpha}2^{i\alpha}(-K_2\lambda + mce^{\lambda})\big].\end{aligned} \tag{5.47}$$

In the last inequality we used bound (5.46) and the fact that $(1+1/n)^n \leq e$. If K_2 is chosen large enough, the expression in the parentheses is negative and thus the required probability decreases exponentially. The probability of H_1^c satisfies

$$\mathbb{P}[H_1^c] = \mathbb{P}\big[\bigcup_{i=0}^{i_0(n)}\big(\big|T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\big| \geq K_2 n\varepsilon^{-\alpha}2^{i\alpha}\big)\big]. \tag{5.48}$$

Hence, it is bounded by

$$\begin{aligned}\mathbb{P}\big[\bigcup_{i=0}^{i_0(n)}\big(\big|T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\big| \geq K_2 n\varepsilon^{-\alpha}2^{i\alpha}\big)\big] &\leq \sum_{i=0}^{i_0(n)}\exp\big\{n\varepsilon^{-\alpha}2^{i\alpha}(-K_2\lambda+mce^{\lambda})\big\} \\ &\leq i_0(n)\exp\big\{n\varepsilon^{-\alpha}(-K_2\lambda+mce^{\lambda})\big\}.\end{aligned} \tag{5.49}$$

Since $i_0(n) \leq n/\alpha$, the proof is finished. □

5.4 Very deep traps

In this section we estimate the probability of hitting a very deep trap.

Lemma 5.4.1. *For every $\delta > 0$ and m there exists M such that for n large enough and for $\mathbb{P}$-a.e. environment $\boldsymbol{\tau}$*

$$\mathbb{P}\big[X(t) \text{ hits } T_M(n) \text{ before } \Lambda(n)|\boldsymbol{\tau}\big] \leq \delta. \tag{5.50}$$

Proof. The standard large deviation argument gives

$$\mathbb{P}[|T_M(n)| > Cnm/M^{\alpha}] \leq C'\exp(-cnm/M^{\alpha}) \tag{5.51}$$

for some constants C, C' and c. We can thus take $\mathbb{P}$-a.s. n large enough such that $|T_M(n)| \leq Cnm/M^{\alpha}$. Let A be an uniformly chosen random subset of $\mathbb{D}(n)$ with Cnm/M^{α} elements. Then

$$\mathbb{P}\big[\mathbb{P}[X \text{ hits } T_M \text{ before } \Lambda|\boldsymbol{\tau}] > \delta\big] \leq \mathbb{P}\big[\mathbb{P}[X \text{ hits } A \text{ before } \Lambda|A] > \delta\big]. \tag{5.52}$$

Further, let $\{y_i\}$, $i = 1, \ldots, Cnm/M^\alpha$ be a collection of independently, uniformly chosen random points in $\mathbb{D}(n)$. Similarly as in the previous section we can replace A by this collection. The expression (5.52) is then bounded by

$$\leq (1+c)\mathbb{P}\Big[\sum_{i=1}^{Cnm/M^\alpha} \mathbb{P}[X \text{ hits } y_i \text{ before } \Lambda | y_i] \geq \delta\Big] \tag{5.53}$$

for some small positive c. The sum in the brackets is a sum of i.i.d. random variables and we use again the exponential Markov inequality to bound it,

$$\leq (1+c)\exp(-\delta\lambda_n)\mathbb{E}\big[\exp\big(\lambda_n \mathbb{P}[X \text{ hits } y_i \text{ before } \Lambda|y_i]\big)\big]^{Cnm/M^\alpha}. \tag{5.54}$$

The inequality (5.237) from Appendix 5.A applied on the disk $\mathbb{D}(n)$ gives

$$\mathbb{E}\big[\exp\big(-(n\log 2/2 + o(n))\mathbb{P}[X \text{ hits } y_1 \text{ before } \Lambda]\big)\big] \leq C. \tag{5.55}$$

And thus, taking $\lambda_n = \log\sqrt{\pi^{-1}m2^n n^{1-\alpha}}$,

$$\mathbb{P}\big[\mathbb{P}[X \text{ hits } T_M \text{ before } \Lambda|\boldsymbol{\tau}] > \delta\big] \leq \exp\big\{-\delta cn + c'mn/M^\alpha + o(n)\big\}. \tag{5.56}$$

The lemma then follows by taking M large enough and applying Borel-Cantelli argument. □

5.5 *J is large enough*

To justify the approximation of $\bar{S}^n$ by Y^n we should now prove that the index of the first bad part J is large enough. More precisely, we should show that one can choose κ and m such that, with large probability, the index J of the first bad part of the trajectory of X is sufficiently large for our purposes.

Lemma 5.5.1. *For any δ, k, and $\mathbb{P}$-a.e. $\boldsymbol{\tau}$ there exist m and κ not depending on ε and M such that for n large enough*

$$\mathbb{P}\big[J(n)n^{\alpha+\beta-1} \geq k|\boldsymbol{\tau}\big] \geq 1-\delta. \tag{5.57}$$

To prove this lemma we should verify that all events described in Section 5.2 happen with low probability. This is the goal of all following technical lemmas. The proof of Lemma 5.5.1 can be found at the end of this section.

Event (i). The most important part of this section is to show that X does not hit two deep traps during one part of the trajectory. The following lemma is a little bit more precise than is needed to bound J, however, we will need this more precise result later. We use p_ε^M to denote the factor $\varepsilon^{-\alpha} - M^{-\alpha}$.

Lemma 5.5.2. *Let*

$$V_{x_0}(n) = \sum_{y \in T_\varepsilon^M} \mathbb{P}_{x_0}\big[X_d \text{ hits } y \text{ before exiting } D_{x_0}(2^n n^\beta) | \boldsymbol{\tau}\big], \tag{5.58}$$

where $\mathbb{P}_{x_0}$ denotes the law of the simple random walk X_d started at x_0. Then for any δ and $\mathbb{P}$-a.e. $\boldsymbol{\tau}$ there is n_0 such that for all $x_0 \in \mathcal{E}(n)$ (see (5.12) *for definition of $\mathcal{E}(n)$) and for all $n > n_0$*

$$\frac{\mathcal{K}(1-\delta)p_\varepsilon^M}{n^{1-\alpha-\beta}} \le V_{x_0}(n) \le \frac{\mathcal{K}(1+\delta)p_\varepsilon^M}{n^{1-\alpha-\beta}} \tag{5.59}$$

with $\mathcal{K} = (\log 2)^{-1}$.

To prove this lemma we should describe the distribution of the deep traps inside $\mathbb{D}(n)$. This description is contained in Lemmas 5.5.3 and 5.5.4.

First, we will show that the deep traps are distributed almost homogeneously around the disk. Let $\nu < \beta < 1 - \alpha$ and let $H_2 = H_2(n, \delta, \varepsilon, M)$ be the set of configurations of the environment satisfying the "homogeneity" condition:

$$\begin{aligned} H_2 = \big\{\boldsymbol{\tau} \colon \big|T_\varepsilon^M \cap B_x(2^n n^\nu)\big| \in \big[(1-\delta)p_\varepsilon^M n^{\nu+\alpha}, (1+\delta)p_\varepsilon^M n^{\nu+\alpha}\big] \\ \text{for all } x \text{ such that } B_x(2^n n^\nu) \subset \mathbb{D}(n).\big\} \end{aligned} \tag{5.60}$$

Lemma 5.5.3. *For any ε, M, and δ there exist positive constants C and c such that for n large enough*

$$\mathbb{P}[H_2] \ge 1 - Cn^{1-\alpha-\nu}\delta^{-2}\exp(-cn^{\nu+\alpha}). \tag{5.61}$$

Proof. We divide the complement of H_2 into two parts. First, we treat the case when there is a region in $\mathbb{D}$ where there are not enough of deep traps. Let A be the event that there is a square of area $2^n n^\nu$ in $\mathbb{D}(n)$ where there are less than $(1-\delta)p_\varepsilon^M n^{\nu+\alpha}$ sites from $T_\varepsilon^M(n)$,

$$A = \{\exists x \in \mathbb{D} : \big|T_\varepsilon^M \cup B_x(2^n n^\nu)\big| < (1-\delta)p_\varepsilon^M n^{\nu+\alpha}, D_x(2^n n^\nu) \subset \mathbb{D}\}. \tag{5.62}$$

We use G to denote the grid $\lfloor 2^{n/2} n^{\nu/2}\delta/5\rfloor \mathbb{Z}^2$. Every square of area $2^n n^\nu$ contains at least one square of area $2^n n^\nu(1-\delta/2)$ with the centre in G. Hence, if A is true, then there is a square of area $2^n n^\nu(1-\delta/2)$ which has centre $x \in G$, and which contains less than $(1-\delta)p_\varepsilon^M n^{\nu+\alpha}$ sites. We use A_x to denote the last event. We have

$$\mathbb{P}[A] \le \sum_x \mathbb{P}[A_x] = C'\delta^{-2}n^{1-\alpha-\nu}\mathbb{P}[A_x], \tag{5.63}$$

where the sum runs over all $x \in G$ such that $B_x((1-\delta/2)2^n n^\nu) \subset \mathbb{D}$. We used the obvious fact that $\mathbb{P}[A_x]$ does not depend on x. The probability of A_x can be bounded using the standard method. Take $\eta > 0$. For n large enough, the probability p that a site is in $T_\varepsilon^M(n)$ is larger than $(1-\eta)p_\varepsilon^M 2^{-n} n^\alpha$. For $\lambda > 0$ we have

$$\begin{aligned}\mathbb{P}[A_x] &\le \exp(\lambda(1-\delta)n^{\nu+\alpha}p_\varepsilon^M)\big[(1-p)+e^{-\lambda}p\big]^{2^n n^\nu(1-\delta/2)} \\ &\le \exp(\lambda(1-\delta)n^{\nu+\alpha}p_\varepsilon^M)\Big[1+(e^{-\lambda}-1)\frac{(1-\eta)n^\alpha p_\varepsilon^M}{2^n}\Big]^{2^n n^\nu(1-\frac{\delta}{2})}.\end{aligned} \tag{5.64}$$

If n is large enough, the last expression is bounded by

$$\mathbb{P}[A_x] \le \exp\big[n^{\nu+\alpha}p_\varepsilon^M\big(\lambda(1-\delta)+(e^{-\lambda}-1)(1-\eta)^2(1-\delta/2)\big)\big]. \tag{5.65}$$

It is not difficult to show that for any δ there exist η and λ such that the exponent is negative. Hence, we have

$$\mathbb{P}[A] \le C' n^{1-\alpha-\nu}\delta^{-2}\exp(-c' n^{\nu+\alpha}). \tag{5.66}$$

In the second part of the proof we exclude the possibility that there are places in $\mathbb{D}$ where the deep traps are too dense. Let B be the event that there is a square of area $2^n n^\nu$ intersecting $\mathbb{D}(n)$ where is more than $(1+\delta)\varepsilon^{-\alpha}n^{\nu+\alpha}$ sites from $T_\varepsilon^M(n)$. The probability of B can be bounded exactly in the same way as the probability of A, one should only look at squares with area $2^n n^\nu(1+\delta/2)$ and centres in G. We thus have

$$\mathbb{P}[H_2(n)^c] \le \mathbb{P}[A \cup B] \le C n^{1-\alpha-\nu}\delta^{-2}\exp(-c n^{\nu+\alpha}). \tag{5.67}$$

This finishes the proof. □

The lemma we have just proved is not precise enough to bound the probability of hitting traps that are closer than $\sqrt{2^n n^\nu}$ to the starting point. The following lemma will serve us for that bound. Again, it describes some sort of homogeneity of the environment

We consider the events $H_3(i) = H_3(i,n,\varepsilon,M)$,

$$H_3(i) = \big\{\exists x \in \mathbb{D}(n) : \big|B_x(2^{n+i}n^{-\kappa}) \cap T_\varepsilon^M\big| \le 4\log^2 n(1 \vee 2^i n^{\alpha-\kappa}\varepsilon^{-\alpha})\big\}, \tag{5.68}$$

where $a \vee b$ denotes the maximum of a, b. We define H_3 by

$$H_3 = \bigcap_{i=-1}^{\infty} H_3(i). \tag{5.69}$$

Observe that $2^{n+i}n^{-\kappa} \ll 2^n n^\nu$ for fixed i and n large enough. So, we study here much smaller squares than in the previous lemma. Hence, the description of the homogeneity is more precise in this direction. On the other hand, we prove only the upper bound on the number of the deep traps in these squares and this bound is also "weaker" than the previous bound.

Lemma 5.5.4. *There exist a constant C such that*

$$\mathbb{P}[H_3] \geq 1 - Cn^{-3}. \tag{5.70}$$

Proof. Fix some i and consider the lattice $G_i = \mathbb{Z}^2\sqrt{2^{n+i}n^{-\kappa}}$. If there is x such that $|B_x(2^{n+i}n^{-\kappa}) \cap T_\varepsilon^M| \geq 4\log^2 n(1 \vee 2^i n^{\alpha-\kappa}\varepsilon^{-\alpha})$, then there is a point $y \in G_i$ such that $B_y(4 \cdot 2^{n+i}n^{-\kappa})$ contains more than $4\log^2 n(1 \vee 2^i n^{\alpha-\kappa}\varepsilon^{-\alpha})$ sites from T_ε^M. The number of squares with area $4 \cdot 2^{n+i}n^{-\kappa}$ and centres in G_i that intersect $\mathbb{D}(n)$ is bounded by $Cn^{1-\alpha+\kappa}2^{-i}$.

Consider now one such square. The probability that it contains too many sites from T_ε^M can be bounded by standard argument

$$\begin{aligned}\mathbb{P}\big[|B(4 \cdot 2^{n+i}n^{-\kappa}) \cap T_\varepsilon^M| \geq 4\log^2 n(1 \vee 2^i n^{\alpha-\kappa}\varepsilon^{-\alpha})\big]\\ \leq c\exp\big(- \lambda 4\log^2 n(1 \vee 2^i n^{\alpha-\kappa}\varepsilon^{-\alpha}) + 4p_\varepsilon^M(e^\lambda - 1)2^i n^{\alpha-\kappa}\varepsilon^{-\alpha}\big).\end{aligned} \tag{5.71}$$

Since $\alpha - \kappa < 0$, we can choose λ such that for n large enough the last expression is bounded by $(1/2)^{\log^2 n}$. Summation over i and over all squares that intersect $\mathbb{D}(n)$ gives us

$$\mathbb{P}[H_6^c] \leq \sum_{i=-1}^{\infty} C2^{-i}n^{1-\alpha+\kappa}(1/2)^{\log^2 n} \leq Cn^{-3}. \tag{5.72}$$

This completes the proof. ∎

We now have all ingredients to prove Lemma 5.5.2.

Proof of Lemma 5.5.2. We can suppose that x_0 is the origin. We use ξ to denote the exit time from $D(2^n n^\beta)$. Let β' be a constant satisfying $\nu < \beta' < \beta$. We divide the sum $V_0(n)$ into two parts. First, we sum over all deep traps that are far enough from the origin. Precisely, we consider the deep traps that are in $D(2^n n^\beta) \setminus D(2^n n^{\beta'})$. Let I_1 denotes the sum over such traps. We use I_2 to denote the sum over the remaining deep traps.

To show the upper bound on I_1, we cover the set $D(2^n n^\beta) \setminus D(2^n n^{\beta'})$ by squares of area $2^n n^\nu$ and centres in $\sqrt{2^n n^\nu}\mathbb{Z}^2$. Let $x_1, \dots, x_R$ denote the set of centres of such squares that intersect $D(2^n n^\beta) \setminus D(2^n n^{\beta'})$. Since $\nu < \beta'$, the size of each such square is negligible with respect to its distance to the origin. All deep traps in such square have thus almost the same chance to be hit. We use expression (5.234) from Appendix 5.A to estimate probability that X hits some point before exiting from $D(2^n n^\beta)$. Let r_n be the radius of this disk, $r_n = \sqrt{\pi^{-1} 2^n n^\beta}$.

$$I_1 \leq \sum_{i=1}^{R} \sum_{\substack{y_j \in B_{x_i}(2^n n^\nu) \\ y \in T_\varepsilon^M}} \Big(1 - \frac{\log|y_j|}{\log r_n} + O(n^{-2})\Big)$$
$$= \sum_{i=1}^{R} \big|B_{x_i}(2^n n^\nu) \cap T_\varepsilon^M\big| \Big(1 - \frac{\log|x_i|}{\log r_n} + O(n^{-1+(\nu-\beta')/2})\Big), \quad (5.73)$$

where we use the estimate

$$\frac{\log|y_j|}{\log r_n} - \frac{\log|x_i|}{\log r_n} = O(n^{-1+(\nu-\beta')/2}) \quad (5.74)$$

that is valid for any $y_j \in B_{x_i}(2^n n^\nu)$.

From Lemma 5.5.3 we know that for n large enough $|B_{x_i}(2^n n^\nu) \cap T_\varepsilon^M| \leq n^{\nu+\alpha} p_\varepsilon^M (1+\delta/2)$ and thus

$$I_1 \leq \sum_{i=1}^{R} n^{\nu+\alpha} p_\varepsilon^M (1+\delta/2) \Big(1 - \frac{\log|x_i|}{\log r_n} + O(n^{-1+(\nu-\beta')/2})\Big). \quad (5.75)$$

We now replace the summation by integration making again an error of order $O(n^{-1+(\nu-\beta')/2})$. I_1 is thus bounded from above by

$$\int_{D(2^n n^\beta)\setminus D(2^n n^{\beta'})} \frac{n^{\nu+\alpha} p_\varepsilon^M}{2^n n^\nu} \Big(1 + \frac{\delta}{2}\Big)\Big(1 - \frac{\log|x|}{\log r_n} + O(n^{-1+(\nu-\beta')/2})\Big)\, dx. \quad (5.76)$$

The integration gives

$$I_1 \leq \frac{n^{\alpha+\beta-1} p_\varepsilon^M}{\log 2}\Big(1 + \frac{\delta}{2}\Big)(1+o(1)) \leq \frac{n^{\alpha+\beta-1} p_\varepsilon^M}{\log 2}\Big(1 + \frac{3\delta}{4}\Big) \quad (5.77)$$

for n large enough. This finishes the proof of the upper bound for I_1. The proof of the lower bound is analogous. After a very similar calculation we get

$$I_1 \geq \frac{n^{\alpha+\beta-1} p_\varepsilon^M}{\log 2}\Big(1 - \frac{3\delta}{4}\Big). \quad (5.78)$$

We should now estimate the sum I_2 over all sites $x \in T_\varepsilon^M \cap (D(2^n n^{\beta'}) \setminus D(2^n n^{-\kappa}))$. The disk $D(2^n n^{-\kappa})$ can be excluded since by the assumptions of the lemma there are no deep traps in this disk. We cover the domain by objects composed by eight squares of area $2^{n+i} n^{-\kappa}$ composing together the square of nine times larger area with the middle square cut off. The centre of these object is the origin. The parameter i takes values in the set

$\{-1,0,1,\dots,(\beta'+\kappa)\log_2 n\}$. We use this covering because if the trap is too close to origin, we should know more precisely its position to estimate its hitting probability. Our covering becomes clearly finer when the origin is approached.

Any point inside the i-th object from the previous paragraph has distance from origin at least $\sqrt{2^{n+i}n^{-\kappa}}/2$. In each of the eight squares there is, by Lemma 5.5.4, at most $4\log^2 n(1\vee 2^i n^{\alpha-\kappa}\varepsilon^{-\alpha})$ sites from T_ε^M. By formula (5.234) for the hitting probability of a point in $\mathbb{D}(2^n n^\beta)$ we have

$$I_2 \le 8 \sum_{i=-1}^{(\beta'+\kappa)\log_2 n} \Big[1-\frac{\log(\sqrt{2^{n+i}n^{-\kappa}}/2)}{\log r_n} + O\Big(\frac{2^{-n-i}n^\kappa}{\log r_n}\Big) + O(\log^{-2} r_n)\Big] \cdot 4\log^2 n(1\vee 2^i n^{\alpha-\kappa}\varepsilon^{-\alpha}). \tag{5.79}$$

The expression in the brackets can be easily bounded by $Cn^{-1}\log n$ with some large constant C. Hence,

$$I_2 \le C \sum_{i=-1}^{(\beta'+\kappa)\log_2 n} \frac{\log n}{n} \log^2 n(1\vee 2^i n^{\alpha-\kappa}\varepsilon^{-\alpha}). \tag{5.80}$$

Since the expression inside of the summation is increasing in i, the last display can be trivially estimated by $(\beta'+\kappa)\log_2 n$ times the last term. This gives

$$I_2 \le Cn^{\alpha+\beta'-1}\log^4 n \ll \frac{n^{\alpha+\beta-1}p_\varepsilon^M}{\log 2}\Big(1+\frac{\delta}{4}\Big). \tag{5.81}$$

Putting together (5.77), (5.78), and (5.81) we get

$$\frac{n^{\alpha+\beta-1}p_\varepsilon^M}{\log 2}(1-\delta) \le I_1 \le V_0(n) = I_1 + I_2 \le \frac{n^{\alpha+\beta-1}p_\varepsilon^M}{\log 2}(1+\delta). \tag{5.82}$$

This finishes the proof of Lemma 5.5.2. ☐

The following lemma is an easy consequence of Lemma 5.5.2. It is the actual estimate of the probability of hitting of a deep trap.

Lemma 5.5.5. *For any $\delta>0$ and $\mathbb{P}$-a.e. $\boldsymbol{\tau}$, there exists n_0 such that for $n>n_0$ and for all $x\in\mathcal{E}(n)$, the probability that the simple random walk started at x hits exactly one site from $T_\varepsilon^M(n)$ before exiting $D_x(2^n n^\beta)$ is in interval*

$$\big(\mathcal{K}(1-\delta)p_\varepsilon^M n^{\alpha+\beta+1}, \mathcal{K}(1+\delta)p_\varepsilon^M n^{\alpha+\beta+1}\big). \tag{5.83}$$

The probability that it hits more than one deep trap is bounded by

$$\mathbb{P}[X \text{ hits at least two sites from } T_\varepsilon^M] \le Cn^{2(\alpha+\beta-1)}(p_\varepsilon^M)^2 \tag{5.84}$$

for some positive constant C.

Proof. Let $T_\varepsilon^M \cap D_x(2^n n^\beta) = \{x_1, \ldots, x_L\}$. Assume that some point x_i was hit by X before the exit from $D(2^n n^\beta)$. We would like to apply the strong Markov property together with Lemma 5.5.2 at the moment of the first visit of x_i. To this end we should prove that there is no other deep trap in $D_{x_i}(2^n n^{-\kappa})$.

Let $H_4 = H_4(n, \varepsilon)$ be the event

$$H_4(n,\varepsilon) = \big\{\boldsymbol{\tau} : \min\{|x-y| : x, y \in T_\varepsilon(n)\} \geq 2\sqrt{\pi^{-1} 2^n n^{-\kappa}}\big\}. \tag{5.85}$$

The constant 2 before the square root is not necessary for the current application, but it will be used later.

Lemma 5.5.6. *There exists constant $C = C(\varepsilon, m)$ such that*

$$\mathbb{P}[H_4] \geq 1 - Cn^{1+\alpha-\kappa}. \tag{5.86}$$

Proof. Let $B(x)$ be the event

$$B(x) = \big\{x \in T_\varepsilon(n) \wedge \exists y \in T_\varepsilon(n), |y - x| \leq 2\sqrt{\pi^{-1} 2^n n^{-\kappa}}\big\}. \tag{5.87}$$

Then

$$\mathbb{P}[B(x)] \leq C\frac{n^{2\alpha-\kappa}}{2^n}\varepsilon^{-2\alpha}. \tag{5.88}$$

and the result follows by summation over all $x \in \mathbb{D}(n)$. □

We apply now Lemma 5.5.2 on $D_{x_i}(4 \cdot 2^n n^\beta)$. We should make two comments to its application. First, the site x_i is not strictly speaking in $\mathcal{E}(n)$ because $x_i \in T_\varepsilon^M(n)$. However, we are not interested in the successive returns to x_i and we can thus ignore this fact. Note also the appearance of the additional factor 4 in the area of the disk. We did not prove Lemma 5.5.2 with this factor. However, the proof can be easily modified to establish similar claim as (5.59) with this additional constant. Its appearance changes only the constant $\mathcal{K}$ and not the asymptotic behaviour. We thus have

$$\sum_{j \neq i} \mathbb{P}[X \text{ hits } x_j | X \text{ hit } x_i] \leq Cn^{\alpha+\beta-1} p_\varepsilon^M. \tag{5.89}$$

The Bonferroni inequalities give

$$\begin{aligned}
&\mathbb{P}[X \text{ hits } T_\varepsilon^M] \leq \sum_i \mathbb{P}[X \text{ hits } x_i] \leq \mathcal{K}(1+\delta) p_\varepsilon^M n^{\alpha+\beta-1} \\
&\mathbb{P}[X \text{ hits } T_\varepsilon^M] \geq \sum_i \mathbb{P}[X \text{ hits } x_i] - \frac{1}{2}\sum_i \sum_{j\neq i} \mathbb{P}[X \text{ hits } x_i \text{ and } x_j] \\
&\quad \geq \mathcal{K}(1-\delta) p_\varepsilon^M n^{\alpha+\beta-1} - C(p_\varepsilon^M)^2 n^{2(\alpha+\beta-1)} \geq \mathcal{K}(1-2\delta) p_\varepsilon^M n^{\alpha+\beta-1}
\end{aligned} \tag{5.90}$$

for n large enough. Similarly we get

$$\mathbb{P}[X \text{ hits at least two points from } T_\varepsilon^M] \le C(p_\varepsilon^M)^2 n^{2(\alpha+\beta-1)}. \tag{5.91}$$

This finishes the proof of Lemma 5.5.5. □

Event (iv). To find a lower bound for J, we should further verify that the probability that a part of the trajectory ends too close to some deep trap is small.

Lemma 5.5.7. *For $\mathbb{P}$-a.e. $\boldsymbol{\tau}$, the probability that the simple random walk started at arbitrary $x \in \mathbb{D}(n)$ exits $D_x(2^n n^\beta)$ at some point that is in $\mathbb{D}(n)\backslash\mathcal{E}(n)$ is smaller than $Cn^{2-\kappa/2-\beta/2}$.*

Proof. We start again with the description of the properties of the environment. Let r_n be the radius of the disk $D(2^n n^\beta)$. We use $A_x(2^n n^\beta)$ to denote the annular ring with the centre x, the inner radius $r_n - \sqrt{\pi^{-1}2^n n^{-\kappa}}$, and the outer radius $r_n + \sqrt{\pi^{-1}2^n n^{-\kappa}}$. Let $H_5 = H_5(n,\varepsilon,M)$ be the event

$$H_5 = \big\{\boldsymbol{\tau} : |T_\varepsilon^M(n) \cap A_x(2^n n^\beta)| \le n^2 \text{ for all } x \in \mathbb{D}(n)\big\}. \tag{5.92}$$

Lemma 5.5.8. *For n large there exist constants C and c such that*

$$\mathbb{P}[H_5] \ge 1 - C2^n n^{1-\alpha} \exp(-cn^2). \tag{5.93}$$

Proof. There is less than $C2^n n^{\beta/2-\kappa/2}$ points in the annulus $A_x(2^n n^\beta)$. The probability that a trap is in $T_\varepsilon^M(n)$ is of order $p_\varepsilon^M 2^n n^{-\alpha}$. The standard application of Markov inequality gives

$$\mathbb{P}\big[|A_x(2^n n^\nu) \cap T_\varepsilon^M(n)| > n^2\big] \le \exp(-c(\varepsilon,M)n^2). \tag{5.94}$$

The result follows by summation over all $x \in \mathbb{D}(n)$. □

We can now finish the proof of Lemma 5.5.7. We use the fact that probability of exiting the disk of radius R in a particular point at its border is $O(1/R)$ (see [Law91] Lemma 1.7.4). From Lemma 5.5.8 we know that there is less than n^2 deep traps in annulus $A_x(2^n n^\beta)$. This implies that there is at most $cn^2\sqrt{2^n n^{-\kappa}}$ points on the border of $D_x(2^n n^\beta)$ that are close to some deep trap. The required probability is thus bounded from above by

$$C\sqrt{2^{-n}n^{-\beta}}\,n^2\sqrt{2^n n^{-\kappa}} = Cn^{2-\kappa/2-\beta/2}. \tag{5.95}$$

This completes the proof. □

Event (v). The next lemma excludes the possibility of hitting a deep trap that is too close to the border of disk with area $2^n n^\beta$ around the starting point.

Lemma 5.5.9. *For any $x \in \mathbb{D}$, the probability that the random walk started at x hits a deep trap in $A_x(2^n n^\beta)$ before the exit from $D_x(2^n n^\beta)$ is smaller than $Cn^{2-\beta/2-\kappa/2}$.*

Proof. We need to estimate the probability that we hit some point y that is in the distance smaller than $\sqrt{\pi^{-1}2^n n^{-\kappa}}$ from the border of $D_x(2^n n^\beta)$. We use (5.235) to estimate this probability. The advantage of (5.235) against (5.234) is that the error terms are much smaller. Since for any disk D centred at x

$$G_D(x,y) = \mathbb{P}_x[X \text{ hits } y \text{ before exit from } D]G_D(y,y) \tag{5.96}$$

and $G_D(y,y) \geq 1$, we know that $\mathbb{P}_x(X \text{ hits } y) \leq G_D(x,y)$. According to Lemma 5.5.8 there is at most n^2 deep traps in $A_x(2^n n^\beta)$. We thus have

$$\begin{aligned}
&\mathbb{P}_x[X \text{ hits } T_\varepsilon^M \cap A_x(2^n n^\beta) \text{ before exiting } D_x(2^n n^\beta)] \\
&\leq \frac{2n^2}{\pi}\Big[\log\sqrt{\pi^{-1}2^n n^\beta} - \log\big(\sqrt{\pi^{-1}2^n n^\beta}(1-n^{-\beta/2-\kappa/2})\big) + O(2^{-n/2})\Big] \\
&\leq -cn^2\log(1-n^{-\beta/2-\kappa/2}) \leq Cn^{2-\beta/2-\kappa/2}.
\end{aligned} \tag{5.97}$$

This finishes the proof. ∎

Event (iii). Finally, we need to show that X almost never returns to a deep trap after exiting a disk of area $2^n n^{-\kappa}$ around it. We do not need to consider the traps that are closer than $\sqrt{\pi^{-1}2^n n^{-\kappa}}$ to the border of $\mathbb{D}$ because hitting such trap is due to conditions (ii) and (v) from Section 5.2 the bad event.

Lemma 5.5.10. *There exists a constant C such that for any x satisfying $D_x(2^n n^{-\kappa}) \cap \mathbb{D}(n)^c = \emptyset$, the probability that X returns to x before Λ after exiting disk $D_x(2^n n^{-\kappa})$ is smaller than $Cn^{-1}\log n$.*

Proof. Let p_{ret} denotes the required probability and let ξ be the first time when X exits $D_x(2^n n^{-\kappa})$. Obviously, $\xi < \Lambda$. By Markov property

$$\begin{aligned}
G_{\mathbb{D}}(x,x) &= \sum_{i=0}^{\Lambda}\mathbb{P}_x[X_d(i)=x] = \sum_{i=0}^{\xi}\mathbb{P}_x[X_d(i)=x] + \sum_{i=\xi+1}^{\Lambda}\mathbb{P}_x[X_d(i)=x] \\
&= G_{D(2^n n^{-\kappa})}(0,0) + p_{\text{ret}}G_{\mathbb{D}}(x,x).
\end{aligned} \tag{5.98}$$

Hence,

$$p_{\text{ret}} = 1 - \frac{G_{D(2^n n^{-\kappa})}(0,0)}{G_{\mathbb{D}}(x,x)} \leq 1 - \frac{G_{D(2^n n^{-\kappa})}(0,0)}{G_{2\mathbb{D}}(0,0)}, \tag{5.99}$$

where $2\mathbb{D}$ denotes two times larger disk than $\mathbb{D}$. Using the expression (5.236) we get

$$p_{\mathrm{ret}} \le 1 - \frac{\log(2^n n^{-\kappa}) + O(1)}{\log(2\cdot 2^n n^{1-\alpha}) + O(1)} \le Cn^{-1}\log n. \tag{5.100}$$

This finishes the proof. $\square$

Proof of Lemma 5.5.1. We have now all ingredients to prove Lemma 5.5.1. We should prove that the probability that some of the events (i)–(v) from Section 5.2 happen during first $Cn^{1-\alpha-\beta}$ parts can be made very small. We will use $J_{(i)},\dots,J_{(v)}$ to denote the first part where (i), …, resp. (v) occurs.

The simplest condition is (ii). This condition requires that X cannot exit $\mathbb{D}$ during the good part of the trajectory. That means that starting point of a part of the trajectory satisfying (ii) should be in the annular ring with the outer radius $\sqrt{\pi^{-1}m2^n n^{1-\alpha}}$ (which is the radius of $\mathbb{D}$) and the inner radius $\sqrt{\pi^{-1}m2^n n^{1-\alpha}} - \sqrt{\pi^{-1}2^n n^{\beta}}$. The sequence of starting points x_i^n is a random walk on $\mathbb{Z}^2$. Every step of this walk has length approximately $\sqrt{\pi^{-1}2^n n^{\beta}}$ and its direction is almost uniformly chosen. It follows from standard properties of random walks that the law of $J_{(ii)}n^{\alpha+\beta-1}m^{-1/2}$ converges as $n\to\infty$ to some distribution not depending on m. It is thus possible to fix m large enough such that

$$\mathbb{P}[J_{(ii)}n^{\alpha+\beta-1} \ge k|\boldsymbol{\tau}] \ge 1-\delta/4. \tag{5.101}$$

From the same reason we can choose $K > k$ such that

$$\mathbb{P}[J_{(ii)}n^{\alpha+\beta-1} \le K|\boldsymbol{\tau}] \ge 1-\delta/4. \tag{5.102}$$

Hence, outside a set of probability $\delta/2$ the number of parts before $J_{(ii)}$ is in interval $(kn^{1-\alpha-\beta}, Kn^{1-\alpha-\beta})$. We use A to denote this event.

Conditionally on A, we will show that

$$\mathbb{P}\big[\min(J_{(i)},J_{(iii)},J_{(iv)},J_{(v)}) \le J_{(ii)}\big|\boldsymbol{\tau},A\big] \to 0 \quad \text{as} \quad n\to\infty. \tag{5.103}$$

The claim of the lemma is then an easy consequence of this fact and the previous paragraph. Observe that (5.103) means that in the majority of cases the first bad event that happens is the possibility of exit from $\mathbb{D}$. The probability of all other events is negligible.

We start with condition (iv). According to it, the part is bad if its end is not in $\mathcal{E}(n)$. Lemma 5.5.7 states that the probability that this happens during a particular part of trajectory is of order $n^{2-\kappa/2-\beta/2}$. Since the number of parts before $J_{(ii)}$ is bounded by $Kn^{1-\alpha-\beta}$, the probability that (iv) happens is bounded by $Kn^{3-\alpha-\beta/2-\kappa/2}$. However, κ can be chosen large enough to assure that this bound converges to 0. We thus have

$$\mathbb{P}[J_{(iv)} < J_{(ii)}|\boldsymbol{\tau},A] \to 0. \tag{5.104}$$

Using a very similar reasoning and Lemma 5.5.9 we get exactly the same estimate for condition (v). Hence,

$$\mathbb{P}[J_{(v)} < J_{(ii)}|\boldsymbol{\tau}, A] \to 0. \tag{5.105}$$

Condition (i) requires that X does not visit two deep traps during one part of the trajectory. We use B to denote the event $A \cap \{J_{(iv)} > J_{(ii)}\}$. We show

$$\mathbb{P}[J_{(i)} < J_{(ii)}|B, \boldsymbol{\tau}] \to 0. \tag{5.106}$$

Since we assume that $J_{(iv)} \geq J_{(ii)}$, we can apply Lemma 5.5.5. It claims that probability of hitting two deep traps during one part is of order $n^{2(\alpha+\beta-1)}$. By the same argument as before we can bound the probability in (5.106) by $Kn^{\alpha+\beta-1}$ and it tends to 0 as $n \to \infty$.

The last condition (iii) demands that X does not return to a deep trap after exiting the disk of area $2^n n^{-\kappa}$ around it. For one particular trap probability of such event can be bounded by $cn^{-1}\log n$ by Lemma 5.5.10. According to Lemma 5.5.2, the probability of visiting a deep trap during one part of the trajectory is of order $n^{\alpha+\beta-1}$. Let N denotes the number of visited deep traps before Λ. Conditionally on B, it is not difficult to show using Markov inequality that

$$\mathbb{P}[N \geq n^{1/2}|B, \boldsymbol{\tau}] \leq Cn^{-1/2}. \tag{5.107}$$

We have thus

$$\begin{aligned}
&\mathbb{P}[J_{(iii)} < J_{(ii)}|B, \boldsymbol{\tau}] \\
&\quad \leq \mathbb{P}[J_{(iii)} < J_{(ii)}|B, \boldsymbol{\tau}, N \leq n^{1/2}]\mathbb{P}[N \leq n^{1/2}|B, \boldsymbol{\tau}] + \mathbb{P}[N \geq n^{1/2}|B, \boldsymbol{\tau}] \\
&\quad \leq cn^{-1/2}\log n + Cn^{-1/2} \to 0 \qquad \text{as } n \to \infty.
\end{aligned} \tag{5.108}$$

The claim (5.103) that follows easily from (5.104)–(5.108). This finishes the proof of Lemma 5.5.1. ☐

5.6 Properties of the score

In this section we will prove the convergence of the sequence of processes Y^n to a Lévy process. Recall that Y^n was defined as a well rescaled sum of scores. Hence, we should first study the properties of the score.

The score of the i-th part of the trajectory depends on the history only through its starting point x_i^n. We thus associate to every point $x \in \mathcal{E}(n)$ the random variable s_x, which has the same distribution as the part of the trajectory of X that is started at x. We can ignore the points in $\mathbb{D}(n) \setminus \mathcal{E}(n)$

because we do not consider the parts of trajectory started in this set (see definition of J). We have got already some information which can help us to describe the distribution of the random variable s_x. According to Lemma 5.5.5, the probability of hitting two deep traps in the disk $D_x(2^n n^\beta)$ is of order $n^{2(\alpha+\beta-1)}$, and the probability of hitting one deep trap is with high precision $\mathcal{K}p_\varepsilon^M n^{\alpha+\beta-1}$. Otherwise X does not hit any deep trap. In the last case $s_x = 0$ (if none from (i)–(v) of Section 5.2 happen).

We want now to study more precisely the distribution of s_x conditionally on $s_x < \infty$. To achieve it we should gain more information about the depth of the trap that X hits as the first. The idea behind the proof is that as n increases the density of deep traps becomes lower, and the hitting measure of T_ε^M charges more and more sites. The distribution of the depth of the first visited trap should be thus close to the original distribution of the depth of the traps conditioned on being between $\varepsilon 2^{n/\alpha}/n$ and $M2^{n/\alpha}/n$.

To prove this heuristics we divide the set of deep traps to several parts and we estimate the probability of hitting each of them. Let $h(x)$ be a function satisfying

$$h(x) \geq (\log x)^{-1}, \qquad \lim_{x\to\infty} h(x) = 0, \tag{5.109}$$

and

$$L(2^{n/\alpha}n^{-1}x) - 1 = o(h(n)) \quad \text{for all } x \geq \varepsilon. \tag{5.110}$$

Such function exists because $\lim_{x\to\infty} L(x) = 1$. Let $z_n(i)$ satisfy $\varepsilon = z_n(0) < z_n(1) < \cdots < z_n(R) = M$ and $z_n(i+1) - z_n(i) \in (h(n), 2h(n))$ for all $i \in \{0, \dots R-1\}$.

We now estimate the probability of hitting a trap in $T_{z_n(i)}^{z_n(i+1)}$. We use p_i^n to denote

$$p_i^n = z_n(i)^{-\alpha} - z_n(i+1)^{-\alpha}. \tag{5.111}$$

Lemma 5.6.1. *For any $\delta > 0$ there exists n_0 such that for all $n > n_0$, for all $x \in \mathcal{E}(n)$, and for all $i = \{0, \dots, R-1\}$ the probability that the simple random walk started at x hits a trap in $T_{z_n(i)}^{z_n(i+1)}$ before the exit from $D_x(2^n n^\beta)$ is in the interval*

$$\big[\mathcal{K}(1-\delta)n^{\alpha+\beta-1}p_i^n, \mathcal{K}(1+\delta)n^{\alpha+\beta-1}p_i^n\big]. \tag{5.112}$$

Proof. The proof is very similar to the proof of Lemma 5.5.2. We should first improve the bounds on the homogeneity of the environment that we have proved in Lemma 5.5.3.

Let $H_6 = H_6(n, \delta, \varepsilon, M)$ be the event that for every square $B_x(2^n n^\nu)$ in $\mathbb{D}(n)$ and for every $i \in \{0, \dots R-1\}$ the number of sites in $T_{z_n(i)}^{z_n(i+1)} \cap B_x(2^n n^\nu)$ is in the interval

$$\big[(1-\delta)n^{\alpha+\nu}p_i^n, (1+\delta)n^{\alpha+\nu}p_i^n\big]. \tag{5.113}$$

We prove that H_6 occurs $\mathbb{P}$-a.s. for n large enough.

Lemma 5.6.2. *For any δ there exist constants c and C such that for n large enough*

$$\mathbb{P}[H_6] \geq 1 - C\log(n) n^{1-\alpha-\nu}\delta^{-2}\exp\big(-cn^{\nu+\alpha}h(n)\big). \tag{5.114}$$

Using this lemma it is not difficult to finish the proof of 5.6.1. We will not give the detailed reasoning, because the proof follows the same line as the proof of Lemma 5.5.2. The only change is that the previous lemma should be used instead of Lemma 5.5.3. □

Proof of Lemma 5.6.2. To show that H_6 occurs $\mathbb{P}$-a.s. for n large enough we will need one technical lemma that estimates the probability that a trap is in $T_{z_n(i)}^{z_n(i+1)}$.

Lemma 5.6.3. *For any $\eta > 0$ there exist n_0 such that for all $n \geq n_0$ and all $i = 0, \dots, R-1$*

$$\mathbb{P}\big[0 \in T_{z_n(i)}^{z_n(i+1)}\big] \in \Big((1-\eta)\frac{n^\alpha}{2^n}p_i^n, (1+\eta)\frac{n^\alpha}{2^n}p_i^n\Big). \tag{5.115}$$

Proof. Let $g(x) = L(x) - 1$. Then by (5.13) we have

$$\begin{aligned}\mathbb{P}\big[0 \in T_{z_n(i)}^{z_n(i+1)}\big] &= \mathbb{P}\Big[\tau_0 \in \Big[z_n(i)\frac{2^{n/\alpha}}{n}, z_n(i+1)\frac{2^{n/\alpha}}{n}\Big)\Big]\\ &= \frac{n^\alpha}{2^n}\Big[p_i^n + \frac{g(2^{n/\alpha}n^{-1}z_n(i))}{z_n(i)^\alpha} - \frac{g(2^{n/\alpha}n^{-1}z_n(i+1))}{z_n(i+1)^\alpha}\Big]. \end{aligned}\tag{5.116}$$

We should thus show that

$$\frac{g(2^{n/\alpha}n^{-1}z_n(i))}{z_n(i)^\alpha} - \frac{g(2^{n/\alpha}n^{-1}z_n(i+1))}{z_n(i+1)^\alpha} = o(p_i^n). \tag{5.117}$$

However, this is obviously true since

$$p_i^n = (z_n(i))^{-\alpha} - (z_n(i+1))^{-\alpha} \geq ch(n) \tag{5.118}$$

for some c not depending on i and n, and $g(2^{n/\alpha}n^{-1}z_j^n) = o(h(n))$ by (5.110). □

The remaining part of the proof of Lemma 5.6.2 is analogous to the proof of Lemma 5.5.3. We only explain the appearance of the additional factors $\log(n)$ and $h(n)$ that are in (5.114) but not in (5.61). The logarithm before the exponential is due to the summation over all possible i and (5.109). The factor $h(n)$ inside the exponent comes from Lemma 5.6.3 which replaces the bound on p before (5.64) and the existence of constants c, C such that

$$ch(n) \leq \frac{1}{z_n(i)^\alpha} - \frac{1}{z_n(i+1)^\alpha} \leq Ch(n). \tag{5.119}$$

This finishes the proof. □

Using Lemma 5.6.1 we can now describe the behaviour of variables s_x. We are interested only in the distribution of s_x conditionally on $s_x < \infty$, because if $s_x = \infty$, we do not include it into the definition of Y^n and we use at its place an artificial random variable $\tilde{s}_i^n$. Due to condition (ii) from Section 5.2, all good parts of the trajectory starts at sites that are in the distance larger than $\sqrt{\pi^{-1}2^n n^\beta}$ from the border of $\mathbb{D}(n)$. That is why we introduce $\mathcal{E}_0(n) = \{x \in \mathcal{E}(n) : D_x(2^n n^\beta) \cap \mathbb{D}(n)^c = \emptyset\}$. The random variables s_x then satisfy

Lemma 5.6.4. *For $\mathbb{P}$-a.e. random environment $\boldsymbol{\tau}$*

$$\begin{aligned}
\lim_{n\to\infty} \max_{x\in\mathcal{E}_0(n)} \frac{1-\mathbb{E}[\exp(-\frac{\lambda s_x}{2^{n/\alpha}})|s_x<\infty,\boldsymbol{\tau}]}{n^{\alpha+\beta-1}} &= F(\lambda),\\
\lim_{n\to\infty} \min_{x\in\mathcal{E}_0(n)} \frac{1-\mathbb{E}[\exp(-\frac{\lambda s_x}{2^{n/\alpha}})|s_x<\infty,\boldsymbol{\tau}]}{n^{\alpha+\beta-1}} &= F(\lambda),
\end{aligned} \tag{5.120}$$

with

$$F(\lambda) = F(\lambda;\varepsilon,M,\alpha) = \mathcal{K}\Big(p_\varepsilon^M - \int_\varepsilon^M \frac{\alpha}{1+\mathcal{K}'\lambda z}\cdot\frac{1}{z^{\alpha+1}}dz\Big) \tag{5.121}$$

and $\mathcal{K}' = \pi^{-1}\log 2$.

Proof. If the process X hits deep trap y in $D_x(2^n n^\beta)$ and nothing unusual happens, then the random variable s_x is a sum of a geometrically distributed number of exponential random variables with mean τ_y. The mean of the geometrical distribution is easy to calculate since it is equal to $G_{D(2^n n^{-\kappa})}(0,0)$. By (5.236) it equals

$$G_{D(2^n n^{-\kappa})}(0,0) = \frac{2}{\pi}\log\sqrt{\pi^{-1}2^n n^{-\kappa}} + O(1) = \frac{n}{\pi}\log 2 + O(\log n). \tag{5.122}$$

The geometrically long sum of exponential variables is again exponentially distributed. Hence, the score s_x is in this case an exponential random variable with mean $\tau_y(n\log 2/\pi + O(\log n))$. This implies that conditionally on hitting a trap with the depth τ_y the Laplace transform of $s_x/2^{n/\alpha}$ equals

$$\mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}}\Big)\Big|\tau_y\Big] = \frac{1}{1+\lambda\tau_y 2^{-n/\alpha}(n\log 2/\pi + O(\log n))}. \tag{5.123}$$

By Lemmas 5.5.2, 5.5.7, and 5.5.9 we know that if κ is large enough, $\mathbb{P}[s_x = \infty] = O(n^{2(\alpha+\beta-1)})$. Since this probability is much smaller than any other probability that will be used in the following computation, the conditioning on $s_x < \infty$ has almost no effect. Actually,

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}}\Big)\Big|s_x<\infty,\boldsymbol{\tau}\Big]&\\ &=\mathbb{P}[s_x<\infty|\boldsymbol{\tau}]^{-1}\mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}}\Big)\chi\{s_x<\infty\}\Big|\boldsymbol{\tau}\Big]\\ &=\mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}}\Big)\Big|\boldsymbol{\tau}\Big]\big(1+O(n^{2(\alpha+\beta-1)})\big).\end{aligned}\tag{5.124}$$

We now use Lemmas 5.5.5, 5.6.1, and expression (5.123) to estimate the Laplace transform. We start with a lower bound. Choose $\delta > 0$. Then for n large enough

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}}\Big)\Big|\boldsymbol{\tau}\Big]&\geq\Big(1-(1+\delta)\mathcal{K}p_\varepsilon^M n^{\alpha+\beta-1}\Big)\\ &+\mathcal{K}n^{\alpha+\beta-1}\sum_{i=1}^{R}\frac{1-\delta}{1+\lambda\frac{z_n(i)}{2^{n/\alpha}}\frac{2^{n/\alpha}}{n}\frac{n}{\pi}\log 2+o(1)}\Big(\frac{1}{(z_{i-1}^n)^\alpha}-\frac{1}{(z_n(i))^\alpha}\Big).\end{aligned}\tag{5.125}$$

The last expression can be bounded from bellow by

$$1-\mathcal{K}n^{\alpha+\beta-1}\bigg(p_\varepsilon^M-\int_\varepsilon^M\frac{\alpha}{1+\mathcal{K}'\lambda z}\frac{1}{z^{\alpha+1}}dz\bigg)-\delta C n^{\alpha+\beta-1}p_\varepsilon^M,\tag{5.126}$$

with C being a constant not depending on δ. The last expression together with (5.124) give

$$\begin{aligned}\limsup_{n\to\infty}\max_{x\in\mathcal{E}_0(n)}&\frac{1-\mathbb{E}[\exp(-\frac{\lambda s_x}{2^{n/\alpha}})|s_x<\infty,\boldsymbol{\tau}]}{n^{\alpha+\beta-1}}\\ &\leq\mathcal{K}\bigg(p_\varepsilon^M-\int_\varepsilon^M\frac{\alpha}{1+\mathcal{K}'\lambda z}\frac{1}{z^{\alpha+1}}dz\bigg)+C\delta p_\varepsilon^M.\end{aligned}\tag{5.127}$$

Since δ can be taken arbitrarily small, we have finished the proof of the upper bound for the first expression in (5.120). The proof of the lower bound for the second expression in (5.120) is completely similar. □

We can finally show the convergence of the sequence Y^n to a Lévy process. The following proposition will be used later to prove aging.

Proposition 5.6.5. *For $\mathbb{P}$-a.e. realisation of the environment, the sequence of processes $Y^n(t)$ converges weakly in the Skorokhod topology on $D([0,\infty))$ to the Lévy process $Y(t)$ with Lévy measure*

$$\rho(dx)=\frac{\alpha\mathcal{K}}{\mathcal{K}'}\int_\varepsilon^M\frac{1}{z^{\alpha+2}}\exp\Big(-\frac{x}{\mathcal{K}'z}\Big)\,dz\,dx.\tag{5.128}$$

Proof. We first prove the weak convergence of finite dimensional distributions. Let $0 = t_0 < t_1 < \cdots < t_\ell$. We will show the convergence of Laplace transforms. By definition of Y^n

$$\mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{\ell}\lambda_i\big(Y^n(t_i)-Y^n(t_{i-1})\big)\Big)\Big] = \mathbb{E}\Big[\prod_{i=1}^{\ell}\prod_{j\in B(n,i)}\exp\Big(-\frac{\lambda_i}{2^{n/\alpha}}s_j^n\Big)\Big], \tag{5.129}$$

where $B(n,i) = \{\lfloor n^{1-\alpha-\beta}t_{i-1}\rfloor + 1, \dots, \lfloor n^{1-\alpha-\beta}t_i\rfloor\}$.

If $j < J$, then the random variables s_j^n have the same distribution as s_x, otherwise they are equal to $\tilde{s}_j^n$. Since the variables $\tilde{s}_j^n$ are independent of everything we can write

$$= \mathbb{E}\Big[\prod_{i=1}^{\ell}\prod_{\substack{j\in B(n,i)\\ j<J}}\exp\Big(-\frac{\lambda_i}{2^{n/\alpha}}s_j^n\Big)\Big]\mathbb{E}\Big[\prod_{i=1}^{\ell}\prod_{\substack{j\in B(n,i)\\ j\geq J}}\exp\Big(-\frac{\lambda_i}{2^{n/\alpha}}\tilde{s}_j^n\Big)\Big]. \tag{5.130}$$

At this place it is necessary to define the distribution of $\tilde{s}_j^n$. We require that $\tilde{s}_i^n$ satisfies the same relation as s_x in the limit, that is

$$\mathbb{E}\Big[\exp\Big(-\frac{\lambda}{2^{n/\alpha}}\tilde{s}_j^n\Big)\Big] = 1 - F(\lambda)n^{\alpha+\beta-1}. \tag{5.131}$$

We have obviously chosen the $\tilde{s}_j^n$'s in the way that the second part of (5.130) does not pose any problems. We should thus control only the first part.

Let $\boldsymbol{y} = \{y_0, \dots, y_J\} \in \mathcal{E}(n)^J$. We use $\boldsymbol{x}_n$ to denote the sequence $x_0^n, \dots, x_J^n$ of starting points of the parts of the trajectory. We have

$$\begin{aligned}\mathbb{E}\Big[\prod_{i=1}^{\ell}\prod_{\substack{j\in B(n,i)\\ j<J}}&\exp\Big(-\frac{\lambda_i}{2^{n/\alpha}}s_j^n\Big)\Big]\\ &= \sum_{\boldsymbol{y}}\mathbb{P}[\boldsymbol{x}_n = \boldsymbol{y}]\mathbb{E}\Big[\prod_{i=1}^{\ell}\prod_{\substack{j\in B(n,i)\\ j<J}}\exp\Big(-\frac{\lambda_i}{2^{n/\alpha}}s_j^n\Big)\Big|\boldsymbol{x}_n = \boldsymbol{y}\Big].\end{aligned} \tag{5.132}$$

Only the last term of the product depends on y_J. We can thus sum over all possible values of the endpoint of the last part. Let $\boldsymbol{x}'_n$, resp. $\boldsymbol{y}'$, denote the sequences $\boldsymbol{x}_n$ and $\boldsymbol{y}$ without the last element. We get

$$= \sum_{\boldsymbol{y}'}\mathbb{P}[\boldsymbol{x}'_n = \boldsymbol{y}']\mathbb{E}\Big[\prod_{i=1}^{\ell}\prod_{\substack{j\in B(n,i)\\ j<J}}\exp\Big(-\frac{\lambda_i}{2^{n/\alpha}}s_j^n\Big)\Big|\boldsymbol{x}'_n = \boldsymbol{y}'\Big]. \tag{5.133}$$

Conditionally on the value x^n_{J-1}, the random variable s^n_{J-1} is independent of the rest. The expectation in the last formula can be thus written as

$$\mathbb{E}\Big[\prod_{i=1}^{\ell}\prod_{\substack{j\in B(n,i)\\ j<J-1}}\exp\Big(-\frac{\lambda_i}{2^{n/\alpha}}s^n_j\Big)\Big|\boldsymbol{x}'_n=\boldsymbol{y}'\Big]\mathbb{E}\Big[\exp\Big(-\frac{\lambda_k}{2^{n/\alpha}}s_{x^n_{J-1}}\Big)\Big|s_{x^n_{J-1}}<\infty\Big], \tag{5.134}$$

where the index k satisfies $J-1\in B(n,k)$. According to Lemma 5.6.4, the second expectation can be bounded from above by

$$1-(1-\delta)F(\lambda_k)n^{\alpha+\beta-1} \tag{5.135}$$

if n is large enough.

We can now repeat the same manipulation with the last but one value of j, etc. At the end we get

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{\ell}\lambda_i\big(Y^n(t_i)-Y^n(t_{i-1})\big)\Big)\Big]&\\ \le\prod_{i=1}^{\ell}\big(1-(1-\delta)F(\lambda_i)n^{\alpha+\beta-1}\big)^{\lfloor n^{1-\alpha-\beta}(t_i-t_{i-1})\rfloor}&.\end{aligned} \tag{5.136}$$

Taking the limits we obtain

$$\begin{aligned}\limsup_{n\to\infty}\mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{\ell}\lambda_i\big(Y^n(t_i)-Y^n(t_{i-1})\big)\Big)\Big]&\\ \le\exp\Big[-\sum_{i=1}^{\ell}(1-\delta)F(\lambda_i)(t_i-t_{i-1})\Big]&.\end{aligned} \tag{5.137}$$

In the same way we obtain upper bound. Since δ was arbitrary we have

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{\ell}\lambda_i\big(Y^n(t_i)-Y^n(t_{i-1})\big)\Big)\Big]&\\ =\exp\Big[-\sum_{i=1}^{\ell}F(\lambda_i)(t_i-t_{i-1})\Big]&.\end{aligned} \tag{5.138}$$

The corresponding Laplace transform of $Y(t)$ is easy to calculate. We have

$$\mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{\ell}\lambda_i\big(Y(t_i)-Y(t_{i-1})\big)\Big)\Big]=\exp\Big[-\sum_{i=1}^{\ell}\Psi(\lambda_i)(t_i-t_{i-1})\Big], \tag{5.139}$$

where $\Psi(\lambda)$ is the Laplace exponent of Y. By Lévy-Khintchine formula it is equal to

$$\Psi(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\rho(dx). \tag{5.140}$$

An easy integration gives the same result as (5.138).

To prove the weak convergence it remains to verify that the sequence Y_n is tight. We use Theorem 16.8 from [Bil99]. We should show that for any N and δ_1, δ_2 there exist a, n_0, and η such that

$$\text{(i)} \qquad \mathbb{P}[\sup_{t\in[0,N]} |Y_n(t)| \geq a] < \delta_1 \quad \text{for all } n > n_0$$

$$\text{(ii)} \qquad \mathbb{P}[w(Y^n, \eta, N) \geq \delta_2] < \delta_1 \quad \text{for all } n > n_0,$$

where

$$w(f, \eta, N) = \inf_{\{t_i\}} \max_{0<i\leq r} \sup\{|f(s) - f(t)| : s, t \in [t_{i-1}, t_i)\} \tag{5.141}$$

and the infimum runs over all finite collections $\{t_i\}$ such that $0 < t_i - t_{i-1} < \eta$, $t_0 = 0$, and $t_r = N$.

Proof of (i) Since Y^n are increasing, (i) is equivalent to the tightness of the sequence $Y^n(N)$. From convergence of finite dimensional distribution we know that the Laplace transforms of $Y^n(N)$ converge to $\mathcal{L}_{Y(N)}(\lambda) = \mathbb{E}[\exp(-\lambda Y(N))]$. It is sufficient to verify that this Laplace transform satisfies $\lim_{\lambda\to 0} \mathcal{L}_{Y(N)}(\lambda) = 1$. However, $\mathcal{L}_{Y(N)}$ is continuous and

$$\mathcal{L}_{Y(N)}(0) = \exp(-NF(0)) = \exp\Big[-N\mathcal{K}\Big(p_\varepsilon^M - \int_\varepsilon^M \frac{\alpha}{z^{\alpha+1}}dz\Big)\Big] = 1. \tag{5.142}$$

Proof of (ii) According to Lemma 5.5.5, the expected number of jumps of Y^n in the interval $[0, N]$ can be bounded by some constant C not depending on n. Markov inequality then gives the existence of some C' such that the probability that the number of jumps of Y^n exceeds C' is smaller than $\delta_1/2$ for all n large enough. If the number of jumps is finite, we can take $\{t_i\}$ being the superset of the set of all jumps. The process Y^n is then constant on any interval $[t_{i-1}, t_i)$ and thus $w(Y^n, \eta, N) = 0$. This completes the proof of Proposition 5.6.5. □

5.7 Proof of aging

We prove here the following proposition that is the more precise version of Theorem 5.1.1.

Proposition 5.7.1. *For $\mathbb{P}$-a.e. realisation of the environment $\boldsymbol{\tau}$ and for every $0<\theta<\infty$*

$$\lim_{t\to\infty} R(t,t+\theta t)=\int_0^{1/1+\theta}\frac{\sin\alpha\pi}{\pi}u^{\alpha-1}(1-u)^{-\alpha}\,du\equiv R(\theta). \tag{5.143}$$

An easy calculation gives

Corollary 5.7.2. *The function $R(\theta)$ satisfies*

$$\lim_{\theta\to 0} R(\theta)=1 \qquad \text{and} \qquad \lim_{\theta\to\infty} R(\theta)=0. \tag{5.144}$$

Proof. **I.** We introduce some additional notation. Let $Z(t)=Z(t;\varepsilon,M)$ be the Lévy process with the Lévy measure

$$\rho'(dx)=\frac{\alpha\mathcal{K}}{\mathcal{K}'}\Big(\int_0^{\varepsilon}+\int_M^{\infty}\Big)\frac{1}{z^{\alpha+2}}\exp\Big(-\frac{x}{\mathcal{K}'z}\Big)\,dz\,dx. \tag{5.145}$$

We define the new family of processes,

$$\tilde{Y}^n(t)=Y^n(t)+Z(t) \qquad \text{and} \qquad \tilde{Y}(t)=Y(t)+Z(t). \tag{5.146}$$

The advantage of this new class is that the Lévy measure of $\tilde{Y}$ satisfies

$$\begin{aligned}\rho(dx)+\rho'(dx)&=\frac{\alpha\mathcal{K}}{\mathcal{K}'}\int_0^{\infty}\frac{1}{z^{\alpha+2}}\exp\Big(-\frac{x}{\mathcal{K}'z}\Big)\,dz\,dx\\&=\frac{\alpha^2\Gamma(\alpha)\mathcal{K}(\mathcal{K}')^{\alpha}}{x^{\alpha+1}}\,dx,\end{aligned} \tag{5.147}$$

and thus $\tilde{Y}$ is an α-stable subordinator. As an easy consequence of the previous section, the sequence $\tilde{Y}^n$ converges weakly to $\tilde{Y}$. Let $\mathcal{R}_n=\mathcal{R}(\tilde{Y}^n)$, $\mathcal{R}=\mathcal{R}(\tilde{Y})$ denote the range of $\tilde{Y}^n$, resp. of $\tilde{Y}$.

Fix $\theta>0$. Let δ_1, $\delta_2>0$ small. We will now fix the values of M, m, ε and n as functions of t and δ_1, δ_2. First, let $n(t)$ be an integer satisfying

$$1\le\frac{t}{2^{n(t)/\alpha}}<2^{1/\alpha}. \tag{5.148}$$

Obviously, $n(t)\to\infty$ as $t\to\infty$. In this section $n=n(t)$ is always connected with t using (5.148). We use $s=s(t)$ to denote the rescaled value of t, $s=t2^{-n(t)/\alpha}$. By (5.148) s satisfies $1\le s<2^{1/\alpha}$. In the same way we rescale the value $(1+\theta)t$. The process $\tilde{Y}^n$ that we will use to approximate the time change $\bar{S}^n$ should be thus relevant until the level $(1+\theta)s<(1+\theta)2^{1/\alpha}$. Let t_0 be such that

$$\mathbb{P}[\tilde{Y}(t_0)<(1+\theta)2^{1/\alpha}]<\delta_1. \tag{5.149}$$

By the weak convergence of $\tilde{Y}^n$ to $\tilde{Y}$ we can take t (and so n) large enough such that

$$\mathbb{P}[\tilde{Y}^{n(t)}(t_0) \geq (1+\theta)2^{1/\alpha}] > 1 - 2\delta_1. \tag{5.150}$$

There are $J(n)$ relevant parts of the trajectory of the process X. For every time unit we need $n^{1-\alpha-\beta}$ parts. So, we should choose m in the way that

$$\mathbb{P}[J(n)n^{\alpha+\beta-1} \geq t_0] > 1 - \delta_1. \tag{5.151}$$

By Lemma 5.5.1, this can be done independently of ε and M. Let A_1 be the event $\big\{\tilde{Y}^n(t_0) \geq (1+\theta)s$ and $J(n) \geq t_0 n^{1-\alpha-\beta}\big\}$. Then, by (5.150) and (5.151),

$$\mathbb{P}[A_1] \geq 1 - 3\delta_1. \tag{5.152}$$

We can now fix the values of ε and M. Later, we want to work with the processes $\tilde{Y}^n$ instead of Y^n. We should thus guarantee that the artificial addition of process Z is not relevant. We take ε_1 and M_1, such that

$$\mathbb{P}[Z(t_0; \varepsilon_1, M_1) > \delta_2] < \delta_1. \tag{5.153}$$

We want also safely ignore the error introduced by the very deep and the shallow traps. By Lemma 5.4.1, we can take M_2 such that

$$\mathbb{P}[X(t) \text{ hits } T_{M_2} \text{ before } \Lambda_n] < \delta_1. \tag{5.154}$$

Further, by Lemma 5.3.1, we know that there is a constant K_1, such that

$$\mathbb{E}\Big[\frac{1}{2^{n/\alpha}} \cdot \text{time spent in } T^\varepsilon \,\Big|\boldsymbol{\tau}\Big] \leq K_1 \varepsilon^{1-\alpha}, \tag{5.155}$$

and thus

$$\mathbb{P}\Big[\frac{1}{2^{n/\alpha}} \cdot \text{time spent in } T^\varepsilon \; > \delta_2 \Big|\boldsymbol{\tau}\Big] \leq \delta_2^{-1} \varepsilon^{1-\alpha} K_1. \tag{5.156}$$

Let us take ε_2 such that $\delta_2^{-1}\varepsilon_2^{1-\alpha}K_1 < \delta_1$. The constants ε and M are then defined by

$$\varepsilon = \min(\varepsilon_1, \varepsilon_2) \qquad \text{and} \qquad M = \max(M_1, M_2). \tag{5.157}$$

This choice of constants ensures that the distance between the rescaled time change $\bar{S}^n$ and the process $\tilde{Y}^n$ is small. Precisely, let

$$A_2 = \big\{|\bar{S}^n(t) - \tilde{Y}^n(t)| \leq 2\delta_2 \,\forall t \leq t_0\big\}. \tag{5.158}$$

Then our choice of constants gives

$$\mathbb{P}\big[A_2 | A_1\big] \geq 1 - 3\delta_1. \tag{5.159}$$

Let $A = A_1 \cap A_2$. Then from (5.152) and (5.159) follows that for t large enough

$$\mathbb{P}[A] \geq 1 - 6\delta_1. \tag{5.160}$$

II. Later we will take the limit $n \to \infty$ for fixed value of $s \in [1, 2^{1/\alpha}]$ instead of taking the limit $t \to \infty$. We will show that the limit $n \to \infty$ does not depend on s. To be able to show the existence of the limit $t \to \infty$ we will need the uniformity of convergence in s. To establish it we will use the following auxiliary lemma.

Lemma 5.7.3. *Let $\mathcal{P}_u(s, Y) = \mathbb{P}\big[[s, s+u] \cap \mathcal{R}(Y) \neq \emptyset\big]$ for Y being $\tilde{Y}^n$ or $\tilde{Y}$. Then for any $u < \theta 2^{1/\alpha}$*

$$\lim_{n \to \infty} \mathcal{P}_u(s, \tilde{Y}^n) = \mathcal{P}_u(s, \tilde{Y}) \tag{5.161}$$

uniformly for $s \in [1, \theta 2^{1/\alpha}]$.

Proof. Since

$$\mathcal{P}_u(s, Y) = \mathbb{E}\big[\chi\big\{[s, s+u] \cap \mathcal{R}(Y) \neq \emptyset\big\}\big], \tag{5.162}$$

we will first show that the functional $Y \to \chi\big\{[s, s+u] \cap \mathcal{R}(Y) \neq \emptyset\big\}$ is continuous in Skorokhod topology on $D([0,\infty))$ at almost all sample points of $\tilde{Y}$. The pointwise convergence in (5.161) then follows from weak convergence of $\tilde{Y}^n$ (using e.g. Corolary 2, page 447 of [GS69]). The above functional is discontinuous at $Y \in D([0,\infty))$ only if s or $s+u$ are boundary points of $\mathcal{R}(Y)$. However, $\mathbb{P}[\{s, s+u\} \cap \partial\mathcal{R}(\tilde{Y}) \neq \emptyset] = 0$ since $\tilde{Y}$ is a stable subordinator.

To show the uniform convergence, we will first verify sort of equicontinuity property of $\mathcal{P}_u(s, \tilde{Y}^n)$. Choose $\eta > 0$ small. Let $s_1, s_2 \in [1, \theta 2^{1/\alpha}]$ such that $0 < s_2 - s_1 < \eta$. Then,

$$\big|\mathcal{P}_u(s_1, \tilde{Y}^n) - \mathcal{P}_u(s_2, \tilde{Y}^n)\big| \leq \mathcal{P}_\eta(s_1, \tilde{Y}^n) + \mathcal{P}_\eta(s_2 + u, \tilde{Y}^n). \tag{5.163}$$

Take now $1 = t_0 < t_1 < \cdots < t_R = \theta 2^{1+1/\alpha}$ such that $t_{i+1} - t_i \in [\eta/2, \eta]$. By pointwise convergence it is possible to take n_0, such that for $n \geq n_0$ and for all $i \in \{0, \dots R\}$, $\big|\mathcal{P}_{2\eta}(t_i, \tilde{Y}^n) - \mathcal{P}_{2\eta}(t_i, \tilde{Y})\big| \leq \eta$. Since $\tilde{Y}$ is a stable subordinator, the probability $\mathcal{P}_\eta(t, \tilde{Y})$ can be bounded from above, uniformly for all $t \in [1, \theta 2^{1+1/\alpha}]$ by some constant $h(\eta)$ satisfying $h(\eta) \to 0$ as $\eta \to 0$. We have thus for n large enough $\mathcal{P}_{2\eta}(t_i, \tilde{Y}^n) \leq \eta + h(\eta)$. Since any interval $[s, s+\eta]$ is contained in some of $[t_i, t_i + 2\eta]$, we can bound (5.163) by $2\eta + 2h(\eta)$ for $n \geq n_0$.

The uniform convergence then follows by the following reasoning. Take $\eta > 0$ and t_i as in the previous paragraph. From pointwise convergence we know that there exist n_1, such that for $n \geq n_1$ $\left|\mathcal{P}_u(t_i, \tilde{Y}^n) - \mathcal{P}_u(t_i, \tilde{Y})\right| \leq \eta$. Then for any $s \in [t_i, t_{i+1}]$,

$$\begin{aligned}|\mathcal{P}_u(s, \tilde{Y}^n) - \mathcal{P}_u(s, \tilde{Y})| \leq &|\mathcal{P}_u(s, \tilde{Y}^n) - \mathcal{P}_u(t_i, \tilde{Y}^n)| \\ &+ |\mathcal{P}_u(t_i, \tilde{Y}^n) - \mathcal{P}_u(t_i, \tilde{Y})| + |\mathcal{P}_u(t_i, \tilde{Y}) - \mathcal{P}_u(s, \tilde{Y})|. \quad (5.164)\end{aligned}$$

For $n \geq n_0 \vee n_1$ we can bound the first term by $2\eta + 2h(\eta)$ as follows from the previous paragraph. The second term is smaller than η, and the third is smaller than $2h(\eta)$. The uniform convergence follows. ☐

III. We now study the event $G(t) = \{X(t) = X((1+\theta)t)\}$ for t large. To simplify the reasoning we suppose for the moment that the event A occurs. We divide the probability space into three disjoint parts,

$$\begin{aligned}E_1(n,s) &= \{\operatorname{dist}(s, \mathcal{R}_n) \leq 2\delta_2 \text{ or } \operatorname{dist}((1+\theta)s, \mathcal{R}_n) \leq 2\delta_2\}, \\ E_2(n,s) &= \{\operatorname{dist}(s, \mathcal{R}_n) > 2\delta_2, \operatorname{dist}((1+\theta)s, \mathcal{R}_n) > 2\delta_2 \text{ and} \\ &\qquad (s, (1+\theta)s) \cap \mathcal{R}_n \neq \emptyset\}, \qquad (5.165)\\ E_3(n,s) &= \{\operatorname{dist}(s, \mathcal{R}_n) > 2\delta_2, \operatorname{dist}\big((1+\theta)s, \mathcal{R}_n\big) > 2\delta_2 \text{ and} \\ &\qquad (s, (1+\theta)s) \cap \mathcal{R}_n = \emptyset\}.\end{aligned}$$

This division has the following reason. Any of the intervals that do not intersect $\mathcal{R}_n$ corresponds to a time period that X spent in $D_y(2^n n^{-\kappa})$ around some deep trap y. The points of the range correspond to times when the walk did not meet any deep trap for a long time. However, both these claims should be taken only with precision $2\delta_2$ because of definition of A_2.

The requested probability equals

$$\mathbb{P}[G(t)] = \sum_{i=1}^{3} \mathbb{P}[G(t)|E_i(n,s)]\mathbb{P}[E_i(n,s)]. \tag{5.166}$$

We should thus estimate all quantities in last display. When E_1 occurs, at least one of the values s, $(1+\theta)s$ is too close to $\mathcal{R}_n$. Hence, we cannot know precisely what happens with the process X in this situation. However, the probability of E_1 is very small. Indeed,

$$\mathbb{P}[E_1] \leq \mathbb{P}[\operatorname{dist}(s, \mathcal{R}_n) \leq 2\delta_2] + \mathbb{P}[\operatorname{dist}((1+\theta)s, \mathcal{R}_n) \leq 2\delta_2]. \tag{5.167}$$

If n is large, we can bound the first term in the last expression by

$$\mathbb{P}[\operatorname{dist}(s, \mathcal{R}_n) \leq 2\delta_2] \leq \delta_1 + 1 - \mathbb{P}[\mathcal{R} \cap [s - 2\delta_2, s + 2\delta_2] = \emptyset]. \tag{5.168}$$

The constant δ_1 comes from the approximation of $\mathcal{R}_n$ by $\mathcal{R}$ and by Lemma 5.7.3 can be chosen independent of s. Since $\tilde{Y}$ is a stable subordinator, the probability $\mathbb{P}[\mathcal{R} \cap [s-2\delta_2, s+2\delta_2] = \emptyset]$ can be evaluated using formulas from Lemma 5.B.1,

$$\begin{aligned}\mathbb{P}[\mathrm{dist}(s,\mathcal{R}_n) \le 2\delta_2] &\le \delta_1 + 1 - \mathbb{P}[g(s+2\delta_2) < s-2\delta_2] \\ &= \delta_1 + 1 - \int_0^{\frac{s-2\delta_2}{s+2\delta_2}} \frac{\sin\alpha\pi}{\pi} u^{\alpha-1}(1-u)^{-\alpha}\,du \le C\delta_1 + C'\delta_2^{1-\alpha} \end{aligned} \quad (5.169)$$

for some constants C, C' independent of s. In the same way we can estimate the second probability from (5.167). We have thus

$$\mathbb{P}[E_1] \le C\delta_1 + C'\delta_2^{1-\alpha}. \quad (5.170)$$

If A is true, then the realisation of E_2 means that $X(t)$ is in disk $D_{y_1}(2^n n^{-\kappa})$ and $X\big((1+\theta)t\big)$ is in $D_{y_2}(2^n n^{-\kappa})$ for some $y_1, y_2 \in T_\varepsilon^M$. By definition of J we have necessarily $y_1 \neq y_2$, and thus by Lemma 5.5.6

$$\mathbb{P}[G(t) \cap E_2(n,s) \cap A] = 0. \quad (5.171)$$

Hence,

$$\mathbb{P}[G(t)|E_2(n,s)] = \frac{\mathbb{P}[G \cap E_2 \cap A] + \mathbb{P}[G \cap E_2 \cap A^c]}{\mathbb{P}[E_2]} \le \frac{6\delta_1}{\mathbb{P}[E_2]}. \quad (5.172)$$

The denominator $P[E_2]$ increases to $\mathcal{P}_{\theta s}(s, \tilde{Y}^n)$ as δ_2 decreases. Since

$$\mathcal{P}_{\theta s}(s, \tilde{Y}^n) \ge \mathcal{P}_\theta(s, \tilde{Y}) - \delta_1 \quad (5.173)$$

for n large enough, there exists a constant C depending only on θ, such that

$$P[G(t)|E_2(n,s)] \le C\delta_1. \quad (5.174)$$

The most interesting event is E_3. The probability of E_3 can be calculated in the similar manner as the probability of E_1. For n large enough

$$\begin{aligned}\mathbb{P}[E_3(n,s)] &= \mathbb{P}\big[\mathcal{R} \cap [s-2\delta_2, (1+\theta)s + 2\delta_2] = \emptyset\big] \pm \delta_1 \\ &= \int_0^{1/1+\theta} \frac{\sin\alpha\pi}{\pi} u^{\alpha-1}(1-u)^{-\alpha}\,du \pm (C\delta_1 + C'\delta_2). \end{aligned} \quad (5.175)$$

The constants C and C' can be chosen again independent of s. Note also that the main term does not depend on s.

We will now estimate the probability $\mathbb{P}[G(t)|E_3(n,s) \cap A]$. If $E_3 \cap A$ occurs, then X stays during the time interval $[t, (1+\theta)t]$ in the disk $D_y(2^n n^{-\kappa})$ around $y \in T_\varepsilon^M$. At the end of this section we will show

Lemma 5.7.4. *If X stays during $[t,(1+\theta)t]$ in the disk $D_y(2^n n^{-\kappa})$ around $y \in T_\varepsilon^M$, then for $\mathbb{P}$-a.e. $\boldsymbol{\tau}$*

$$\begin{aligned}\lim_{t\to\infty} \mathbb{P}\big[X(t)=y\big|E_3(n(t),s(t))\cap A\big] \\ = \lim_{t\to\infty} \mathbb{P}\big[X((1+\theta)t)=y\big|E_3(n(t),s(t))\cap A\big] = 1. \quad (5.176)\end{aligned}$$

We use this lemma to finish the proof of Proposition 5.7.1. For t large enough we have

$$\begin{aligned}\mathbb{P}[G(t)|E_3(n,s)] &= \frac{\mathbb{P}[G|E_3\cap A]\mathbb{P}[E_3\cap A]+\mathbb{P}[G\cap E_3\cap A^c]}{\mathbb{P}[E_3]} \\ &= \frac{(1-o(1))\big(\mathbb{P}[E_3]\pm 6\delta_1\big)\pm 6\delta_1}{\mathbb{P}[E_3]} = 1\pm C\delta_1.\end{aligned} \quad (5.177)$$

Putting (5.170), (5.174), (5.175), and (5.177) into (5.166) we get

$$\mathbb{P}[G(t)] \le C\delta_1 + C'\delta_2^{1-\alpha} + \mathbb{P}[E_3(n,s)]. \quad (5.178)$$

Similarly, we obtain the lower bound

$$\mathbb{P}[G(t)] \ge (1-C\delta_1)P[E_3(n,s)] - C\delta_1 - C'\delta_2^{1-\alpha}. \quad (5.179)$$

Since the expression (5.175) for E_3 and also the constants in error terms do not depend on s, and since δ_1 and δ_2 can be taken arbitrarily small, we have

$$\lim_{t\to\infty}\mathbb{P}[G(t)] = \int_0^{1/1+\theta} \frac{\sin\alpha\pi}{\pi} u^{\alpha-1}(1-u)^{-\alpha}\,du. \quad (5.180)$$

This finishes the proof. □

IV. It remains to show Lemma 5.7.4

Proof of Lemma 5.7.4. Let us introduce some notation to describe the movement of X inside $\mathcal{D}\equiv D_y(2^n n^{-\kappa})$. Let t_1 be the time of the first arrival of X to y. After t_1 X stays in y an exponential time U_0 with mean τ_y, then it leaves y, makes an excursion not leaving $\mathcal{D}$ that takes the time V_1, returns to y, stays there U_1, etc. The number of such excursion is geometrically distributed with mean of order n. After the last visit of y, X leaves y at the time t_2 and then it leaves the disk.

We first bound the expected duration of one excursion. It is easy to prove that in the neighbourhood of y there are only traps shallower than $\varepsilon n^{-5/(1-\alpha)}2^{n/\alpha}/n$. Indeed, as in the proof of Lemma 5.5.6, let

$$B(y) = \Big\{y\in T_\varepsilon^M, \exists x\in\mathcal{D}, \tau_x \ge \varepsilon n^{-\frac{5}{1-\alpha}}\frac{2^{n/\alpha}}{n}\Big\}. \quad (5.181)$$

Then,

$$\mathbb{P}[B(y)] \leq C2^n n^{-\kappa} \frac{n^{2\alpha} n^{\frac{5\alpha}{1-\alpha}}}{2^{2n}}. \tag{5.182}$$

The summation over all sites in $\mathbb{D}(n)$ gives

$$\mathbb{P}\Big[\bigcup_{y\in(n)} B(y)\Big] \leq Cn^{1+\alpha-\kappa} n^{\frac{5\alpha}{1-\alpha}} \tag{5.183}$$

and the claim follows by Borel-Cantelli lemma taking κ large enough.

Next, we estimate the expected number of visits of $z \in \mathcal{D} \setminus \{y\}$ during one excursion that does not leave the disk. In is known fact that the expected number of visits of $z \in \mathbb{Z}^2$ by the simple random walk during one excursion from the origin is equal to one. So,

$$\begin{aligned} 1 &= \mathbb{E}[\# \text{ visits of } z] \\ &= \mathbb{E}[\# \text{ visits of } z | X_d \text{ does not leave } \mathcal{D}]\mathbb{P}[X_d \text{ does not leave } \mathcal{D}] \\ &\quad + \mathbb{E}[\# \text{ visits of } z | X_d \text{ leaves } \mathcal{D}]\mathbb{P}[X_d \text{ leaves } \mathcal{D}]. \end{aligned} \tag{5.184}$$

It follows that for n large enough

$$\begin{aligned} &\mathbb{E}[\# \text{ of visits of } z | X_d \text{ does not leave } \mathcal{D}] \\ &\qquad \leq \big(\mathbb{P}[\text{excursion does not leave the disk}]\big)^{-1} \\ &\qquad \leq (1 - G_{D(2^n n^{-\kappa})}(0,0)^{-1})^{-1} \leq 1 + C/n \leq 2. \end{aligned} \tag{5.185}$$

The expected duration of one excursion thus satisfies

$$\mathbb{E}[V_i] \leq 2 \sum_{z\in\mathcal{D}\setminus\{y\}} \tau_z \leq 2 \sum_{z\in\mathbb{D}(n)} \tau_z \chi\{\tau_z \leq n^{-5/(1-\alpha)} \varepsilon 2^{n/\alpha}/n\}. \tag{5.186}$$

The last sum can be bounded using Lemma 5.3.2. Let $i_2(n)$ be such that $2^{-i_2(n)} \leq n^{-5/(1-\alpha)} \leq 2^{-i_2(n)+1}$. Then,

$$\begin{aligned} \mathbb{E}[V_i] &\leq 2 \sum_{z\in\mathbb{D}} \tau_z \chi\{\tau_z \leq 2\} + 2 \sum_{i=i_2(n)}^{i_0(n)} \sum_{z\in T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}} \tau_z \\ &\leq 4 \cdot 2^n n^{1-\alpha} + 2 \sum_{i=i_2(n)}^{i_0(n)} \varepsilon \frac{2^{n/\alpha}}{n} 2^{-i+1} \big|T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\big| \\ &\leq 4 \cdot 2^n n^{1-\alpha} + C 2^{n/\alpha} \sum_{i=i_2(n)}^{i_0(n)} 2^{-(1-\alpha)i} \leq C 2^{n/\alpha} n^{-5}. \end{aligned} \tag{5.187}$$

Since the expected number of excursions is $O(n)$, the mean of the total time spent by X during the excursions can be bounded by

$$\mathbb{E}\Big[\sum V_i\Big] \leq C2^{n/\alpha}n^{-4}. \tag{5.188}$$

Recall that t_1 and t_2 denote the first, resp. last time when $X(t)$ was in y. Conditionally on $E_3 \cap A$ we have $t_1 < t < t_2$. We treat separately two cases. Suppose first that $t_2 - t \leq n^{-2}2^{n/\alpha}$. Since $X(t_2-) = y$, we can write

$$\mathbb{P}[X(t) = y] \geq \exp\Big(-\frac{t_2 - t}{\tau_y}\Big) \geq \exp\Big(-\frac{2^{n/\alpha}n^{-2}}{\varepsilon 2^{n/\alpha}n^{-1}}\Big) \geq 1 - C/\varepsilon n. \tag{5.189}$$

On the other hand, if $t_2 - t \geq n^{-2}2^{n/\alpha}$, we have by Fubini theorem

$$\mathbb{E}\Big[\sum V_i\Big] = \mathbb{E}\Big[\int_{t_1}^{t_2} \chi\{X(u) \neq y\}du\Big] = \int_{t_1}^{t_2} \mathbb{P}[X(u) \neq y]du. \tag{5.190}$$

From (5.188) and (5.190) it is easy to see that there exist $u \in [t, t + n^{-2}2^{n/\alpha}]$ such that

$$\mathbb{P}[X(u) \neq y] \leq Cn^{-2}. \tag{5.191}$$

Let us now define the process X' that is coupled with X. It has the same trajectory as X, at all sites it stays the same time with the exception of the first visit of y. Let the duration of the first visit satisfies $U_0' = U_0 + (u - t)$. That means that if $X(t) \neq y$, then also $X'(u) \neq y$. Probability that X stays at y the additional time $u - t$ can be bounded from bellow by

$$\exp\Big(-\frac{u - t}{\tau_y}\Big) \geq \exp\Big(-\frac{2^{n/\alpha}n^{-2}}{\varepsilon 2^{n/\alpha}n^{-1}}\Big) \geq 1 - C'/\varepsilon n. \tag{5.192}$$

Hence,

$$Cn^{-2} \geq \mathbb{P}[X(u) \neq y] \geq (1 - C'/\varepsilon n)\mathbb{P}[X(t) \neq y] \tag{5.193}$$

and thus

$$\mathbb{P}[X(t) \neq y] \leq Cn^{-2}. \tag{5.194}$$

The proof then follows from (5.189) and (5.194). $\square$

5.8 Proof of subaging

In this section we prove the subaging behaviour of the function $\Pi(t_w, t_w + t)$. Recall that this function has been defined as the probability that X does not jump between t_w and $t_w + t$. If we know that at time t_w the process X is in a trap y with depth τ_y, then this probability is easy to obtain. The Markov property gives

$$\mathbb{P}\big[X(t') = X(t)\,\forall t' \in [t_w, t_w + t]\big|\tau_{X(t_w)}\big] = \exp\Big(-\frac{t}{\tau_{X(t_w)}}\Big). \tag{5.195}$$

We should thus gain an information about the depth $\tau_{X(t_w)}$. We would like to deduce its distribution from the behaviour of processes $\tilde{Y}^n$ and $\tilde{Y}$, because these are the only objects we really control. It should be obvious that the depth of the trap where X is at time t_w depends on the size of the jump of $\tilde{Y}^n$ that intersects the level $t_w/2^{n/\alpha}$. Hence, to find an expression for the function $\Pi(t_w, t_w + t)$ we should control two basic objects. First, the distribution of the size of the jump that intersect certain level, and second, the conditional distribution of $\tau_{X(t_w)}$ knowing the size of this jump.

We start by controlling the size of the jump. Let $\ell_n = \ell_n(s)$ be the size of the jump of $\tilde{Y}^n$ that intersect the level s,

$$\ell_n(s) = \inf\{x \in \mathcal{R}_n : x > s\} - \sup\{x \in \mathcal{R}_n : x \le s\}, \tag{5.196}$$

and let $\ell = \ell(s)$ be the same size for the limiting process $\tilde{Y}$. We use μ_s^n, resp. μ_s to denote the distributions of $\ell_n(s)$ and $\ell(s)$.

Lemma 5.8.1. *The sequence μ_s^n converges weakly uniformly in $s \in [1, 2^{1/\alpha}]$ to μ_s, that is for every bounded continuous function g*

$$\int g(\ell)\mu_s^n(d\ell) \xrightarrow{n\to\infty} \int g(\ell)\mu_s(d\ell) \qquad \textit{uniformly in } s \in [1, 2^{1/\alpha}]. \tag{5.197}$$

Proof. The proof is very similar to the proof of Lemma 5.7.3. The pointwise convergence follows from the $\mathbb{P}$-a.s. continuity of the functional $Y \to \inf\{x \in \mathcal{R}(Y) : x > s\} - \sup\{x \in \mathcal{R}(Y) : x \le s\}$ in the Skorokhod topology on $D([0, \infty))$. Take now $\eta > 0$. Let s_1, $s_2 \in [1, 2^{1/\alpha}]$, $0 < s_2 - s_1 < \eta$. Then,

$$\begin{aligned}\Big|\int g(\ell)\mu_{s_2}^n(d\ell) - \int g(\ell)\mu_{s_1}^n(d\ell)\Big|& \\ \le \|g\|_\infty \mathbb{P}[\mathcal{R}_n \cap [s_1, s_2] \ne \emptyset] &= \|g\|_\infty \mathcal{P}_\eta(s_1, \tilde{Y}^n).\end{aligned} \tag{5.198}$$

As follows from Lemma 5.7.3, the value of $\mathcal{P}_\eta(s_1, \tilde{Y}^n)$ converges uniformly to $\mathcal{P}_\eta(s_1, \tilde{Y})$. Further, $\mathcal{P}_\eta(s_1, \tilde{Y}) \le h(\eta)$ for some $h(\eta)$ independent of s satisfying $h(\eta) \to 0$ as $\eta \to 0$. Therefore, we can bound (5.198) for n large enough by $\|g\|_\infty(\eta + h(\eta))$. The uniform convergence then follows by the same reasoning as in the proof of Lemma 5.7.3. $\square$

As a consequence of the scaling invariance of $\tilde{Y}$ (recall that $\tilde{Y}$ is a stable subordinator) we get the following relation between the measures μ_s,

$$\mu_s([a,b]) = \mu_1([a/s, b/s]) \tag{5.199}$$

for any interval $[a,b] \subset (0,\infty)$.

The control of $\tau_{X(t_w)}$ knowing the size of the jump is more complicated. It occupies the majority of the proof of the following proposition that is a refined version of Theorem 5.1.2.

Proposition 5.8.2. *For $\mathbb{P}$-a.e. realisation of the environment $\boldsymbol{\tau}$,*

$$\lim_{t\to\infty} \Pi\Big(t, t + \frac{\theta t}{\log t}\Big) = \int_0^\infty \Big(\frac{\ell\pi}{\ell\pi + \theta\alpha}\Big)^{1+\alpha} \mu_1(d\ell) \equiv \Pi(\theta). \tag{5.200}$$

By an easy application of dominated convergence theorem we get

Corollary 5.8.3. *The function $\Pi(\theta)$ satisfies*

$$\lim_{\theta\to 0} \Pi(\theta) = 1 \qquad \textit{and} \qquad \lim_{\theta\to\infty} \Pi(\theta) = 0. \tag{5.201}$$

Proof of Proposition 5.8.2. We proceed similarly as in the proof of aging. We take $n(t)$ as in (5.148) and we define $s = s(t) = t/2^{n(t)/\alpha}$. Next, we choose δ_1 and δ_2, and we set the constants ε, M and m in the same manner as before. We thus know that the process $\tilde{Y}^n$ is a good approximation of the rescaled time change $\bar{S}^n$. That means that $\mathbb{P}[A] = \mathbb{P}[A_1 \cap A_2] \geq 1 - C\delta_1$ with A_1, A_2 defined as in the previous section. For the following discussion we will suppose that A occurs and we take account of the remaining part of the probability space at the end of the proof.

As we have already noted, it is necessary to obtain the conditional distribution of $\tau_{X(t)}$ knowing $\ell_n(s)$. Similarly as in the proof of aging not much can be done if the distance between s and $\mathcal{R}_n$ is smaller than $2\delta_2$, because the approximation is not sufficiently precise. However, the probability of this bad case can be bounded by $C\delta_1 + C'\delta_2^{1-\alpha}$ uniformly in s in the same way as in (5.170).

Let $E = E(n,s)$ denote the event $\mathrm{dist}(s, \mathcal{R}_n) > 2\delta_2$. If E occurs, then the situation is more favourable. We know that X was at time t inside a disk $D_y(2^n n^{-\kappa})$ around some deep trap $y = y(n,s)$. Moreover, similarly as in Lemma 5.7.4, we can show

$$\mathbb{P}[X(t) = y(n,s) | E(n,s)] \to 1 \quad \text{ as } \quad t \to \infty. \tag{5.202}$$

The last expression allows us to compute the conditional distribution of $\tau_{y(n,s)}$ knowing $\ell_n(s)$ instead of the distribution of $\tau_{X(t)}$. As we have already discussed in the proof of Lemma 5.6.4, the size ℓ of the jump that is the result of the visit of y satisfies

$$2^{n/\alpha}\ell = \tau_y \sum_{i=1}^{\xi} e_i', \tag{5.203}$$

where ξ is a geometrically distributed random variable with mean

$$G_{D(2^n n^{-\kappa})}(0,0) = n \log 2/\pi + o(n) = \mathcal{K}'n + o(n), \tag{5.204}$$

and e_i' are i.i.d., exponential random variables with mean one. It is convenient to introduce the rescaled depth of trap, $\sigma_x = \tau_x n/2^{n/\alpha}$. Equation (5.203) then becomes

$$\ell = \frac{\sigma_y}{n} \sum_{i=1}^{\xi} e_i'. \tag{5.205}$$

As can be seen from Lemma 5.6.1, the distribution ν_n of σ_y converges weakly to the distribution ν,

$$\nu(dx) = \frac{\alpha}{\varepsilon^{-\alpha} - M^{-\alpha}} \cdot \frac{1}{x^{\alpha+1}}\, dx \quad \text{for} \quad \varepsilon \le x \le M. \tag{5.206}$$

The random variable $n^{-1}\sum_{i=1}^{\xi} e_i'$ is an exponential random variable with mean $\mathcal{K}'+o(1)$. Let f_n denote its density, and let f denote the density of the limiting distribution,

$$f(x) = \exp\big(-x/\mathcal{K}'\big)/\mathcal{K}'. \tag{5.207}$$

We use F_ℓ^n to denote the distribution function of $\sigma_{y(n,s)}$ conditionally on $\ell_n(s) = \ell$,

$$F_\ell^n(a) = \mathbb{P}[\sigma_{y(n,s)} \le a | \ell_n(s) = \ell]. \tag{5.208}$$

Lemma 5.8.4. *The function F_ℓ^n can be written as*

$$F_\ell^n(a) = \frac{\int_\varepsilon^a \frac{1}{x} f_n(\frac{\ell}{x})\nu_n(dx)}{\int_\varepsilon^M \frac{1}{x} f_n(\frac{\ell}{x})\nu_n(dx)}. \tag{5.209}$$

Proof. We should verify that for any event B that is measurable with respect to the σ-algebra generated by the random variable $\ell_n(s)$

$$\int_B \chi\{\sigma_y \le a\} d\mathbb{P} = \int_B F_\ell^n(a)\, d\mathbb{P}. \tag{5.210}$$

It is sufficient to verify the last expression for an event B that has the form $\{\ell_n(s) \in I\}$ for some interval $I \subset [0,\infty)$. The left hand side of (5.210) can be then written as

$$\int_B \chi\{\sigma_y \le a\} d\mathbb{P} = \int_\varepsilon^a \int_{I/x} f_n(z)\, dz\, \nu_n(dx). \tag{5.211}$$

To compute the right hand side we should first find the distribution of $\ell_n(s)$

$$\mathbb{P}[\ell_n(s) \le u] = \int_\varepsilon^M \int_0^{u/x} f_n(z)\, dz\, \nu_n(dx). \tag{5.212}$$

The right hand side of (5.210) then equals

$$\begin{aligned}\int_I \frac{\int_\varepsilon^a \frac{1}{x} f_n(\frac{\ell}{x})\nu_n(dx)}{\int_\varepsilon^M \frac{1}{x} f_n(\frac{\ell}{x})\nu_n(dx)}\, d\Big(\int_\varepsilon^M \int_0^{\ell/x} f_n(z)dz\, \nu_n(dx)\Big)& \\ = \int_I \frac{\int_\varepsilon^a \frac{1}{x} f_n(\frac{\ell}{x})\nu_n(dx)}{\int_\varepsilon^M \frac{1}{x} f_n(\frac{\ell}{x})\nu_n(dx)} \Big(\int_\varepsilon^M \frac{1}{x} f_n(\ell/x)\, \nu_n(dx)\Big)\, d\ell.&\end{aligned} \tag{5.213}$$

Making the substitution $z = \ell/x$ and changing the order of integration it is easy to get the same expression as in (5.211). This finishes the proof. □

As an consequence of the previous lemma we get

Lemma 5.8.5. *For any bounded continuous function g*

$$\int g(a) dF_\ell^n(a) \xrightarrow{n\to\infty} \int g(a) dF_\ell(a), \tag{5.214}$$

where

$$F_\ell(a) = \frac{\int_\varepsilon^a z^{-\alpha-2} \exp(\ell/\mathcal{K}' z) dz}{\int_\varepsilon^M z^{-\alpha-2} \exp(\ell/\mathcal{K}' z) dz}. \tag{5.215}$$

Moreover, if $K \subset (0,\infty)$ compact and g has bounded first derivative, then the convergence is uniform in $\ell \in K$.

Proof. First we show that the sequence $F_\ell^n(a)$ converges to $F_\ell(a)$ for any $a \in [\varepsilon, M]$. We will verify separately the convergence of the nominator and the denominator in (5.209). For the nominator we have

$$\begin{aligned}&\Big|\int_\varepsilon^a \frac{1}{x} f_n\Big(\frac{\ell}{x}\Big) d\nu_n - \int_\varepsilon^a \frac{1}{x} f\Big(\frac{\ell}{x}\Big) d\nu\Big| \\ &\quad \le \Big|\int_\varepsilon^a \frac{1}{x}\Big(f_n\Big(\frac{\ell}{x}\Big) - f\Big(\frac{\ell}{x}\Big)\Big) d\nu_n\Big| + \Big|\int_\varepsilon^a \frac{1}{x} f\Big(\frac{\ell}{x}\Big) d\nu_n - \int_\varepsilon^a \frac{1}{x} f\Big(\frac{\ell}{x}\Big) d\nu\Big|.\end{aligned} \tag{5.216}$$

The second difference converges to zero, because ν_n converges weakly to ν. The first term goes to zero too, because ν_n have compact support $[\varepsilon, M]$ and the sequence f_n converges to f uniformly on this interval. The proof of the convergence of the denominator can be done in the same manner. By substitution from (5.206) and (5.207) we can obtain the expression (5.215),

$$\int_{\varepsilon}^{a} \frac{1}{x} f\Big(\frac{\ell}{x}\Big)\, \nu(dx) = \int_{\varepsilon}^{a} z^{-\alpha-2} \exp(\ell/\mathcal{K}' z)\, dz, \tag{5.217}$$

and similarly for the denominator.

To show the uniform convergence note that by integration by parts

$$\int_{\varepsilon}^{M} g(a) dF_\ell^n(a) - \int_{\varepsilon}^{M} g(a) dF_\ell(a) \le \|g'\|_\infty \int_{\varepsilon}^{M} \big(F_\ell(a) - F_\ell^n(a)\big) da. \tag{5.218}$$

We should thus prove that F_ℓ^n converges uniformly in ℓ. We will show that for all $a \in [\varepsilon, M]$ and any K the family $F_\ell^n(a)$ is uniformly equicontinuous in $\ell \in K$. It will then together with the pointwise convergence imply the uniform convergence.

Take $\eta > 0$, $\ell_1, \ell_2 \in K$, $0 < \ell_2 - \ell_1 \le \eta$. Let $J_\ell^n(a)$ denote $\int_\varepsilon^a \frac{1}{x} f_n(\frac{\ell}{x}) \nu_n(dx)$. Then,

$$\big|F_{\ell_1}^n(a) - F_{\ell_2}^n(a)\big| = \left| \frac{J_{\ell_1}^n(a) J_{\ell_2}^n(M) - J_{\ell_1}^n(M) J_{\ell_2}^n(a)}{J_{\ell_1}^n(M) J_{\ell_2}^n(M)} \right|. \tag{5.219}$$

The denominator is decreasing in ℓ_1 and ℓ_2 and it can be bounded from bellow by some constant depending only on K. From the uniform equicontinuity of f_n then follows that the nominator can be bounded from above by some $h(\eta)$ satisfying $h(\eta) \to 0$ as $\eta \to 0$. Moreover, for n large enough $h(\eta)$ can be chosen to be dependent only on K. This proves uniform equicontinuity. $\square$

We have now all ingredients to finish the proof of Proposition 5.8.2. Let $G = G(t)$ denote the event

$$G = \big\{X(t') = X(t)\, \forall t' \in [t, t + \theta t/\log t]\big\}. \tag{5.220}$$

Then,

$$\begin{aligned} \mathbb{P}[G] &= \int_0^\infty \mathbb{P}[G|\ell_n(s) = \ell] \mu_s^n(d\ell) \\ &= \int_0^\infty \mathbb{P}[G|\ell \cap (A \cap E)] \big(1 - \mathbb{P}\big[(A \cap E)^c|\ell\big]\big) \mu_s^n(d\ell) \\ &\quad + \int_0^\infty \mathbb{P}[G|\ell \cap (A \cap E)^c] \mathbb{P}[(A \cap E)^c|\ell] \mu_s^n(d\ell). \end{aligned} \tag{5.221}$$

The second integral can be bounded by $\mathbb{P}[(A\cap E)^c]\leq C\delta_1+C'\delta_2^{1-\alpha}$. The first one can be bounded from above by

$$\int_0^\infty \mathbb{P}[G|\ell\cap(A\cap E)]\mu_s^n(d\ell)\equiv I(t) \tag{5.222}$$

and from bellow by $I(t)-C\delta_1-C'\delta_2^{1-\alpha}$. We should thus compute the value of $I(t)$. Using (5.195) we get

$$I(t)=\int_0^\infty\int_\varepsilon^M \exp\Big(-\frac{\theta tn}{a2^{n/\alpha}\log t}\Big)dF_\ell^n(a)\,\mu_s^n(d\ell). \tag{5.223}$$

Taking $t=s2^{n/\alpha}$ we get

$$I(s2^{n/\alpha})=\int_0^\infty\int_\varepsilon^M \exp\Big(-\frac{\theta s\alpha}{a\log 2+cn^{-1}\log s}\Big)dF_\ell^n(a)\,\mu_s^n(d\ell). \tag{5.224}$$

We show that

$$\lim_{n\to\infty} I(s2^{n/\alpha})=\int_0^\infty\int_\varepsilon^M \exp\Big(-\frac{\theta s\alpha}{a\log 2}\Big)dF_\ell(a)\,\mu_s(d\ell)\equiv I_\infty(s). \tag{5.225}$$

Moreover, this convergence is uniform in $s\in[1,2^{1/\alpha}]$. Indeed,

$$\begin{aligned}
&\big|I(s2^{n/\alpha})-I_\infty(s)\big|\leq\\
&\leq\int_0^\infty\int_\varepsilon^M\Big|\exp\Big(\frac{-\theta s\alpha}{a\log 2+o(n^{-1})}\Big)-\exp\Big(\frac{-\theta s\alpha}{a\log 2}\Big)\Big|dF_\ell^n(a)\,\mu_s^n(d\ell)\\
&\quad+\int_0^\infty\int_\varepsilon^M\exp\Big(-\frac{\theta s\alpha}{a\log 2}\Big)\big|dF_\ell^n(a)-dF_\ell(a)\big|\,\mu_s^n(d\ell)\\
&\quad+\int_0^\infty\int_\varepsilon^M\exp\Big(-\frac{\theta s\alpha}{a\log 2}\Big)dF_\ell(a)\big|\mu_s^n(d\ell)-\mu_s(d\ell)\big|.
\end{aligned} \tag{5.226}$$

The first term converges to 0 due to the uniform convergence of exponentials inside of the integral. Using Lemma 5.8.5 and the tightness of sequence μ^n (see Lemma 5.8.1), it is possible to prove the convergence to 0 of the second term. The convergence of third term follows from Lemma 5.8.1 and continuity of the integrand. Note, that all these convergences can be proved to be uniform in s.

Inserting (5.215) into (5.225) we get

$$I_\infty(s)=\int_0^\infty\int_\varepsilon^M \exp\Big(-\frac{\theta s\alpha}{a\log 2}\Big)\frac{a^{-\alpha-2}\exp(-\ell/\mathcal{K}'a)\,da}{\int_\varepsilon^M z^{-\alpha-2}\exp(\ell/\mathcal{K}'z)\,dz}\,\mu_s(d\ell). \tag{5.227}$$

For any $c > 0$ the integral $\int_0^\infty \exp(-c/z) z^{-\alpha-2}\, dz = c^{-\alpha-1}\Gamma(\alpha+1)$. We introduce the following notation. Let

$$g_c(\varepsilon, M) = \frac{1}{\Gamma(\alpha+1)}\Big(\int_0^\varepsilon + \int_M^\infty\Big) e^{-c/z} z^{-\alpha-2}\, dz, \tag{5.228}$$

and

$$d_1 = \frac{\theta s \alpha}{\log 2} + \frac{\ell}{\mathcal{K}'} \qquad \text{and} \qquad d_2 = \frac{\ell}{\mathcal{K}'}. \tag{5.229}$$

Then

$$I_\infty(s) = \int_0^\infty \frac{d_1^{-\alpha-1} - g_{d_1}(\varepsilon, M)}{d_2^{-\alpha-1} - g_{d_2}(\varepsilon, M)} \mu_s(d\ell). \tag{5.230}$$

The difference between $I_\infty(s)$ and $J(s) \equiv \int_0^\infty (d_2/d_1)^{1+\alpha} \mu_s(d\ell)$ is small for ε small and M large. To see it consider

$$\begin{aligned} &\lim_{\substack{\varepsilon\to 0 \\ M\to\infty}} I_\infty(s) \\ &\quad = \lim_{\substack{\varepsilon\to 0 \\ M\to\infty}} \Big[\int \frac{d_1^{-\alpha-1}}{d_2^{-\alpha-1} - g_{d_2}(\varepsilon, M)} \mu_s(d\ell) - \int \frac{g_{d_1}(\varepsilon, M)}{d_2^{-\alpha-1} - g_{d_2}(\varepsilon, M)} \mu_s(d\ell) \Big]. \end{aligned} \tag{5.231}$$

Both terms converge due to the monotone convergence theorem, first one to $J(s)$ and second to 0 uniformly in s. From the scaling relation (5.199) we get that $J(s)$ actually does not depend on s. It equals

$$J(1) = \int_0^\infty \Big(\frac{\ell\pi}{\ell\pi + \theta\alpha} \Big)^{1+\alpha} \mu_1(d\ell). \tag{5.232}$$

Since $\varepsilon \to 0$ and $M \to \infty$ when $\delta_1, \delta_2 \to 0$, there exists a function $h(\delta_1, \delta_2)$ such that $h(\delta_1, \delta_2) \to 0$ as $\delta_1, \delta_2 \to 0$ satisfying $|I_\infty(s) - J(1)| \le h(\delta_1, \delta_2)$ for all s. By (5.225), (5.231), the bounds in the paragraph after (5.221), and the last paragraph we get that for n larger than some $n(\delta_1, \delta_2)$ and for any $s \in [1, 2^{1/\alpha}]$

$$\mathbb{P}[G(s 2^{n/\alpha})] = J(1) \pm \big(C\delta_1 + C'\delta_2^{1-\alpha} + h(\delta_1, \delta_2) \big). \tag{5.233}$$

Since δ_1 and δ_2 can be taken arbitrarily small, the proof is finished. □

Appendix 5.A Some properties of the simple random walk

We summarise here some known properties of Green's function and hitting probabilities of the simple random walk on $\mathbb{Z}^2$ that is killed when it exits the disk D with radius r. Let ξ denote the exit time from this disk.

The most important formula that we use repeatedly is

$$\mathbb{P}[X \text{ hits } x \text{ before } \xi] = 1 - \frac{\log|x|}{\log r} + O\Big(\frac{|x|^{-2}}{\log r}\Big) + O(\log^{-2} r). \tag{5.234}$$

The proof of it can be found for example in Lawler [Law91], Proposition 1.6.7. We use also a similar expansion for the Green's function,

$$G_D(0,x) = \frac{2}{\pi}(\log r - \log|x|) + O(|x|^{-2}) + O(r^{-1}). \tag{5.235}$$

For $G_D(0,0)$ there is the following formula ([Law91], Theorem 1.6.6)

$$G_D(0,0) = \frac{2}{\pi}\log r + k + O(r^{-1}). \tag{5.236}$$

As an easy consequence of formula (5.234) we get following lemma:

Lemma 5.A.1. *Let y be an uniformly chosen point in D. Then there exists constant C independent of r such that*

$$\mathbb{E}\big[\exp\big(\log r\mathbb{P}[X \textit{ hits } y \textit{ before } \xi]\big)\big] \le C. \tag{5.237}$$

Proof. Let a be a positive constant and let D_a denotes the disk with radius a. Then by (5.234) we have

$$\begin{aligned} &\mathbb{E}\big[\exp\big(\log r\mathbb{P}[X \text{ hits } y \text{ before } \xi]\big)\big] \\ &\quad\le \frac{1}{\pi r^2}\sum_{y\in D_a}\exp(\log r) + \frac{1}{\pi r^2}\sum_{D\setminus D_a}\exp\big(\log r\mathbb{P}[X \text{ hits } y \text{ before } \xi]\big) \\ &\quad\le \frac{C}{r} + \frac{1}{\pi r^2}\sum_{y\in D\setminus D_a}\exp\{\log r - \log|y| + O(|y|^{-2}) + O(\log^{-1} r)\} \\ &\quad\le \frac{C}{r} + \frac{1}{\pi r^2}\sum_{y\in D\setminus D_a}\frac{Cr}{y} \le C. \end{aligned} \tag{5.238}$$

This finishes the proof. □

Similarly we get

Lemma 5.A.2. *There exist $\lambda > 0$ and C independent of r such that*

$$\mathbb{E}[\exp(\lambda G_D(0,y))] \le C. \tag{5.239}$$

Appendix 5.B Some properties of stable subordinators

Let Y be a stable subordinator with the Lévy measure

$$\pi(dx) = kx^{-\alpha-1}\chi\{x \geq 0\}\, dx, \qquad k > 0. \tag{5.240}$$

We use $\mathcal{R} = \mathcal{R}(Y)$ to denote the range of this process. Let $U(dx)$ denote its potential measure that is defined by

$$U(A) = \int_0^\infty \mathbb{P}(Y(t) \in A)\, dt \quad \text{for any } A \in \mathcal{B}(\mathbb{R}). \tag{5.241}$$

For every $x > 0$, let

$$g(x) = \sup\{y \in \mathcal{R} : y \leq x\}, \tag{5.242}$$

and let

$$d(x) = \inf\{y \in \mathcal{R} : y \geq x\}. \tag{5.243}$$

Then it follows from Bertoin [Ber96], Theorems III.2, III.6, and the discussion following the second theorem that

Lemma 5.B.1. *(i) For each fixed $x \geq 0$ and every $0 \leq y \leq x < z$, we have*

$$\mathbb{P}(g(x) \in dy, d(x) \in dz) = U(dy)\pi(dz - y). \tag{5.244}$$

(ii) For every $x > 0$ the random variable $x^{-1}g(x)$ has the distribution

$$\frac{s^{\alpha-1}(1-s)^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}ds = \frac{\sin \alpha\pi}{\pi}s^{\alpha-1}(1-s)^{-\alpha}ds \qquad (0 < s < 1). \tag{5.245}$$

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6. AGING FOR BOUCHAUD'S MODEL FOR DIMENSION LARGER THAN TWO

6.1 Introduction

We prove here the aging behaviour of Bouchaud's model on $\mathbb{Z}^d$, $d \geq 3$. We can apply here a very similar method as we have used to prove the aging in the two-dimensional case. In general, the proof becomes slightly simpler because the random walk is transient if $d \geq 3$, and, moreover, all interesting quantities (like Green's function, hitting probabilities, etc.) depends only polynomially on the radius of the ball that we use as the relevant part of the environment. There is no need for logarithmic corrections and all expressions become slightly simpler.

On the other hand, there is one additional difficulty. It comes from the fact that the volume of d-dimensional ball is much larger than the volume of two-dimensional ball with the same radius. That means that somewhere in this large volume can exist regions where some of the conditions H_1, H_2, etc. that we used to describe the environment are violated. However, since the trajectory of the random walk occupies only a very small part of the volume of the ball, it can be proved that the probability that the random walk visits such "not usual" places is very small.

We will prove very similar results about aging as in $d = 2$. We use the same two-point functions as before. Namely, let $R(t_w, t_w + t)$ be the probability that X is at the same site at both times t, $t + t_w$, and let $\Pi(t_w, t_w + t)$ be the probability that X does not jump between this two times. The first two-point function has exactly the same behaviour as in dimensions one and two. The scale for the second one is changed because in $d \geq 3$ random walk returns to any site only finitely many times (it was of order $\log n$ after n steps in $d = 2$). Therefore, the successive returns have no importance for scaling. That means that the good scale for the two-point function $\Pi(t_w, t_w + t)$ is the same as for $R(t_w, t_w + t)$. On the other hand, it is evident that the successive returns should influence the value of the limiting functions $R(\theta)$ and $\Pi(\theta)$ whose existence is proved in the following theorem.

Theorem 6.1.1. *There exist functions $R(\theta)$ and $\Pi(\theta)$ independent of $\boldsymbol{\tau}$, such* *that for* $\mathbb{P}$*-a.e. realisation of the environment* $\boldsymbol{\tau}$

$$\begin{aligned}\lim_{t_w\to\infty} R(t_w, t_w+\theta t_w) &= R(\theta),\\ \lim_{t_w\to\infty} \Pi(t_w, t_w+\theta t_w) &= \Pi(\theta).\end{aligned} \tag{6.1}$$

The functions $R(\theta)$ *and* $\Pi(\theta)$ *can be explicitly calculated (see Sections 6.6 and 6.7) and they satisfy*

$$\lim_{\theta\to 0} R(\theta) = \lim_{\theta\to 0} \Pi(\theta) = 1 \quad \textit{and} \quad \lim_{\theta\to\infty} R(\theta) = \lim_{\theta\to\infty} \Pi(\theta) = 0. \tag{6.2}$$

The complete proof of this theorem would be quite long and boring. We prefer here to describe roughly the strategy of the proof and its differences with respect to the two-dimensional case. We will show only the lemmas whose proofs depended on the fact that we have worked on the two-dimensional lattice, and we will comment briefly the parts that need only cosmetic changes. The understanding of the proof for $d = 2$ is necessary for reading the following pages.

We start with the coarse-graining construction and the definition of bad events. We change slightly the notation, we use $D_x(m)$, $B_x(m)$ to denote the ball (cube) with centre x and radius (edge length) m. The relevant part of the environment will be this time the ball

$$\mathbb{D}(n) = D(m2^n). \tag{6.3}$$

We cut the trajectory using the balls with radius $2^{n\beta}$, $\beta < 1$. Formally, let $j_0^n = 0$ and let for $i = 1, 2, \dots$

$$j_i^n = \min\big\{k \ge j_{i-1}^n : X_d(k) \notin D_{X_d(j_{i-1}^n)}(2^{n\beta})\big\}. \tag{6.4}$$

We write $x_i^n = X_d(j_i^n)$ for the starting points of the parts of the trajectory.

There is another important distance in the proof for $d = 2$. It specifies the close neighbourhood of traps, where the successive returns are not considered as bad and all contribute to one jump of the Lévy process Y^n. We choose here $2^{n\gamma}$, $\gamma < \beta$, as this distance. The constants β, γ cannot take an arbitrary value, their precise values will be specified in Section 6.4.

The most important role is again played by traps that have the depth in some properly chosen interval. We use $T_\varepsilon^M(n)$ to denote the set of deep traps,

$$T_\varepsilon^M(n) = \big\{x \in \mathbb{D}(n) : \varepsilon 2^{2n/\alpha} \le \tau_x < M 2^{2n/\alpha}\big\}, \tag{6.5}$$

where as before we suppose $\varepsilon \ll 1 \ll M$. If $x \in T_M$, then it is referred to as the very deep trap, if $x \in T^\varepsilon$, then it is the shallow trap. We also introduce the set of "external" sites

$$\mathcal{E}(n) = \big\{x \in \mathbb{D}(n) : \mathrm{dist}(x, T_\varepsilon^M) \ge 2^{n\gamma}\big\}. \tag{6.6}$$

It is now possible to define which parts of the trajectory will be considered as bad. The part between x_i^n and x_{i+1}^n is called bad if

(i) X can exit $\mathbb{D}(n)$, that is $\operatorname{dist}\big(x_i^n, \mathbb{D}(n)^c\big) \le 2^{n\beta}$.

(ii) X hits more than two deep traps during this part of the trajectory, that is $\big|X_d[j_i^n, j_{i+1}^n) \cap T_\varepsilon^M\big| \ge 2$.

(iii) X returns during this part to some deep trap y after exiting the ball $D_y(2^{n\gamma})$ around it.

(iv) The endpoint x_{i+1}^n of this part is closer than $2^{n\gamma}$ to some deep trap, that is $x_{i+1}^n \notin \mathcal{E}(n)$.

(v) X hits a deep trap that is closer than $2^{n\gamma}$ to the border of $D_{x_i^n}(2^{n\beta})$.

These five bad events are simple rephrasing of the corresponding events from $d = 2$. We should add one additional bad event that comes from the fact that in dimension larger than three the deep traps are not distant. (Actually, it can be proved that for d large enough there exist two deep traps that are neighbours in $\mathbb{Z}^d$, however, it will be not important for the future discussion.) We introduce the set of bad (deep) traps,

$$\mathcal{B}(n) = \big\{x \in T_\varepsilon^M : \exists y \in D_x(2^{n\gamma}), \tau_y \ge n^{-5/(1-\alpha)}\varepsilon 2^{2n/\alpha}\big\}. \tag{6.7}$$

That means that a deep trap is bad if it has some "quite deep" site in its neighbourhood. The last condition for the bad part of the trajectory is

(vi) X hits a bad trap, that is $X[j_i^n, j_{i+1}^n) \cap \mathcal{B}(n) \neq \emptyset$.

For every part $X[j_i^n, j_{i+1}^n)$ we define the score

$$s_i^n = \begin{cases} \sum_{k=j_i^n}^{j_{i+1}^n} e_k \tau_{X_d(k)} \chi\{X_d(k) \in T_\varepsilon^M\} & \text{if the part is good,} \\ \infty & \text{if the part is bad.} \end{cases} \tag{6.8}$$

As in $d = 2$ we define an auxiliary sequence of i.i.d. random variables $\tilde{s}_i^n$, whose distribution will be specified later (see Section 6.5). Let $J(n)$ be the index of the first bad part of the trajectory. We set

$$\bar{s}_i^n = \begin{cases} s_i^n & \text{if } i < J, \\ \tilde{s}_i^n & \text{if } i \ge J. \end{cases} \tag{6.9}$$

Then we define the *score process* by

$$Y^n(t) = \frac{1}{2^{2n/\alpha}} \sum_{i=0}^{\lfloor t 2^{2n(1-\beta)} \rfloor} \bar{s}_i^n. \tag{6.10}$$

The processes Y^n will be used as an approximation of rescaled *time change* process

$$\bar{S}^n(t) = \frac{1}{2^{2n/\alpha}} \sum_{i=0}^{j^n_{\lfloor t2^{2n(1-\beta)} \rfloor}} e_i \tau_{X_d(i)}. \tag{6.11}$$

In Sections 6.2, 6.3, and 6.4 we will prove that this approximation is reasonable. Namely, in Section 6.2 we show that the time that is spent in the shallow traps is negligible with respect to the time spent in the deep traps. In Section 6.3 we bound the probability of hitting a very deep trap. Finally, in Section 6.4 we prove that we can choose m large to ensure that J is large enough.

Further, we prove the convergence of Y^n to a certain Lévy process (Section 6.5). This convergence will be then used to prove aging in the same way as in $d = 2$ (Sections 6.6 and 6.7).

6.2 The shallow traps

We should prove that the time spent in the shallow traps is negligible with respect to expected time spent in $\mathbb{D}(n)$. Since X_d needs approximately 2^{2n} steps to leave $\mathbb{D}(n)$ and every site in $\mathbb{D}(n)$ is visited only finitely many times, the order of time spend by X in $\mathbb{D}(n)$ is close to the order of the sum of 2^{2n} α-stable variables, that is X spends in $\mathbb{D}(n)$ time that grows like $2^{2n/\alpha}$. Recall that Λ_d denotes the exit time of X_d from $\mathbb{D}$. We will prove

Lemma 6.2.1. *There exists constant K_1 independent of ε such that for $\mathbb{P}$-a.e. realisation of environment $\boldsymbol{\tau}$ and for n large enough*

$$\mathbb{E}\Big[\sum_{i=0}^{\Lambda_d - 1} e_i \tau_{X_d(i)} \chi\{X_d(i) \in T^\varepsilon\} \Big| \boldsymbol{\tau} \Big] \le K_1 \varepsilon^{1-\alpha} 2^{2n/\alpha}. \tag{6.12}$$

Proof. In the two-dimensional case the proof of this lemma was separated into two parts—the description of the environment and the actual proof. Here we choose a little bit different approach where both parts are mixed together. We start with the upper bound on the time spend in really very shallow traps, that is in traps with $\tau_x \le 1$.

$$\begin{aligned} \mathbb{E}\Big[\sum_{i=0}^{\Lambda_d - 1} e_i \tau_{X_d(i)} \chi\{\tau_{X_d(i)} \le 1\} \Big| \boldsymbol{\tau} \Big] &= \sum_{x \in \mathbb{D}} \tau_x G_{\mathbb{D}}(0,x) \chi\{\tau_x \le 1\} \\ &\le \sum_{x \in \mathbb{D}} G_{\mathbb{D}}(0,x) = \mathbb{E}(\Lambda_d) = O(2^{2n}) \ll 2^{2n/\alpha}. \end{aligned} \tag{6.13}$$

Next, we divide the remaining part of T^ε into disjoint sets $T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}$, with $i \in \{1, \dots, i_{\max}\}$, where $i_{\max}$ is an integer satisfying

$$1/2 \le 2^{-i_{\max}} \varepsilon 2^{2n/\alpha} < 1. \tag{6.14}$$

The probability that a fixed site x is in $T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}$ can be bounded by

$$\begin{aligned} p_{n,i} &\equiv \mathbb{P}\big[x \in T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\big] \\ &= \varepsilon^{-\alpha} 2^{-2n} 2^{i\alpha} \big[L(\varepsilon 2^{2n/\alpha - i}) - 2^{-\alpha} L(\varepsilon 2^{2n/\alpha - i + 1})\big] \\ &\le c \varepsilon^{-\alpha} 2^{-2n} 2^{i\alpha}, \end{aligned} \tag{6.15}$$

since L is a bounded function.

For any fixed $i \in \{1, \dots, i_{\max}\}$ and K' large we can write

$$\begin{aligned} &\mathbb{P}\Big[\mathbb{E}\Big[\sum_{i=0}^{\Lambda_d - 1} e_i \tau_{X_d(i)} \chi\big\{X_d(i) \in T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\big\} \Big| \boldsymbol{\tau}\Big] \ge K' \varepsilon^{1-\alpha} 2^{i(\alpha-1)} 2^{2n/\alpha}\Big] \\ &\quad = \mathbb{P}\Big[\sum_{x \in \mathbb{D}} G_{\mathbb{D}}(0,x) \tau_x \chi\big\{x \in T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\big\} \ge K' \varepsilon^{1-\alpha} 2^{i(\alpha-1)} 2^{2n/\alpha}\Big] \\ &\quad \le \mathbb{P}\Big[\sum_{x \in \mathbb{D}} G_{\mathbb{D}}(0,x) \chi\big\{x \in T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\big\} \ge K' \varepsilon^{-\alpha} 2^{i\alpha - 1}\Big]. \end{aligned} \tag{6.16}$$

Using Markov inequality (with $\lambda_n > 0$) this can be bounded by

$$\begin{aligned} &\le \exp(-\lambda_n K' \varepsilon^{-\alpha} 2^{i\alpha - 1}) \prod_{x \in \mathbb{D}} \big[(1 - p_{n,i}) + p_{n,i} e^{\lambda_n G_{\mathbb{D}}(0,x)}\big] \\ &\le \exp(-\lambda_n K' \varepsilon^{-\alpha} 2^{i\alpha - 1}) \prod_{x \in \mathbb{D}} \big[1 + c 2^{i\alpha - 2n} \varepsilon^{-\alpha} \big(e^{\lambda_n G_{\mathbb{D}}(0,x)} - 1\big)\big]. \end{aligned} \tag{6.17}$$

We should find an upper bound on the logarithm of the product in the last equation. Since $x \ge \log(1+x)$, we have

$$\begin{aligned} &\log \prod_{x \in \mathbb{D}} \big[1 + c 2^{i\alpha - 2n} \varepsilon^{-\alpha} \big(e^{\lambda_n G_{\mathbb{D}}(0,x)} - 1\big)\big] \\ &\qquad \le \sum_{x \in \mathbb{D}} c 2^{i\alpha - 2n} \varepsilon^{-\alpha} \big(e^{\lambda_n G_{\mathbb{D}}(0,x)} - 1\big). \end{aligned} \tag{6.18}$$

Let $\lambda_n = n G_{\mathbb{D}}(0,0)^{-1}$. We divide the last sum into two parts. First, we will sum over the sites that are close to the origin, $|x| \le n^{2/(d-2)}$. Since $G_{\mathbb{D}}(0,x) \le G_{\mathbb{D}}(0,0)$, we have

$$\begin{aligned} &\sum_{x \in D(n^{2/(d-2)})} c 2^{i\alpha - 2n} \varepsilon^{-\alpha} \big(e^{\lambda_n G_{\mathbb{D}}(0,x)} - 1\big) \\ &\qquad \le C n^{2d/(d-2)} 2^{i\alpha - 2n} \varepsilon^{-\alpha} e^{\lambda_n G_{\mathbb{D}}(0,0)} \le C n^{2d/(d-2)} 2^{i\alpha - 2n} \varepsilon^{-\alpha} e^n. \end{aligned} \tag{6.19}$$

The last expression tends to 0 as $n \to \infty$.

In the second part of the sum we cannot simply replace $G_{\mathbb{D}}(0,x)$ by $G_{\mathbb{D}}(0,0)$. We use the following formula for the Green's function (see (6.127) in Appendix)

$$G_{D(n)}(0,x) = a_d\big(|x|^{2-d} - n^{2-d}\big) + O\big(|x|^{1-d}\big), \tag{6.20}$$

with a_d being a constant depending only on dimension. It follows that for any $x \in \mathbb{D}(n) \setminus D(n^{2/(d-2)})$ we have $G_{\mathbb{D}}(0,x) \leq cn^{-2}$. Therefore, the argument of the exponential in (6.18) is smaller than $c'n^{-1}$. Using the fact that $e^x - 1 \leq 2x$ for x sufficiently close to 0 we get

$$e^{\lambda_n G_{\mathbb{D}}(0,x)} - 1 \leq cnG_{\mathbb{D}}(0,x) \tag{6.21}$$

and thus

$$\begin{aligned}\sum_{x\in\mathbb{D}\setminus D(n^{2/(d-2)})} & c2^{i\alpha-2n}\varepsilon^{-\alpha}\big(e^{\lambda_n G_{\mathbb{D}}(0,x)} - 1\big) \\ &\leq \sum_{x\in\mathbb{D}\setminus D(n^{2/(d-2)})} Cn2^{i\alpha}\varepsilon^{-\alpha}2^{-2n}G_{\mathbb{D}}(0,x) \leq C2^{i\alpha}\varepsilon^{-\alpha}n, \end{aligned}\tag{6.22}$$

where we again used the fact that $\sum_{x\in\mathbb{D}} G_{\mathbb{D}}(0,x) = O(2^{2n})$. From (6.19) and (6.22) it follows that the expression in (6.17) can be bounded from above by

$$\exp(-K'cn\varepsilon^{-\alpha}2^{i\alpha})\exp(Cn\varepsilon^{-\alpha}2^{i\alpha}). \tag{6.23}$$

Therefore, it is possible to choose K' large enough such that this bound decreases exponentially with n for all $i \in \{0,\dots,i_{\max}\}$.

Summation over all possible values of i gives

$$\begin{aligned}\mathbb{P}\Big[\bigcup_{i=0}^{i_{\max}}\Big(\mathbb{E}\Big[\sum_{i=0}^{\Lambda_d-1} e_i\tau_{X_d(i)}\chi(X_d(i) \in T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}})\Big|\boldsymbol{\tau}\Big] \geq K'\varepsilon^{1-\alpha}2^{i(\alpha-1)}2^{2n/\alpha}\Big)\Big] \\ \leq cn\exp(-c'n). \end{aligned}\tag{6.24}$$

By Borel-Cantelli argument we obtain

$$\mathbb{E}\Big[\sum_{i=0}^{\Lambda_d-1} e_i\tau_{X_d(i)}\chi(X_d(i) \in T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}})\Big|\boldsymbol{\tau}\Big] \leq K'\varepsilon^{1-\alpha}2^{i(\alpha-1)}2^{2n/\alpha} \tag{6.25}$$

$\mathbb{P}$-a.s. for all i and for n large enough. Combining together (6.13) and (6.25) we get easily the claim of the lemma. $\square$

6.3 The very deep traps

Similarly as in $d = 2$, we show that the very deep traps can be safely ignored.

Lemma 6.3.1. *For every δ and m there exist M such that for n large enough and of $\mathbb{P}$-a.e. $\boldsymbol{\tau}$*

$$\mathbb{P}\big[X \text{ hits } T_M(n) \text{ before } \Lambda(n)|\boldsymbol{\tau}\big] \le \delta. \tag{6.26}$$

Proof. We use Borel-Cantelli argument to prove the lemma. Since

$$\begin{aligned}\mathbb{P}\big[\mathbb{P}[&X \text{ hits } T_M(n) \text{ before } \Lambda(n)|\boldsymbol{\tau}] \ge \delta\big] \\ &\le e^{-\lambda_n\delta}\mathbb{E}\big[\exp\big\{\lambda_n\mathbb{P}[X \text{ hits } T_M(n) \text{ before } \Lambda(n)|\boldsymbol{\tau}]\big\}\big],\end{aligned} \tag{6.27}$$

we should bound the last expectation. We will use a very similar argument as in the previous proof.

$$\begin{aligned}\log\mathbb{E}\big[&\exp\big\{\lambda_n\mathbb{P}[X \text{ hits } T_M(n) \text{ before } \Lambda(n)|\boldsymbol{\tau}]\big\}\big] \\ &\le \log\mathbb{E}\big[\exp\big\{\lambda_n\sum_{x\in\mathbb{D}}\mathbb{P}[X \text{ hits } x \text{ before } \Lambda]\chi(x\in T_M)\big\}\big].\end{aligned} \tag{6.28}$$

Since $\mathbb{P}[x \in T_M] \le cM^{-\alpha}2^{-2n}$, we get

$$\begin{aligned}&\le \sum_{x\in\mathbb{D}}\log\big\{1 + cM^{-\alpha}2^{-2n}\big(\exp\{\lambda_n\mathbb{P}[X \text{ hits } x \text{ before } \Lambda]\} - 1\big)\big\} \\ &\le \sum_{x\in\mathbb{D}} cM^{-\alpha}2^{-2n}\big\{\exp(\lambda_n\mathbb{P}[X \text{ hits } x \text{ before } \Lambda]) - 1\big\}.\end{aligned} \tag{6.29}$$

Let $\lambda_n = n$. We again divide the sum into two parts. For $|x| \le n^{2/(d-2)}$ we use $\mathbb{P}[X \text{ hits } x \text{ before } \Lambda] \le 1$. Hence,

$$\begin{aligned}\sum_{x\in D(n^{2/(d-2)})} &cM^{-\alpha}2^{-2n}\big\{\exp(\lambda_n\mathbb{P}[X \text{ hits } x \text{ before } \Lambda]) - 1\big\} \\ &\le cn^{2d/(d-2)}2^{-2n}e^n\end{aligned} \tag{6.30}$$

and this decreases to 0 as $n \to \infty$.

For $|x| \ge n^{2/(d-2)}$ we use the following simple bound on the hitting probability of a point in $D(n)$. Its proof can be found in Appendix 6.A, Proposition 6.A.2.

$$\mathbb{P}\big[X_d \text{ hits } x \text{ before } D(n)^c\big] \le a_d\big(|x|^{2-d} - n^{2-d}\big) + O\big(|x|^{1-d}\big). \tag{6.31}$$

By the last formula the argument of the exponential in (6.28) is smaller than cn^{-1} and thus

$$\exp\big(\lambda_n\mathbb{P}[X \text{ hits } x \text{ before } \Lambda]\big) - 1 \le cn|x|^{2-d} \tag{6.32}$$

for some large c. We have thus

$$\sum_{x\in\mathbb{D}\setminus D(n^{2/(d-2)})} cM^{-\alpha}2^{-2n}\big\{\exp(\lambda_n\mathbb{P}[X \text{ hits } x \text{ before } \Lambda])-1\big\}$$

$$\leq cM^{-\alpha}2^{-2n}n \sum_{x\in\mathbb{D}\setminus D(n^{2/(d-2)})} |y|^{2-d} \leq cM^{-\alpha}n. \quad (6.33)$$

Inserting (6.30) and (6.33) into (6.27) we get

$$\mathbb{P}\big[\mathbb{P}[X \text{ hits } T_M(n) \text{ before } \Lambda(n)|\boldsymbol{\tau}] \geq \delta\big] \leq c\exp(-n\delta + c'M^{-\alpha}n). \quad (6.34)$$

The proof is finished by taking M large enough. □

6.4 *J is large enough*

We show here that the index of the first bad part of the trajectory is large enough. Since we need that the process $Y^n(t)$ is relevant for our model (in the sense that it reflects the real time change) up to some constant time level t_0 that will be specified later, it is necessary that there is at least $t_0 2^{2n(1-\beta)}$ good parts.

Lemma 6.4.1. *For any δ, k, and for $\mathbb{P}$-a.e. $\boldsymbol{\tau}$ there exist constants m, γ, and β not depending on ε and M such that*

$$\mathbb{P}[J(n) \geq k2^{2n(1-\beta)}|\boldsymbol{\tau}] \geq 1-\delta. \quad (6.35)$$

Proof. We should show that the probability that any of events (i)–(vi) occurs during the first $k2^{2n(1-\beta)}$ parts is very small. Actually, we prove that in the majority of cases it is event (i)—the possibility of exit from $\mathbb{D}(n)$—that occurs as the first. All other events occur with probabilities that are negligible with respect to the probability of (i).

We use $J_{(i)}, \ldots, J_{(vi)}$ to denote the indices of the first part where (i), ..., (vi) occur. By the same reasoning as in $d=2$, we first choose m large enough such that

$$\mathbb{P}[J_{(i)} \geq k2^{2n(1-\beta)}|\boldsymbol{\tau}] \geq 1-\delta/4. \quad (6.36)$$

Further, there exists a constant K satisfying

$$\mathbb{P}[J_{(i)} \leq K2^{2n(1-\beta)}|\boldsymbol{\tau}] \geq 1-\delta/4. \quad (6.37)$$

Therefore, the event $A = \big\{J_{(i)} \in [k2^{2n(1-\beta)}, K2^{2n(1-\beta)}]\big\}$ has the probability that is larger than $1-\delta/2$.

Conditionally on A, we will prove that it is possible to choose β and γ in the way that, for $\mathbb{P}$-a.e. $\boldsymbol{\tau}$,

$$\mathbb{P}\big[\min\{J_{(ii)},\dots,J_{(vi)}\} \le J_{(i)}\big|A,\boldsymbol{\tau}\big] \to 0 \quad \text{as } n\to\infty. \tag{6.38}$$

To achieve this we should prove that probability that any of (ii)–(vi) occurs during one part of the trajectory is much smaller than $2^{2n(\beta-1)}$. This is the result of Lemmas 6.4.2–6.4.8. We will use the following values of the constants γ and β,

$$\gamma = \frac{1}{d} \quad \text{and} \quad \beta = 1-\frac{1}{3d}. \tag{6.39}$$

Event (iv). We first bound the probability of (iv)—exiting a part outside $\mathcal{E}(n)$—because the next lemmas rely on the fact that the parts starts in $\mathcal{E}$. We treat separately the origin since it is the starting point of the first part.

Lemma 6.4.2. *For $\mathbb{P}$-a.e. $\boldsymbol{\tau}$ and for n large enough $0\in\mathcal{E}(n)$.*

Proof.

$$\mathbb{P}[0\notin\mathcal{E}(n)] = \mathbb{P}[\exists x : x\in T_\varepsilon^M \cap D(2^{n\gamma})] \le c2^{nd\gamma}2^{-2n}\varepsilon^{-\alpha}. \tag{6.40}$$

Since $\gamma = 1/d$ the lemma follows by Borel-Cantelli argument. □

The principal part of the estimation of the probability of (iv) is

Lemma 6.4.3. *Let $P_1(n,x)$ be the probability that the simple random walk started at x exits $D_x(2^{n\beta})$ at some site that is not in $\mathcal{E}$. Then for $\mathbb{P}$-a.e. $\boldsymbol{\tau}$ and for every $x\in\mathbb{D}$, $P_1(n,x)\le C2^{n(\gamma d-2)}$.*

Proof. Let $A_x(n)$ denotes the annulus

$$A_x(n) = D_x(2^{n\beta}+2^{n\gamma}) \setminus D_x(2^{n\beta}-2^{n\gamma}). \tag{6.41}$$

We first show that $\mathbb{P}$-a.s. for K_2 large enough

$$|A_x\cap T_\varepsilon^M| \le K_2 2^{n(\beta(d-1)+\gamma-2)} \qquad \text{for all } x\in\mathbb{D}. \tag{6.42}$$

Indeed, since the number of the sites in A_x is bounded by $|A_x|\le c'2^{n(\beta(d-1)+\gamma)}$, we have

$$\begin{aligned}\mathbb{P}\big[|A_x\cap T_\varepsilon^M| \ge K_2 2^{n(\beta(d-1)+\gamma-2)}\big]&\\ \le \exp\big(-\lambda K_2 2^{n(\beta(d-1)+\gamma-2)}\big)&\big\{1+c2^{-2n}\varepsilon^{-\alpha}(e^\lambda-1)\big\}^{c'2^{n(\beta(d-1)+\gamma)}}\\ \le \exp\big\{2^{n(\beta(d-1)+\gamma-2)}&[-\lambda K_2 + c(e^\lambda-1)]\big\}.\end{aligned} \tag{6.43}$$

An easy calculation gives that $\beta(d-1)+\gamma-2\geq 0$ if $d\geq 3$ for our choice of constants. The fact (6.42) then follows by Borel-Cantelli argument.

If (6.42) is true, then there is at most $cK_2 2^{n(\beta(d-1)+\gamma-2)}2^{n\gamma(d-1)}$ points on the border of $D_x(2^{n\beta})$ that are not in $\mathcal{E}$. The probability that X exits $D_x(2^{n\beta})$ in any of such points is $O(2^{-n\beta(d-1)})$ (see [Law91] Lemma 1.7.4). Hence,

$$\begin{aligned}\mathbb{P}_x\big[X \text{ exits } D_x(2^{n\beta}) \text{ in } \mathcal{E}^c\big]& \\ &\leq cK_2 2^{n(\beta(d-1)+\gamma-2)}2^{n\gamma(d-1)}2^{-n\beta(d-1)}\leq C2^{n(\gamma d-2)}. \qquad (6.44)\end{aligned}$$

This finishes the proof. □

Event (v). Next, we bound the probability that (v) happens.

Lemma 6.4.4. *Let $P_2(n,x)$ be the probability that the simple random walk started at x hits a deep trap in $D_x(2^{n\beta})\cap A_x(n)$ before exiting $D_x(2^{n\beta})$. Then for $\mathbb{P}$-a.e. $\boldsymbol{\tau}$ and for all $x\in\mathbb{D}$, $P_2(n,x)\leq C2^{n(\beta+\gamma-2)}$.*

Proof. According to (6.42) there is $\mathbb{P}$-a.s. at most $K_2 2^{n(\beta(d-1)+\gamma-2)}$ deep traps in $D_x(2^{n\beta})\cap A_x(n)$. The probability that the walk hits one particular such trap y is by (6.129) from Appendix 6.A bounded from above by $c|x-y|^{2-d}$. There exists constant C such that for all $y\in A_x(n)$, $|x-y|^{2-d}\leq C2^{n\beta(2-d)}$. The required probability is then bounded by

$$CK_2 2^{n(\beta(d-1)+\gamma-2)}2^{n\beta(2-d)}=C2^{n(\beta+\gamma-2)}. \qquad (6.45)$$

This completes the proof. □

Events (ii) and (vi). The next preparatory lemma will serve to bound (ii)—hitting of two deep traps in one part—and also in some sense to bound (vi). It is more precise than is needed here, but this more precise result will be used later.

Lemma 6.4.5. *Let*

$$V_x(n)=\sum_{y\in T_\varepsilon^M}\mathbb{P}_x\big[X_d \textit{ hits } y \textit{ before exiting } D_x(2^{n\beta})|\boldsymbol{\tau}\big]. \qquad (6.46)$$

Then for any δ and $\mathbb{P}$-a.e. $\boldsymbol{\tau}$ there exists n_0 such that for $n\geq n_0$ and for all $x\in\mathcal{E}(n)$

$$(1-\delta)\mathcal{K}p_\varepsilon^M 2^{2n(\beta-1)}\leq V_x(n)\leq(1+\delta)\mathcal{K}p_\varepsilon^M 2^{2n(\beta-1)} \qquad (6.47)$$

for some constant $\mathcal{K}$ depending only on the dimension.

Proof. We follow a similar procedure as in the proof of Lemma 5.5.2 in the two-dimensional case. Without lost of generality we can suppose that x is the origin. First, we define another constant ν,

$$\gamma < \nu = 1 - 2/(3d) < \beta. \tag{6.48}$$

We divide the sum in (6.46) into three parts. We use Σ_1 to denote the sum over $y \in T_\varepsilon^M \cup (D(2^{n\beta} - 2^{n\nu+1}) \setminus B(2^{n\nu}))$, Σ_2 to denote the sum over $y \in T_\varepsilon^M \cap B(2^{n\nu})$, and Σ_3 to denote the sum over $y \in T_\varepsilon^M \cap (D(2^{n\beta}) \setminus D(2^{n\beta} - 2^{n\nu+1}))$. The reason why we introduce the third sum is the error term in (6.130), which is too large for the traps that are too close to the border of $D(2^{n\beta})$.

As in $d = 2$ the main contribution comes from Σ_1, so we treat it first. We cover the ball $D(2^{n\beta})$ by cubes whose edge-length is $2^{n\nu}$. Let $p_\varepsilon^M = \varepsilon^{-\alpha} - M^{-\alpha}$. It is not difficult to show that $\mathbb{P}$-a.s.

$$\big|B_x(2^{n\nu}) \cap T_\varepsilon^M\big| \in \big((1-\delta)2^{nd\nu} p_\varepsilon^M 2^{-2n}, (1+\delta)2^{nd\nu} p_\varepsilon^M 2^{-2n}\big) \tag{6.49}$$

for all $x \in \mathbb{D}$. Indeed, let

$$F_x = \big\{|B_x(2^{n\nu}) \cap T_\varepsilon^M| \geq (1+\delta)2^{nd\nu} p_\varepsilon^M 2^{-2n}\big\}. \tag{6.50}$$

Then for any small η and for n large enough

$$\begin{aligned} \mathbb{P}[F_x] &\leq \exp\big(-\lambda(1+\delta)2^{n(d\nu-2)} p_\varepsilon^M\big)\big\{1 + (e^\lambda - 1)(1+\eta) p_\varepsilon^M 2^{-2n}\big\}^{2^{nd\nu}} \\ &\leq \exp\big\{2^{n(d\nu-2)} p_\varepsilon^M [-\lambda(1+\delta) + (e^\lambda - 1)(1+\eta)]\big\}. \end{aligned} \tag{6.51}$$

Since for any δ one can choose λ and η small enough such that the exponent in the last expression is negative, we have

$$\mathbb{P}[F_x] \leq \exp(-c 2^{n(d\nu-2)}) \tag{6.52}$$

for n large enough. Summation over all x gives

$$\mathbb{P}\Big[\bigcup_x F_x\Big] \leq 2^{nd} \exp\big(-c 2^{n(d\nu-2)}\big). \tag{6.53}$$

Since $d\nu - 2 > 0$, the upper bound for (6.49) is finished. The proof of the lower bound is completely analogous.

We can now actually estimate Σ_1. Let $G = 2^{n\nu}\mathbb{Z}^d$ and let

$$H = \big\{x \in G \setminus \{0\} : B_x(2^{n\nu}) \cap D(2^{n\beta} - 2^{n\nu+1}) \neq \emptyset\big\}. \tag{6.54}$$

We use $\mathcal{D}$ to denote $D\big(2^{n\beta} - 2^{n\nu+1}\big) \setminus B\big(2^{n\nu}\big)$. Using estimation (6.130) from Appendix 6.A we get

$$\begin{aligned}\Sigma_1 &\le \sum_{y\in T_\varepsilon^M\cap\mathcal{D}} a_d\big\{|y|^{2-d} - 2^{n\beta(2-d)} + O(|y|^{1-d})\big\}\big(1+O(2^{n\beta}-|y|)^{2-d}\big)\\ &\le \sum_{x\in H}\sum_{\substack{y\in T_\varepsilon^M\\ y\in B_x(2^{n\nu})}} a_d\big\{|y|^{2-d} - 2^{n\beta(2-d)} + O(|y|^{1-d})\big\}\big(1+O(2^{n\beta}-|y|)^{2-d}\big).\end{aligned} \tag{6.55}$$

For any $y \in B_x(2^{n\nu})$, $x \in H$,

$$\big||y|^{2-d} - |x|^{2-d}\big| \le c2^{n\nu}|x|^{1-d}, \tag{6.56}$$

By the last fact and (6.49) the sum Σ_1 can be bounded by

$$\Sigma_1 \le \sum_{x\in H}(1+\delta)2^{nd\nu}p_\varepsilon^M 2^{-2n}a_d\big\{|x|^{2-d} - 2^{n\beta(2-d)} + c2^{n\nu}|x|^{1-d}\big\} + \mathcal{R}, \tag{6.57}$$

where

$$\mathcal{R} = \sum_{x\in H}\sum_{\substack{y\in T_\varepsilon^M\\ y\in B_x(2^{n\nu})}} a_d\big\{|y|^{2-d} - 2^{n\beta(2-d)} + O(|y|^{1-d})\big\}O(2^{n\beta}-|y|)^{2-d}. \tag{6.58}$$

Every site y from the last summation satisfies $|y| \le 2^{n\beta} - 2^{n\nu}$. Therefore, $O(2^{n\beta}-|y|)^{2-d} = O(2^{n\nu})^{2-d}$. Hence, the error term $\mathcal{R}$ is much smaller than the sum in (6.57).

We now estimate the sum in (6.57). Replacing the summation by integration and making again an error of order $2^{n\nu}|x|^{1-d}$ we get

$$\begin{aligned}\Sigma_1 &\le (1+\delta)p_\varepsilon^M 2^{-2n}\int_{\mathcal{D}} a_d\big\{|x|^{2-d} - 2^{n\beta(2-d)} + c2^{n\nu}|x|^{1-d}\big\}dx + \mathcal{R}\\ &\le (1+\delta)\mathcal{K}p_\varepsilon^M 2^{2n(\beta-1)}(1+o(1)).\end{aligned} \tag{6.59}$$

The constant $\mathcal{K}$ can be explicitly calculated, $\mathcal{K} = a_d\omega_d/2 - 1/d$, with ω_d being the surface of d-dimensional unit sphere.

The lower bound for Σ_1 can be obtained in the same way. It is actually much simpler, because the lower bound (6.128) on hitting probability is less complicated than the upper bound (6.130). Therefore, there is no complication with the error term $\mathcal{R}$. We get

$$\Sigma_1 \ge (1-2\delta)\mathcal{K}p_\varepsilon^M 2^{2n(\beta-1)}. \tag{6.60}$$

We should still bound Σ_2 and Σ_3. To estimate Σ_2 we need a finer description of the homogeneity of the environment. Let $i_{\max}$ be the smallest integer satisfying $2^{n\gamma+i} \geq 2^{n\nu}$, i.e. $i_{\max} \sim n(\nu-\gamma)$. Then for $\mathbb{P}$-a.e. $\boldsymbol{\tau}$, for all $i \in \{-1, 0, \dots, i_{\max}\}$, and for all $x \in \mathbb{D}$

$$\big|B_x(2^{n\gamma+i}) \cap T_\varepsilon^M\big| \leq n^5\big(1 \vee 2^{nd\gamma+id}2^{-2n}\big). \tag{6.61}$$

To see this, we first fix $i \in \{-1, \dots, i_{\max}\}$. Then for any $x \in \mathbb{D}$ we have

$$\begin{aligned} &\mathbb{P}\big[|B_x(2^{n\gamma+i}) \cap T_\varepsilon^M| \geq (1 \vee 2^{nd\gamma+id}2^{-2n})n^5\big] \\ &\quad \leq \exp\big(-\lambda n^5(1 \vee 2^{nd\gamma+id}2^{-2n})\big)\big\{1 + c(e^\lambda - 1)\varepsilon^{-\alpha}2^{-2n}\big\}^{2^{nd\gamma+id}} \\ &\quad \leq C\exp(-c\lambda n^5). \end{aligned} \tag{6.62}$$

By summation over all x and i we get the upper bound $cn2^{nd}e^{-\lambda n^5}$ for the probability of the complement of the event in (6.61). Therefore, (6.61) is true $\mathbb{P}$-a.s. for n large enough.

To bound Σ_2 we cover the cube $B(2^{n\nu})$ be the same system of objects as in $d = 2$. By (6.129) from Appendix 6.A and by (6.61) we get

$$\begin{aligned} \Sigma_2 \leq C \sum_{i=-1}^{n(\nu-\gamma)} n^5\big(1 \vee 2^{nd\gamma+id}2^{-2n}\big)2^{(\gamma n+i)(2-d)} \\ \leq C \sum_{i=-1}^{n(\nu-\gamma)} n^5(2^{(\gamma n+i)(2-d)} \vee 2^{2n\gamma+2i-2n}). \end{aligned} \tag{6.63}$$

The fist term in the parentheses is decreasing in i and the second one is increasing. Hence, the sum can be bounded by $Cn^6(2^{\gamma n(2-d)} \vee 2^{2n(\nu-1)})$. However, both terms $2^{\gamma n(2-d)}$ and $2^{2n(\nu-1)}$ are much smaller than $2^{2n(\beta-1)}$ for our choice of constants. This means that $\Sigma_2 \ll \Sigma_1$.

It remains to estimate the sum Σ_3, that is the sum over y satisfying $y \in T_\varepsilon^M \cap (D(2^{n\beta}) \setminus D(2^{n\beta} - 2^{n\nu+1}))$. However, this sum can be bounded in a similar way as the probability of hitting a deep trap in annulus $A_x(n)$ was bounded in Lemma 6.4.4. Following the same reasoning (with γ replaced by ν) we get $\Sigma_3 \leq C2^{n(\beta+\nu-2)} \ll 2^{2n(\beta-1)}$. That means again $\Sigma_3 \ll \Sigma_1$. This completes the proof. $\square$

Before finishing the bound on the probability of hitting two deep traps in one part, we should proof another lemma. It will serve not only to bound the probability of (ii), but it also bounds the probability of (vi). Recall that $\mathcal{B}(n)$ denotes the set of bad deep traps (see (6.7)).

Lemma 6.4.6. *Let*

$$W_x(n) = \sum_{y \in \mathcal{B}(n)} \mathbb{P}_x[X_d \textit{ hits } y \textit{ before exiting } D_x(2^{n\beta})|\boldsymbol{\tau}]. \tag{6.64}$$

Then for $\mathbb{P}$*-a.e.* $\boldsymbol{\tau}$ *and for all* $x \in \mathcal{E}(n)$

$$W_x(n) = o(2^{2n(\beta-1)}). \tag{6.65}$$

Proof. The proof is very similar to the previous one. We divide the sum into three parts in the same way as before, we keep the notation $\Sigma_1, \dots, \Sigma_3$ for the parts of the sum. Since $\mathcal{B}(n) \subset T_\varepsilon^M$, it follows from the previous proof that Σ_2 and Σ_3 are $o(2^{2n(\beta-1)})$. Hence, it remains to bound from above the sum Σ_1. This bound will be result of the same calculation as before if we show that

$$\big|B_x(2^{n\nu}) \cap \mathcal{B}(n)\big| = o\big(\big|B_x(2^{n\nu}) \cap T_\varepsilon^M(n)\big|\big) = o\big(2^{n(d\nu-2)}\big) \tag{6.66}$$

for all $x \in \mathbb{D}$ (compare it with (6.49)). We will show that

$$\big|B_x(2^{n\nu}) \cap \mathcal{B}(n)\big| \le 2^{nd\nu} n^{5\alpha/(1-\alpha)+2} 2^{n(\gamma d-4)} 2^{nd(1-\nu)} \equiv f(n). \tag{6.67}$$

This bound is clearly not the optimal one, but it is sufficient for our purposes since the right-hand side of the last expression is clearly $o(2^{n(d\nu-2)})$ as can be easily seen from

$$d\nu + \gamma d - 4 + d(1-\nu) = d\nu - 3 + 2/3 < d\nu - 2. \tag{6.68}$$

To show (6.67) we use the standard strategy. Let G denote the grid $2^{n\nu}\mathbb{Z}^d$. Then, $|G \cap \mathbb{D}| \le c2^{nd(1-\nu)}$. We use A to denote the event that there exists a cube with edge-length $2^{n\nu}$ containing more than $f(n)$ bad sites. If A is true, then there is also a cube with edge-length $2 \cdot 2^{n\nu}$ centred on G that contains more than $f(n)$ bad sites. Therefore,

$$\begin{aligned}\mathbb{P}[A] &\le \sum_{x \in G \cap \mathbb{D}} \mathbb{P}\big[|B_x(2^{n\nu+1}) \cap \mathcal{B}| \ge f(n)\big] \\ &\le C2^{nd(1-\nu)} \mathbb{P}\big[|B(2^{n\nu+1}) \cap \mathcal{B}| \ge f(n)\big].\end{aligned} \tag{6.69}$$

Using the definition of $\mathcal{B}$ it is not difficult to show that

$$\mathbb{P}[x \in \mathcal{B}] \le c\varepsilon^{-2\alpha} 2^{n(\gamma d-4)} n^{5\alpha/(1-\alpha)}. \tag{6.70}$$

Therefore, we have by Markov inequality

$$\begin{aligned}&P\big[|B(2^{n\nu+1}) \cap \mathcal{B}| \ge f(n)\big] \\ &\qquad \le f(n)^{-1}\mathbb{E}\Big[\sum_{x \in B(2^{n\nu+1})} \chi\{x \in \mathcal{B}\}\Big] \le C\varepsilon^{-2\alpha} n^{-2} 2^{-nd(1-\nu)}.\end{aligned} \tag{6.71}$$

Putting this into (6.69) we obtain

$$\mathbb{P}[A] \le Cn^{-2}. \tag{6.72}$$

Therefore, (6.67) follows by Borel-Cantelli argument, and the proof of the lemma is completed. $\square$

We use the previous two lemmas to show

Lemma 6.4.7. *For $\mathbb{P}$-a.e $\boldsymbol{\tau}$ and for all $x \in \mathcal{E}$, the probability that the simple random walk started at x visits two deep traps before exiting $D_x(2^{n\beta})$ is $o(2^{2n(\beta-1)})$.*

Proof. We have

$$\begin{aligned}\mathbb{P}_x\big[X_d \text{ hits two deep traps}\big] \\ \le \mathbb{P}\big[X_d \text{ hits a second deep trap}\big|X_d \text{ hits } T_\varepsilon^M \setminus \mathcal{B}\big]\mathbb{P}_x\big[X_d \text{ hits } T_\varepsilon^M \setminus \mathcal{B}\big] \\ + \mathbb{P}\big[X_d \text{ hits a second deep trap}\big|X_d \text{ hits } \mathcal{B}\big]\mathbb{P}_x\big[X_d \text{ hits } \mathcal{B}\big]. \quad (6.73)\end{aligned}$$

By Lemma 6.4.5,

$$\mathbb{P}_x\big[X_d \text{ hits } T_\varepsilon^M \setminus \mathcal{B}\big] = O\big(2^{2n(1-\beta)}\big). \quad (6.74)$$

Similarly, by Lemma 6.4.6,

$$\mathbb{P}_x\big[X_d \text{ hits } \mathcal{B}\big] = o\big(2^{2n(1-\beta)}\big). \quad (6.75)$$

If the fist deep trap is not bad, we can apply strong Markov property similarly as in $d = 2$ to show

$$\mathbb{P}\big[X_d \text{ hits second deep trap}\big|X_d \text{ hits } T_\varepsilon^M \setminus \mathcal{B}\big] = O\big(2^{2n(1-\beta)}\big). \quad (6.76)$$

Inserting (6.74)–(6.76) into (6.73) it is easy to finish the proof. □

Event (iv). The last bad event that we should treat is (iii)—the possibility of the return to a deep trap x after exiting the ball $D_x(2^{n\gamma})$ around it.

Lemma 6.4.8. *Let $x \in \mathbb{D}$ satisfies $\operatorname{dist}(x, \mathbb{D}^c) \ge 2^{n\gamma}$. We use $p_{\rm ret}(x)$ to denote the probability that the simple random started at x that have exited $D_x(2^{n\gamma})$ returns to x before Λ_d. Then this probability satisfies $p_{\rm ret}(x) \le C2^{n\gamma(2-d)}$.*

Proof. Since

$$G_{\mathbb{D}}(x, x) = G_{D_x(2^{n\gamma})}(x, x) + p_{\rm ret}(x)G_{\mathbb{D}}(x, x), \quad (6.77)$$

we have

$$p_{\rm ret}(x) = 1 - \frac{G_{D(2^{n\gamma})}(0, 0)}{G_{\mathbb{D}}(x, x)} \le 1 - \frac{G_{D(2^{n\gamma})}(0, 0)}{G_{D(2m2^n)}(0, 0)} \le C2^{n\gamma(2-d)} \quad (6.78)$$

by (6.126) from Appendix 6.A. □

We can now finish the proof of (6.38), and consequently also the proof of Lemma 6.4.1. By Lemma 6.4.3 we have

$$\mathbb{P}[J_{(iv)} < J_{(i)}|A, \boldsymbol{\tau}] \\ \leq \sum_{i=0}^{K2^{2n(1-\beta)}} \mathbb{P}\big[\text{(iv) is true for } X[j_i^n, j_{i+1}^n)\big] \leq C2^{2n(1-\beta)}2^{n(\gamma d-2)} \quad (6.79)$$

and it converges to 0 as $n \to \infty$ for our choice of constants. A very similar calculation and Lemma 6.4.4 give

$$\mathbb{P}[J_{(v)} < J_{(i)}|A, \boldsymbol{\tau}] \leq C2^{2n(1-\beta)}2^{n(\gamma d-2)} \to 0 \qquad \text{as } n \to \infty. \quad (6.80)$$

Further, we suppose that $0 \in \mathcal{E}$ which is possible because of Lemma 6.4.2. We use B to denote $A \cap (J_{(iv)} \geq J_{(i)})$. Using Lemma 6.4.6 we get

$$\mathbb{P}[J_{(vi)} < J_{(i)}|B, \boldsymbol{\tau}] \leq \sum_{i=0}^{K2^{2n(1-\beta)}} W_{x_i^n}(n) \to 0 \qquad \text{as } n \to \infty, \quad (6.81)$$

and similarly, by Lemma 6.4.7,

$$\mathbb{P}[J_{(ii)} < J_{(i)}|B, \boldsymbol{\tau}] \to 0 \qquad \text{as } n \to \infty. \quad (6.82)$$

It remains to estimate the probability of (iii). We first bound the number N of visited deep traps before Λ. Lemma 6.4.5 together with Markov inequality give

$$\mathbb{P}[N \geq n^{1/2}|B, \boldsymbol{\tau}] \leq Cn^{-1/2}. \quad (6.83)$$

Further,

$$\mathbb{P}[J_{(iii)} < J_{(i)}|B, \boldsymbol{\tau}] \\ \leq \mathbb{P}[J_{(iii)} < J_{(i)}|N < n^{1/2}, B, \boldsymbol{\tau}]\mathbb{P}[N < n^{1/2}|B, \boldsymbol{\tau}] + \mathbb{P}[N \geq n^{1/2}|B, \boldsymbol{\tau}]. \quad (6.84)$$

However, from Lemma 6.4.8 it follows that

$$\mathbb{P}[J_{(iii)} < J_{(i)}|N < n^{1/2}, B, \boldsymbol{\tau}] \leq Cn^{1/2}2^{n\gamma(2-d)}, \quad (6.85)$$

and so

$$\mathbb{P}[J_{(iii)} < J_{(i)}|B, \boldsymbol{\tau}] \leq Cn^{1/2}2^{n\gamma(2-d)} + n^{-1/2} \to 0. \quad (6.86)$$

The claim (6.38) follows then from (6.79)–(6.86). $\square$

6.5 Properties of the score

As in the proof for $d = 2$, we define the family of random variables s_x indexed by vertices of $\mathbb{Z}^d$. The random variable s_x has the same distribution as the score of the part of X that is started at x. Since all good parts of trajectory start in $\mathcal{E}(n)$ and, more over, in the distance larger than $2^{n\beta}$ from $\mathbb{D}(n)^c$, we are interested only in $x \in \mathcal{E}_0(n)$, where the set $\mathcal{E}_0(n)$ is defined by

$$\mathcal{E}_0(n) = \big\{x \in \mathcal{E}(n) : \operatorname{dist}\big(x, \mathbb{D}(n)^c\big) \geq 2^{n\beta}\big\}. \tag{6.87}$$

To study the distribution of the variables s_x we apply the same strategy as in $d = 2$. Let $h(x)$ be a function satisfying

$$h(x) \geq 1/n, \qquad \lim_{x\to\infty} h(x) = 0, \tag{6.88}$$

and

$$|L(2^{2n/\alpha}x) - 1| = o(h(n)) \quad \text{for all } x \geq \varepsilon. \tag{6.89}$$

Further, let $z_n(i)$ be a sequence satisfying $\varepsilon = z_n(0) < z_n(1) < \dots < z_n(R) = M$, and $z_n(i+1) - z_n(i) \in (h(n), 2h(n))$ for all $i \in \{0, \dots, R-1\}$. Let p_i^n denote the factor

$$p_i^n = \frac{1}{z_n(i)^\alpha} - \frac{1}{z_n(i+1)^\alpha}. \tag{6.90}$$

Lemma 6.5.1. *Let $\mathcal{P}_x(n,i)$ be the probability that the simple random walk started at x hits the set $T_{z_n(i)}^{z_n(i+1)}$ before exiting from $D_x(2^{n\beta})$. Then for any $\delta > 0$ and $\mathbb{P}$-a.e. $\boldsymbol{\tau}$ there is n_0 such that for all $n \geq n_0$, for all $x \in \mathcal{E}_0(n)$, and for all $i \in \{0, \dots, R-1\}$*

$$\mathcal{P}_x(n,i) \in \big(\mathcal{K}(1-\delta)2^{2n(\beta-1)}p_i^n, \mathcal{K}(1+\delta)2^{2n(\beta-1)}p_i^n\big). \tag{6.91}$$

Proof. We will need the following technical claim

Lemma 6.5.2. *For any $\delta > 0$ there exists n_0, such that for $n \geq n_0$*

$$\mathbb{P}\big[0 \in T_{z_n(i)}^{z_n(i+1)}\big] \in \big((1-\delta)2^{-2n}p_i^n, (1+\delta)2^{-2n}p_i^n\big). \tag{6.92}$$

Proof. Let $g(x) = L(x) - 1$. Then we have

$$p_i^n = 2^{-2n}\Big[\Big(\frac{1}{z_n(i)^\alpha} - \frac{1}{z_n(i+1)^\alpha}\Big) + \frac{g\big(2^{2n/\alpha}z_n(i)\big)}{z_n(i)^\alpha} - \frac{g\big(2^{2n/\alpha}z_n(i+1)\big)}{z_n(i+1)^\alpha}\Big]. \tag{6.93}$$

We should thus show that

$$\frac{g\big(2^{2n/\alpha}z_n(i)\big)}{z_n(i)^\alpha} - \frac{g\big(2^{2n/\alpha}z_n(i+1)\big)}{z_n(i+1)^\alpha} = o\Big(\frac{1}{z_n(i)^\alpha} - \frac{1}{z_n(i+1)^\alpha}\Big). \tag{6.94}$$

However, this is obviously true since $z_n(i)^{-\alpha} - z_n(i+1)^{-\alpha} \asymp h(n)$, and, as follows from (6.89), $g\big(2^{2n/\alpha}z_n(i)\big) = o(h(n))$. $\square$

Using the previous lemma we show

Lemma 6.5.3. *For all $x \in \mathbb{D}$ and for all $i \in \{0, \dots, R-1\}$*

$$\big|B_x(2^{n\nu}) \cap T_{z_n(i)}^{z_n(i+1)}\big| \in \big((1-\delta)2^{n(d\nu-2)}p_i^n, (1+\delta)2^{n(d\nu-2)}p_i^n\big). \tag{6.95}$$

Proof. It is easy to show this claim using the same reasoning as in the proof of (6.49). Two additional observation are necessary. First, $p_i^n 2^{n(d\nu-2)} \geq cn^{-1}2^{n(d\nu-2)} \to \infty$ as $n \to \infty$ as follows from (6.88). The modified version of inequality (6.51) is thus suitable for an application of the Borel-Cantelli lemma. Second, the additional summation over i does not create any problem since $R \leq Cn$ for come constant C depending only on ε and M. □

We can now finish the proof of Lemma 6.5.1. We use $V_{x,i}(n)$ to denote

$$V_{x,i}(n) = \sum_{y \in T_{z_n(i)}^{z_n(i+1)}} \mathbb{P}_x\big[X_d \text{ hits } y \text{ before exiting } D_x(2^{n\beta})\big|\boldsymbol{\tau}\big]. \tag{6.96}$$

The same procedure as in the proof of Lemma 6.4.5 together with the previous lemma give

$$(1-\delta)\mathcal{K}p_i^n 2^{2n(\beta-1)} \leq V_{x,i}(n) \leq (1+\delta)\mathcal{K}p_i^n 2^{2n(\beta-1)}. \tag{6.97}$$

The proof is then finished using the previous expression, Lemma 6.4.7, and Bonferroni's inequalities. Indeed,

$$\mathcal{P}_x(n,i) \leq V_{x,i}(n) \leq (1+\delta)\mathcal{K}p_i^n 2^{2n(\beta-1)}. \tag{6.98}$$

Similarly,

$$\begin{aligned} \mathcal{P}_x(n,i) &\geq V_{x,i}(n) - \mathbb{P}\big[X_d \text{ hits two traps from } T_{z_n(i)}^{z_n(i+1)}\big] \\ &\geq V_{x,i}(n) - \mathbb{P}\big[X_d \text{ hits two traps from } T_\varepsilon^M\big] \\ &\geq (1-2\delta)\mathcal{K}p_i^n 2^{2n(\beta-1)}. \end{aligned} \tag{6.99}$$

This completes the proof. □

The next lemma describes the distribution of s_x's.

Lemma 6.5.4. *For $\mathbb{P}$-a.e. random environment $\boldsymbol{\tau}$,*

$$\begin{aligned} \lim_{n\to\infty} \max_{x \in \mathcal{E}_0(n)} \frac{1 - \mathbb{E}[\exp(-\frac{\lambda s_x}{2^{2n/\alpha}})|s_x < \infty, \boldsymbol{\tau}]}{2^{2n(\beta-1)}} &= F(\lambda), \\ \lim_{n\to\infty} \min_{x \in \mathcal{E}_0(n)} \frac{1 - \mathbb{E}[\exp(-\frac{\lambda s_x}{2^{2n/\alpha}})|s_x < \infty, \boldsymbol{\tau}]}{2^{2n(\beta-1)}} &= F(\lambda), \end{aligned} \tag{6.100}$$

with

$$F(\lambda) = F(\lambda; \varepsilon, M, \alpha) = \mathcal{K}\bigg(p_\varepsilon^M - \int_\varepsilon^M \frac{\alpha}{1+\lambda G(0)z} \cdot \frac{1}{z^{\alpha+1}} dz\bigg). \tag{6.101}$$

Proof. When the process X hits a deep trap y during its visit of $D_x(2^{n\beta})$ and when the part started at x is good, then the score of this part is simply the time spent in y before the exit from $D_y(2^{n\gamma})$. The process X hits y a geometrical number times. The mean of this geometrical variable is $G_{D(2^{n\gamma})}(0,0)$. Each visit take an exponential time with mean τ_y. Using the expression (6.126) from Appendix we get the following formula for the conditional Laplace transform of s_x.

$$\mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{2n/\alpha}}\Big)\Big|\tau_y, s_x<\infty\Big] = \frac{1}{1+\lambda\tau_y 2^{-2n/\alpha}G(0)\big(1+o(1)\big)}. \tag{6.102}$$

The probability that $s_x=\infty$ is $o(2^{2n(\beta-1)})$ as follows from the previous section. We have thus by the same calculation as in $d=2$

$$\mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{2n/\alpha}}\Big)\Big|s_x<\infty,\boldsymbol{\tau}\Big] = \mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{2n/\alpha}}\Big)\Big|\boldsymbol{\tau}\Big]\big(1+o(2^{2n(\beta-1)})\big). \tag{6.103}$$

The last expectation can be estimated using Lemma 6.5.4 and (6.102),

$$\mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{2n/\alpha}}\Big)\Big|\boldsymbol{\tau}\Big] \geq$$
$$\big(1-(1+\delta)\mathcal{K}p_\varepsilon^M 2^{2n(1-\beta)}\big)+\mathcal{K}2^{2n(\beta-1)}\sum_{i=1}^{R}\frac{p_i^n(1-\delta)}{1+\lambda z_n(i)G(0)(1+o(1))}. \tag{6.104}$$

The last expression can be bounded from bellow for n large by

$$1-\mathcal{K}2^{2n(\beta-1)}\Big(p_\varepsilon^M-\int_\varepsilon^M\frac{\alpha}{1+\lambda G(0)z}\cdot\frac{1}{z^{\alpha+1}}\,dz\Big)-\delta C 2^{2n(\beta-1)}p_\varepsilon^M. \tag{6.105}$$

The required upper bound on the first expression in (6.100) is then easy to obtain from the last expression and (6.103). The proof of the lower bound for the second expression in (6.100) is very similar. $\square$

Before proving the convergence of sequence Y^n we should define the auxiliary random variables $\tilde{s}_i^n$. We require that they satisfy the same relation as s_x's in the limit, i.e.

$$\mathbb{E}\Big[\exp\Big(-\frac{\lambda}{2^{2n/\alpha}}\tilde{s}_i^n\Big)\Big] = 1-F(\lambda)2^{2n(\beta-1)}. \tag{6.106}$$

Now, we have all ingredients to show the convergence of the sequence of time changes Y^n.

Proposition 6.5.5. *For $\mathbb{P}$-a.e. realisation of the environment, the sequence of processes $Y^n(t)$ converges weakly in the Skorokhod topology on $D([0,\infty))$ to the Lévy process $Y(t)$, whose Lévy measure ρ is given by*

$$\rho(dx) = \frac{\alpha\mathcal{K}}{G(0)} \int_{\varepsilon}^{M} \frac{1}{z^{\alpha+2}} \exp\Big(-\frac{x}{zG(0)}\Big) dz\, dx. \tag{6.107}$$

Proof. The proof is exactly the same as this of Proposition 5.6.5 in the two-dimensional case. The only difference is that all factors $2^{n/\alpha}$ in formulas in $d=2$ should be replaced by $2^{2n/\alpha}$, and similarly $n^{1-\alpha-\beta}$ should be replaced by $2^{2n(1-\beta)}$. □

6.6 Proof of aging for function $R(t_w, t_w + t)$

We use the convergence of Y^n that we have shown in the previous section to prove the first part of Theorem 6.1.1 as in the two-dimensional case. The limiting function $R(\theta)$ can be calculated explicitly using the arcsine law for Lévy processes and is actually the same as in $d=2$. The proof of this result is very similar to that one in $d=2$. We do not repeat it completely.

Proposition 6.6.1. *For $\mathbb{P}$-a.e. realisation of the environment $\boldsymbol{\tau}$*

$$\lim_{t\to\infty} R(t, t+\theta t) = \int_0^{1/1+\theta} \frac{\sin\alpha\pi}{\pi} u^{\alpha-1}(1-u)^{-\alpha}\, du \equiv R(\theta) \tag{6.108}$$

for every $0<\theta<\infty$.

Proof. As before we add to Y^n and Y an auxiliary Lévy process Z with the Lévy measure

$$\rho'(dx) = \frac{\alpha\mathcal{K}}{G(0)} \Big(\int_0^{\varepsilon} + \int_M^{\infty}\Big) \frac{1}{z^{\alpha+2}} \exp\Big(-\frac{x}{zG(0)}\Big) dz\, dx. \tag{6.109}$$

According to the value of t, we choose an integer $n = n(t)$ satisfying

$$1 \le \frac{t}{2^{2n(t)/\alpha}} < 2^{2/\alpha}. \tag{6.110}$$

We rescale the time t. The scaled value of t is defined by $s = s(t) = t2^{-2n(t)/\alpha}$. The choice of constants M and ε can be done in the same way as in Part I of the proof in the two-dimensional case. Only obvious changes as in the proof of Proposition 6.5.5 are needed. All lemmas that are needed for the adaptation have been already proved.

Part II deals only with convergence of the range of $\tilde{Y}^n$ and is completely independent of dimension. Part III of the two-dimensional proof requires only one modification. We should only replace $D_y(2^n n^{-\kappa})$ by $D_y(2^{n\gamma})$.

The last part of the proof in $d = 2$ was the proof of Lemma 5.7.4. In this lemma we proved that if s and $(1+\theta)s$ fall into one jump of the Lévy process, then X is at times t and $(1+\theta)t$ with large probability at the same site. We should prove an equivalent of this lemma here, because the original proof depends on the fact that we have worked in $d = 2$. As before we define $E_3(n, s)$ by

$$\begin{aligned} E_3(n,s) = \big\{ &\operatorname{dist}(s, \mathcal{R}_n) > 2\delta_2, \operatorname{dist}((1+\theta)s, \mathcal{R}_n) > 2\delta_2 \\ &\text{and } (s, (1+\theta)s) \cap \mathcal{R}_n = \emptyset \big\}. \end{aligned} \tag{6.111}$$

We use y to denote the deep trap that caused the jump of $\tilde{Y}^n$ where s and $(1+\theta)s$ are.

Lemma 6.6.2.

$$\begin{aligned} &\lim_{t\to\infty} \mathbb{P}\big[X(t) = y \big| E_3\big(n(t), s(t)\big)\big] \\ &\qquad = \lim_{t\to\infty} \mathbb{P}\big[X\big((1+\theta)t\big) = y \big| E_3\big(n(t), s(t)\big)\big] = 1. \end{aligned} \tag{6.112}$$

Proof. We show the lemma only for $X(t)$, the proof for $X((1+\theta)t)$ is very similar. Let t_1 denote the time when X arrives the first time to y, and let t_2 is the moment of the last visit of y before the exit from $D_y(2^{n\gamma})$. Let U denote the time that X spends during excursions from y between t_1 and t_2. We bound the expected value of U.

$$\begin{aligned} \mathbb{E}[U] \le &\sum_{z \in D_y(2^{n\gamma}) \setminus \{y\}} G_{D_y(2^{n\gamma})}(y,z)\tau_z \\ &\qquad \le \sum_{z\in\mathbb{D}} G_{\mathbb{D}}(0,z)\tau_z \chi\{\tau_z \le n^{-5/(1-\alpha)} 2^{2n/\alpha}\}. \end{aligned} \tag{6.113}$$

In the last inequality we used the fact that $y \notin \mathcal{B}(n)$, because, owing to condition (vi), y is necessarily not bad. The last expression can be bounded using the formulas (6.13) and (6.25). Namely, let $i_{\max}$ be defined as after (6.13), and let $i_{\min}$ be an integer satisfying

$$n^{-5/(1-\alpha)} 2^{2n/\alpha} \le 2^{-i_{\min}} \varepsilon 2^{2n/\alpha} < 2n^{-5/(1-\alpha)} 2^{2n/\alpha}. \tag{6.114}$$

Then (6.113) is smaller than

$$\sum_{z\in\mathbb{D}} G_{\mathbb{D}}(0,z)\tau_z\chi\{\tau_z \le 1\} + \sum_{i_{\min}}^{i_{\max}} G_{\mathbb{D}}(0,z)\tau_z\chi\big\{z \in T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\big\}$$
$$\le C2^{2n} + \sum_{i_{\min}}^{i_{\max}} K'\varepsilon^{1-\alpha}2^{-i(1-\alpha)}2^{2n/\alpha} \le C2^{2n/\alpha}n^{-5}. \quad (6.115)$$

Further, we proceed as in $d=2$. We consider two cases. First, if $t_2 - t \le n^{-2}2^{2n/\alpha}$, we use the fact that $X(t_2-) = y$ and we have

$$\mathbb{P}[X(t)=y] \ge \exp\Big(-\frac{t_2-t}{\tau_y}\Big) \ge \exp\Big(-\frac{2^{2n/\alpha}n^{-2}}{\varepsilon 2^{2n/\alpha}}\Big) \ge 1 - C/n^2. \quad (6.116)$$

Second, if $t_2 - t > n^{-2}2^{2n/\alpha}$, we have by Fubini theorem

$$\mathbb{E}[U] = \mathbb{E}\Big[\int_{t_1}^{t_2}\chi\{X(u)\ne y\}du\Big] = \int_{t_1}^{t_2}\mathbb{P}[X(u)\ne y]du. \quad (6.117)$$

Since $\mathbb{E}[U] \le C2^{2n/\alpha}n^{-5}$, it is easy to see from the previous formula that there exist $u \in [t, t+n^{-2}2^{2n/\alpha}]$ such that

$$\mathbb{P}[X(u)\ne y] \le Cn^{-2}. \quad (6.118)$$

The proof is finished by a construction of a coupled process X' which is the same as in $d=2$. ☐

6.7 Proof of aging for function $\Pi(t_w, t_w+t)$

In the following proposition we present the exact form of the limiting function $\Pi(\theta)$. The second part of Theorem 6.1.1 is then its direct consequence. We use μ_1 to denote the distribution of the size of the jump of $\tilde{Y}$ that intersect level 1.

Proposition 6.7.1. *For $\mathbb{P}$-a.e. realisation of the environment $\boldsymbol{\tau}$*

$$\lim_{t\to\infty}\Pi\big(t,t+\theta t\big) = \int_0^\infty \Big(\frac{\ell}{\ell+\theta G(0)}\Big)^{1+\alpha}\mu_1(d\ell) = \Pi(\theta). \quad (6.119)$$

Proof. There are only small differences between the proof of this proposition and the proof of subaging in the two-dimensional case. We use $\ell_n(s)$ to denote the size of the jump of $\tilde{Y}^n$ that intersects the level $s = t/2^{2n(t)/\alpha}$. Let y be the depth of the trap that is "responsible" for this jump. (If there is one, that means if $\tilde{Y}^n$ is relevant and s is far enough from $\mathcal{R}_n$.) Then we have

$$2^{2n/\alpha}\ell_n(s) = \tau_y \sum_{i=1}^{\xi} e_i', \tag{6.120}$$

where e_i''s are i.i.d. exponential with mean one, and ξ is geometrical random variable with mean $G_{D(2^{n\gamma})}(0,0) = G(0) - O(2^{n\gamma(2-d)})$. Introducing the rescaled depth $\sigma_{y(n)} = \tau_{y(n)}/2^{2n/\alpha}$, we get

$$\ell_n(s) = \sigma_y \sum_{i=1}^{\xi} e_i'. \tag{6.121}$$

As before the distribution ν_n of visited deep traps converges weakly to

$$\nu(dx) = \frac{\alpha}{\varepsilon^{-\alpha} - M^{-\alpha}} \cdot \frac{1}{x^{\alpha+1}}\, dx \quad \text{for} \quad \varepsilon \le x \le M. \tag{6.122}$$

The random variable $\sum_{i=1}^{\xi} e_i'$ with density f_n converges weakly to the distribution whose density is

$$f(x) = G(0)^{-1} \exp\big(- xG(0)^{-1}\big). \tag{6.123}$$

With this notation the Lemmas 5.8.4 and 5.8.5 can be overtaken directly from $d = 2$ without any change. To finish the proof we should only slightly modify the calculation of integral $I(t)$,

$$I(t) = \int_0^\infty \int_\varepsilon^M \exp\Big(- \frac{\theta t}{a 2^{2n/\alpha}}\Big) dF_\ell^n(a) \mu_s^n(d\ell). \tag{6.124}$$

Taking $t = s2^{2n/\alpha}$ we get

$$I(s2^{2n/\alpha}) = \int_0^\infty \int_\varepsilon^M \exp\Big(- \frac{\theta s}{a}\Big) dF_\ell^n(a) \mu_s^n(d\ell). \tag{6.125}$$

Further steps are obvious. The expression $\theta s \alpha/(a \log 2)$ should be everywhere replaced by $\theta s/a$ and the constant $\mathcal{K}'$ by $G(0)$. After an easy computation we get the the claim of the proposition. $\square$

Appendix 6.A Properties of simple random walk

We recall here some known facts about Green's function and hitting probabilities of the simple random walk (SRW) on $\mathbb{Z}^d$. We use $G(0) = G_d(0)$ to denote the mean number of visit of 0 by SRW started at 0.

Proposition 6.A.1. *The Green's function $G_{D(n)}(\cdot,\cdot)$ of SRW killed on exit from disk $D(n)$ satisfies*

$$G_{D(n)}(0,0) = G(0) - O\big(n^{2-d}\big). \tag{6.126}$$

and

$$G_{D(n)}(0,x) = a_d\big(|x|^{2-d} - n^{2-d}\big) + O\big(|x|^{1-d}\big), \tag{6.127}$$

where $a_d = \frac{d}{2}\Gamma(\frac{d}{2}-1)\pi^{-d/2}$.

Proof. For proof see [Law91], Proposition 1.5.9. □

Let $p_n(0,x)$ denote the probability that SRW started at 0 hits x before the exit from $D(n)$.

Proposition 6.A.2. *The function $p_n(0,x)$ satisfies*

$$p_n(0,x) \geq \frac{a_d}{G(0)}\big(|x|^{2-d} - n^{2-d}\big) + O\big(|x|^{1-d}\big), \tag{6.128}$$

$$p_n(0,x) \leq a_d\big(|x|^{2-d} - n^{2-d}\big) + O\big(|x|^{1-d}\big). \tag{6.129}$$

More precisely $p_n(0,x)$ can be bounded from above by

$$p_n(0,x) \leq \frac{a_d}{G(0)}\big(|x|^{2-d} - n^{2-d} + O\big(|x|^{1-d}\big)\big)\big(1 + O\big((n-|x|)^{2-d}\big)\big). \tag{6.130}$$

Proof. The first two claims follows from equation (6.127),

$$G_{D(n)}(0,x) = p_n(0,x)G_{D(n)}(x,x), \tag{6.131}$$

and $1 \leq G_{D(n)}(x,x) < G(0)$. The third fact is a consequence of (6.131) and

$$\begin{aligned} G_{D(n)}(x,x)^{-1} &\leq G_{D_x(n-|x|)}(x,x)^{-1} \\ &= G_{D(n-|x|)}(0,0)^{-1} = G(0)^{-1} + O\big((n-|x|)^{2-d}\big), \end{aligned} \tag{6.132}$$

which is a consequence of (6.126). □

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