TIGHTNESS OF THE MAXIMUM OF BRANCHING RANDOM WALK IN RANDOM ENVIRONMENT AND ZERO-CROSSINGS OF SOLUTIONS TO DISCRETE PARABOLIC DIFFERENTIAL EQUATIONS

JIŘÍ ČERNÝ, FLAVIO DALESSI

ABSTRACT. We study branching random walk on $\mathbb{Z}$ in a bounded i.i.d. random environment. For this process, we prove that, for almost every realization of the environment, the distributions of the maximally displaced particle (re-centered around its median) are tight. This extends the result of [Kri24], where tightness was established in the annealed sense, and of [ČDO25], where a similar quenched result was proved for branching Brownian motion in random environment. Our proof relies on studying certain discrete-space linear PDEs and establishing that the number of zero-crossings of their solutions is non-increasing in time. We observe that our technique requires no additional assumptions on the environment, in contrast to [Kri24, ČDO25].

1. INTRODUCTION

The purpose of this paper is twofold. The first objective is to establish quenched tightness of the maximum of a branching random walk in a random environment (BRWRE), that is, to show that, for almost every realization of the environment, the distributions of the maximally displaced particle at time $t \geq 0$—when re-centered around their respective quenched medians—form a tight family.

The second objective is to analyze the solutions to the discrete parabolic equation

$$\partial_t u(t,x) = \frac{1}{2}\Delta_d u(t,x) - \kappa(t,x)u(t,x), \quad x \in \mathbb{Z},\ t > 0, \tag{1.1}$$

for a given bounded measurable $\kappa : [0,\infty) \times \mathbb{Z} \to \mathbb{R}$, and show that the number of their zero-crossings decreases in time. This analysis is primarily motivated by its role in the proof of the tightness result. By controlling the zero-crossings of the solutions to (1.1), we can adapt the arguments of [ČDO25]—where quenched tightness of the maximum of branching Brownian motion in a random environment (BBMRE) was established—and deduce tightness for the maximum of the BRWRE.

We note that the monotonicity of the number of zero-crossings is a property well known to hold for a broad class of one-dimensional second-order linear parabolic partial differential equations on the *real line* (see, for example, [Ang88], [Nad15]).

To prove that this monotonicity property also holds in the discrete setting, we adapt the probabilistic arguments developed for the real-line setting in [EW99].

We briefly describe the organization of the paper. To make our main objective precise, in Section 2 we introduce the BRWRE and state our first result: tightness of its maximum. In Section 3, we state our second main result, which characterizes the zero-crossings of the solution to (1.1). By using this analytical tool, in Sections 4, 5 and 6, we prove tightness of the maximum of the BRWRE. The proof of the zero-crossings result is given in Section 7. More details on the structure of these proofs will be given in Section 4.

2. Tightness of the maximum of branching random walk in random environment

This section is dedicated to a precise formulation of our main result, which establishes tightness, after appropriate re-centering, of the maximum of the BRWRE. We adopt the same setting as in [ČD20], where an invariance principle for the maximum of the BRWRE was proved. Specifically, we fix a random environment consisting of an i.i.d. family $(\xi(x))_{x\in\mathbb{Z}}$ of random variables satisfying

$$0 < \mathbf{ei} \coloneqq \operatorname{ess\,inf} \xi(0) < \operatorname{ess\,sup} \xi(0) =: \mathbf{es} < \infty. \tag{2.1}$$

Given a fixed starting point $x \in \mathbb{Z}$ and a realization of the environment, we consider the following continuous-time branching process: At time zero we initialize the process by placing a particle at x, which then moves according to a continuous-time simple random walk with jump rate one and, independently, branches into two distinct particles at rate $\xi(y)$ when located at a site $y \in \mathbb{Z}$. Upon branching, the offspring particles evolve as independent and identically distributed copies, following the same diffusion and branching mechanism as the original particle. We write $\mathtt{P}_x^\xi$ for the law of this branching process and $\mathtt{E}_x^\xi$ for the corresponding expectation. The law of the environment and its expectation are denoted by $\mathbb{P}$ and $\mathbb{E}$.

The BRWRE obtained with this construction represents the discrete-space analog of the BBMRE considered in [ČDO25]. While the results in [ČDO25] are presented for a general offspring distribution, we restrict our attention to the binary branching case for simplicity, similarly to [ČD20]. Nevertheless, our analysis can readily be extended to the general setting.

Let $M(t)$ denote the position at time $t \geq 0$ of the rightmost particle of the BRWRE, and $m(t) \coloneqq \sup\{y \in \mathbb{Z} : \mathtt{P}_x^\xi(M(t) \geq y) \geq \frac{1}{2}\}$ be its median under the measure $\mathtt{P}_x^\xi$. Note that the median $m(t)$ is random, as it depends on the realization of the environment ξ. Our tightness result reads as follows.

Theorem 2.1. *For $\mathbb{P}$-almost every realization of the environment, the family $(M(t) - m(t))_{t\geq 0}$ is tight under $\mathtt{P}_0^\xi$.*

Tightness of the maximum of the BRWRE has recently been established in the annealed sense in [Kri24]. We address the (stronger) notion of quenched tightness, which previously had been proved only along subsequences, see [Kri21].

An important feature of Theorem 2.1 is that no assumption is made on the distribution of the random environment, besides (2.1). In particular, in contrast to [Kri24, ČDO25] (but not to [Kri21]), we do not require that the corresponding Lyapunov exponent is strictly concave at the asymptotic speed of the maximum, see Remark 4.2 below for a precise statement of this assumption. For now, note that under this assumption it is possible to define a certain tilted measure under which a single particle moves with the speed of the maximum. This measure, first introduced in [ČD20], is at the core of many results related to the BRWRE, and is likewise central to our arguments.

In the course of proving annealed tightness, Kriechbaum [Kri24] derives formulas for the centering of the maximum and bounds on the decay of the probabilities that the maximum deviates significantly from its median. Our approach does not directly yield estimates of this type.

Finally, we note that it can easily be verified that our observations on the role of the strict concavity of the Lyapunov exponent in the proof of Theorem 2.1 (see Sections 4, 5 and 6) also extend to the continuous-space setting of [ČDO25]. Consequently, Theorem 2.1 in [ČDO25], establishing tightness of the maximum of the BBMRE, remains valid even when Assumption 3 in [ČDO25]—the strict concavity at the asymptotic speed—is removed.

3. Monotonicity of the number of zero-crossings

This section presents our second main result, describing the zero-crossing behavior of solutions to (1.1), which plays a central role in the proof of Theorem 2.1. In order to state this result, we define the number of zero-crossings of a function $f : \mathbb{Z} \to \mathbb{R}$ by

$$\Sigma(f) := 0 \vee \sup\{n \geq 1 : \exists\, x_1 < \cdots < x_{n+1} \text{ s.t. } f(x_i)f(x_{i+1}) < 0 \text{ for } 1 \leq i \leq n\},$$

and recall that the discrete Laplace operator Δ_d is defined, for $f : \mathbb{Z} \to \mathbb{R}$, by

$$\Delta_d f(x) = f(x+1) - 2f(x) + f(x-1), \quad x \in \mathbb{Z}.$$

Theorem 3.1. *For $\kappa \in L^\infty([0,\infty) \times \mathbb{Z})$ and $u_0 \in \ell^1(\mathbb{Z})$, let $u : [0,\infty) \times \mathbb{Z} \to \mathbb{R}$ be a solution of*

$$\begin{aligned} \frac{\partial}{\partial t} u(t,y) &= \frac{1}{2}\Delta_d u(t,y) - \kappa(t,y)u(t,y), && t > 0,\ y \in \mathbb{Z}, \\ u(0,y) &= u_0(y), && y \in \mathbb{Z}. \end{aligned} \tag{3.1}$$

Then, for all $0 \leq s \leq t$,

$$\Sigma(u(t,\cdot)) \leq \Sigma(u(s,\cdot)). \tag{3.2}$$

Moreover, in the special case $\Sigma(u_0) = 1$, if u_0 satisfies

$$\{y : u_0(y) < 0\} = \{y \in \mathbb{Z} : y \leq a_0\} \text{ and } \{y : u_0(y) > 0\} = \{y \in \mathbb{Z} : y \geq b_0\} \quad (3.3)$$

for some $a_0 < b_0 \in \mathbb{Z}$, then, for all $t \geq 0$, the same property is satisfied by $u(t, \cdot)$ for some $a_t < b_t \in \mathbb{Z} \cup \{-\infty, \infty\}$.

Note that the second part of Theorem 3.1 is not an immediate consequence of the first. Indeed, $\Sigma(u(t, \cdot)) \leq 1$ does not preclude the existence of some $y \in \mathbb{Z}$ satisfying $u(t, y) = 0$ and $u(t, y \pm 1) > 0$.

Although in our applications of Theorem 3.1 the initial condition u_0 always satisfies $\Sigma(u_0) = 1$ (see Section 4), we state the result in the more general setting of potentially multiple zero-crossings, as it might be of independent interest.

Continuous-space analogs of Theorem 3.1 have a long history in the analysis literature. Results in this direction were first established in the 19th century by Sturm [Stu36]. His ideas were later revived in the study of linear and nonlinear parabolic equations (see, for example, [Ang88], [Ang91], [DGM14] and [Nad15]); see also [Gal04] for a detailed discussion of the Sturmian principle and its applications. In the context of differential equations arising from a branching Brownian motion, a related result appeared already in the 1930s in the study of the F-KPP equation by Kolmogorov, Petrovskii and Piskunov [KPP37].

A version of Theorem 3.1—describing the evolution of zero-crossings when operators related to certain time-homogeneous Markov processes on the integers are considered—was established already in the 1950s (see [KM57]). The methods employed by the authors highlight, through the Karlin–McGregor determinant formula of coincidence probabilities for multiple particle systems, a probabilistic interpretation of the minors of the transition matrix of the process in question (see [KM59]). This formula was later extended to a larger family of time-inhomogeneous Markov processes (see [Kar88]). We will discuss in Section 7 the connection between these results and the proof of Theorem 3.1.

4. Branching random walk and the randomized F-KPP equation

We now return to the setting of Section 2 and prove tightness of the maximum of the BRWRE. As anticipated, we prove Theorem 2.1 by adapting to the discrete-space framework the ideas in [ČDO25].

The plan for the proof of Theorem 2.1 is as follows. We begin by recalling some results on the BRWRE and its connection to the F-KPP equation. The probabilistic representation of solutions to the F-KPP equation via the BRWRE underlies our proof of Theorem 2.1, and is likewise central to the arguments in [ČDO25]. Next, we present two auxiliary results—Lemmas 4.3 and 4.4—which, together with Theorem 3.1, constitute the key components of the proof of tightness. We conclude Section 4 by explaining how Theorem 2.1 can be deduced from these statements. In Section 5, we analyze a particular change of measure that is essential for studying large deviations of the maximum of the BRWRE. The arguments in this section deviate considerably from their continuous-space counterparts in [ČDO25]. Indeed, since the change of measure behaves quite differently in the discrete and continuous settings, distinct techniques are required. Building on the results of Section 5, Section 6 establishes Lemmas 4.3 and 4.4 by adapting the arguments used in [ČDO25] for their continuous-space analogs (Corollary 3.6 and Lemma 6.1 therein). Finally, the proof of the zero-crossings result, Theorem 3.1, is presented in Section 7. The steps ensuring that Theorem 2.1 holds even without strict concavity of the Lyapunov exponent at the asymptotic speed are contained in Claims 4.5, 4.6 and Lemma 5.4.

4.1. The randomized F-KPP equation

In this section, we study the solution of the randomized F-KPP equation and its relation to the BRWRE.

We begin by recalling that, given an initial condition $w_0 : \mathbb{Z} \to [0,1]$, the randomized F-KPP equation

$$\begin{aligned} \partial_t w(t,x) &= \frac{1}{2}\Delta_d w(t,x) + \xi(x) w(t,x)(1-w(t,x)), && t>0,\ x\in\mathbb{Z}, \\ w(0,x) &= w_0(x), && x \in \mathbb{Z}, \end{aligned} \tag{4.1}$$

admits a unique non-negative solution. Note that the F-KPP equation can be viewed as an instance of (3.1), with $\kappa(t,x) = \xi(x)(w(t,x)-1)$. This observation will later be used to apply Theorem 3.1 to the difference $w - w'$ between two solutions w, w' to (4.1) with different initial conditions.

We now recall a few facts from [ČD20]. As there, we use $N(t,y)$ to denote the number of particles in the BRWRE that are located at $y \in \mathbb{Z}$ at time $t \geq 0$, and write

$$N^{\geq}(t,y) := \sum_{z \geq y} N(t,z)$$

for the number of particles located to the right of y at time t. The next proposition recalls the well-known connection of the solution to (4.1) to the BRWRE. (In its statement we use the convention $0^0 = 1$.)

Proposition 4.1 (Proposition 7.1 in [ČD20])**.** *For each* $w_0 : \mathbb{Z} \to [0,1]$,

$$w(t,x) := 1 - \mathtt{E}_x^{\xi}\Big[\prod_{z\in\mathbb{Z}} (1-w_0(z))^{N(t,z)}\Big] \tag{4.2}$$

solves (4.1)*. In particular, for every* $y \in \mathbb{Z}$,

$$w^y(t,x) := \mathtt{P}_x^{\xi}(M(t) \geq y) \tag{4.3}$$

solves (4.1) *with the initial condition* $w_0^y := \mathbf{1}_{\{y,y+1,\dots\}}$.

In what follows, we will make extensive use of (4.3) to study both the F-KPP equation and the maximum of the BRWRE.

From now on, let $X = (X_t)_{t\geq 0}$ denote a one-dimensional continuous-time simple random walk with rate one, which is started at $x \in \mathbb{Z}$ under the measure P_x. The corresponding expectation is denoted with E_x. By the Feynman–Kac formula, the expected number of particles $\mathtt{E}_x^\xi[N(t,y)]$ can be expressed in terms of the random walk X, namely

$$\mathtt{E}_x^\xi[N(t,y)] = E_x\Big[\exp\Big\{\int_0^t \xi(X_s)ds\Big\}; X_t = y\Big] \tag{4.4}$$

for each $x, y \in \mathbb{Z}$, $t \geq 0$ and ξ. Generalizations of (4.4) can be found in Proposition 3.1 in [ČD20].

Another key object related to our problem is the (quenched) Lyapunov exponent $\lambda : \mathbb{R} \to \mathbb{R}$, given by

$$\lambda(v) \coloneqq \lim_{t\to\infty} \frac{1}{t} \log \mathtt{E}_0^\xi[N(t, \lfloor vt \rfloor)].$$

By Proposition A.3 in [ČD20], λ is well defined, non-random, even and concave. Moreover, there exists a unique $v_0 \in (0,\infty)$ such that

$$\lambda(v_0) = 0, \quad \mathbb{P}\text{-a.s.}$$

Furthermore, there exists a unique

$$v_c \in (0,\infty) \tag{4.5}$$

such that λ is linear on $[0, v_c]$ and strictly concave on the interval (v_c,∞). The role of v_0 and v_c in the proof of Theorem 2.1 will be clarified in the next section.

Remark 4.2. The arguments of [ČD20, Kri24] (and several other recent papers on BRWRE, BBMRE and the F-KPP equation) rely heavily on the assumption

$$v_0 > v_c. \tag{4.6}$$

This assumption enables the introduction of certain tilted measures (see Section 5 below) which are indispensable for studying the behavior of the maximum of the BRWRE. As explained in the introduction, although we also use these tilted measures in this paper, our main result, Theorem 2.1, holds without this assumption and relies solely on the ellipticity condition (2.1).

We conclude this section by presenting two important results which, together with Theorem 3.1, lead to Theorem 2.1. The first is a discrete-space analog of Corollary 3.6 in [ČDO25]. Roughly speaking, it states that if the BRWRE has probability at least ε to reach y by time t, then it also has probability at least $1-\varepsilon$ to reach y by time $t+u_\varepsilon$, with u_ε being independent of y. The result is stated in terms of solutions to the F-KPP equation by means of (4.3).

Lemma 4.3. *For every $\varepsilon \in (0,1/2)$, there exists $u = u(\varepsilon) \in (0,\infty)$ such that, $\mathbb{P}$-a.s, for all $t \geq 0$ and $y \in \mathbb{Z}$ it holds that*

$$w^y(t,0) \geq \varepsilon \textit{ implies } w^y(t+t',0) \geq 1-\varepsilon \textit{ for all } t' \geq u. \tag{4.7}$$

The second result constitutes the backbone of the proof of Theorem 2.1 and can be viewed as a discrete-space analog of Lemma 6.1 in [ČDO25]. Informally, the result asserts that if the BRWRE is started at z, then the probability that the process reaches $z + vt$ by time t is larger than the probability that it reaches $z+vt+\Delta_{u,v}$ by time $t+u$, provided that $\Delta_{u,v} > 0$ is large enough. The interesting feature of the result is that, for given $u, v > 0$, the same $\Delta_{u,v}$ applies to any (sufficiently large) time t and starting point $z \in [-vt, 0]$. Once more, the result is stated using the representation in (4.3).

Lemma 4.4. *There exists $v_2 > \mathbf{es}+2 > 0$ such that for each $u > 0$ and each $v > v_2$ there exist $\Delta_0 = \Delta_0(u, v) \in \mathbb{N}$ and a $\mathbb{P}$-a.s. finite random variable $\mathcal{T} = \mathcal{T}(u, v) \geq 0$ so that, $\mathbb{P}$-a.s., for all $t \geq \mathcal{T}$, $\Delta \in \{\Delta_0, \Delta_0 + 1, \dots\}$ and $y \in \{0, \dots, \lceil vt \rceil\}$,*

$$w^y(t, \lfloor y - vt \rfloor) \geq w^{y+\Delta}(t + u, \lfloor y - vt \rfloor).$$

We postpone the proofs of Lemmas 4.3 and 4.4 to Section 6.

4.2. Proof of Theorem 2.1 assuming Lemmas 4.3 and 4.4

We now show Theorem 2.1 by combining the two lemmas from the previous section with Theorem 3.1. As explained in Remark 4.2, we prove Theorem 2.1 under the minimal assumption (2.1) on the environment. Consequently, the arguments presented in this section differ from those in Section 7 in [ČDO25]. We clarify the nature of these differences as we develop the argument.

Let $\varepsilon \in (0, 1)$. As in [ČDO25] we introduce the quenched ε-quantile of the distribution of $M(t)$,

$$x_t \coloneqq \sup\{y \in \mathbb{Z} : w^y(t, 0) \geq \varepsilon\} = \sup\{y \in \mathbb{Z} : \mathtt{P}_0^\xi(M(t) \geq y) \geq \varepsilon\}, \quad t \geq 0. \tag{4.8}$$

If the environment satisfies (2.1) and $v_0 > v_c$, then, by Theorem 2.1 in [ČD20], the maximum of the BRWRE obeys a functional central limit theorem with speed v_0. Consequently, one deduces that $x_t/t \to v_0$ and, in particular,

$$\liminf_{t\to\infty} x_t = \infty \quad \text{and} \quad \limsup_{t\to\infty} \frac{x_t}{t} < \infty, \quad \mathbb{P}\text{-a.s.} \tag{4.9}$$

It is a priori not clear whether (4.9) remains valid when the assumption $v_0 > v_c$ is removed. (Note also that we cannot directly use the law of large numbers for the maximum of the discrete-time BRWRE established in [CP07], as the law of $M(t)$ has unbounded support for any $t \in (0, \infty)$.) As preparation for the proof of Theorem 2.1, the following two claims show that (2.1) directly implies (4.9).

Claim 4.5. *Let $x, y \in \mathbb{Z}$ and w be a solution to (4.1). If $w(0, y) > 0$ for some $y \in \mathbb{Z}$, then $\lim_{t\to\infty} w(t, x) = 1$, $\mathbb{P}$-a.s. In particular, $\lim_{t\to\infty} w^y(t, x) = 1$, $\mathbb{P}$-a.s. As consequence,*

$$\liminf_{t\to\infty} x_t = \infty, \quad \mathbb{P}\text{-a.s.} \tag{4.10}$$

Proof. By Lemma 6.8 in [ČD20], the number of particles in the BRWRE at the origin grows exponentially over time. More precisely, for all $x \in \mathbb{Z}$,

$$\mathtt{P}_x^\xi(N(t/2,x) \le t^2) \le \mathtt{P}_x^{\mathbf{ei}}(N(t/2,x) \le t^2) \xrightarrow{t\to\infty} 0, \quad \mathbb{P}\text{-a.s.} \tag{4.11}$$

(Here $\mathtt{P}_x^{\mathbf{ei}}$ denotes $\mathtt{P}_x^\xi$ with $\xi \equiv \mathbf{ei}$.) For given x and y, if the number of particles at x at time $t/2$ is at least t^2, then the number of particles reaching y at time t is stochastically bounded from below by a binomial random variable $B_{t,x,y} \sim \text{Bin}\,(\lfloor t^2 \rfloor, p_t)$ with parameters $\lfloor t^2 \rfloor$ and $p_t := P_x(X_{t/2} = y) \in (0,1)$. By (4.11), and since $\lfloor t^2 \rfloor p_t > t^{5/4}$ for $t \ge 0$ large enough,

$$\mathtt{P}_x^\xi(N(t,y) \le t) \le \mathtt{P}_x^\xi(N(t/2,x) \le t^2) + \mathtt{P}_x^\xi(B_{t,x,y} \le t) \xrightarrow{t\to\infty} 0, \quad \mathbb{P}\text{-a.s.}$$

By combining this observation with (4.2), we get

$$w(t,x) \ge 1 - \mathtt{E}_x^\xi\big[(1-w(0,y))^{N(t,y)}\big] \ge 1 - \mathtt{P}_x^\xi(N(t,y) \le t) - (1-w(0,y))^t \xrightarrow[t\to\infty]{\mathbb{P}\text{-a.s.}} 1$$

whenever $w(0,y) > 0$ for some $y \in \mathbb{Z}$. This proves the first part of the claim.

To prove (4.10), assume by contradiction that $\liminf_{t\to\infty} x_t < C - 1$ for some $C = C(\xi) \in \mathbb{N}$. Then there exist $t_n \to \infty$ satisfying $x_{t_n} + 1 \le C$, so that

$$w^C(t_n, 0) = \mathtt{P}_0^\xi(M(t_n) \ge C) \le \mathtt{P}_0^\xi(M(t_n) \ge x_{t_n} + 1) = w^{x_{t_n}+1}(t_n, 0) < \varepsilon.$$

Hence, $\limsup_{n\to\infty} w^C(t_n, 0) < 1$, which contradicts the first part of the claim. □

Claim 4.6. *Let* $\mathbf{es} \in (0,\infty)$ *be as in* (2.1)*. Then* $x_t \le \lceil (\mathbf{es}+2)t \rceil$ *for sufficiently large* $t \ge 0$*,* $\mathbb{P}$*-a.s.*

Proof. By a Chernoff bound on X_t, for all $a > 1$ and $t \ge 0$ one has

$$P_0(X_t \ge at) \le e^{-(a-1)t}. \tag{4.12}$$

Moreover, for $y \in \mathbb{Z}$, $M(t) \ge y$ if and only if $N^{\ge}(t,y) \ge 1$, so that, by (4.4),

$$\mathtt{P}_x^\xi(M(t) \ge y) \le \mathtt{E}_x^\xi[N^{\ge}(t,y)] = E_x\Big[e^{\int_0^t \xi(X_s)ds}; X_t \ge y\Big] \le e^{\mathbf{es}\cdot t} P_x(X_t \ge y) \tag{4.13}$$

for all $x \in \mathbb{Z}$. In particular,

$$\mathtt{P}_0^\xi\big(M(t) \ge \lceil (\mathbf{es}+2)t \rceil\big) \le e^{\mathbf{es}\cdot t} P_0\big(X_t \ge \lceil (\mathbf{es}+2)t \rceil\big), \tag{4.14}$$

which, by (4.12), is bounded by $e^{-t/2}$ for $t \ge 0$ large enough. Since, by assumption, x_t satisfies $\mathtt{P}_0^\xi(M(t) \ge x_t) \ge \varepsilon > 0$, the desired result follows. □

Before turning to the proof of Theorem 2.1, we show a technical result addressing the assumptions required in the second part of Theorem 3.1.

Claim 4.7. *For given* $u \in (0,\infty)$ *and* $y, z \in \mathbb{Z}$*, the function* $u_0 : \mathbb{Z} \to \mathbb{R}$ *given by*

$$x \mapsto u_0(x) := \mathbf{1}_{x \ge z} - \mathtt{P}_x^\xi(M(u) \ge y)$$

satisfies (3.3) *and* $u_0 \in \ell^1(\mathbb{Z})$*.*

Proof. Since $\mathsf{P}_x^\xi(M(u) \geq y) \in (0,1)$ whenever $u \in (0,\infty)$, (3.3) is automatically satisfied. It therefore suffices to prove that $u_0 \in \ell^1(\mathbb{Z})$. Without loss of generality, it is enough to consider the case $z = 0$. Hence, we have to show that

$$x \mapsto 1 - \mathsf{P}_x^\xi(M(u) \geq y) = \mathsf{P}_x^\xi(M(u) < y) \in \ell^1(\mathbb{N}), \quad x \mapsto \mathsf{P}_x^\xi(M(u) \geq y) \in \ell^1(-\mathbb{N}).$$

By symmetry, it suffices to consider the second mapping. Moreover, by (4.13),

$$\mathsf{P}_x^\xi(M(u) \geq y) \leq e^{\mathbf{es}\cdot u} P_x(X_u \geq y) = e^{\mathbf{es}\cdot u} P_0(X_u - \ y \geq -x).$$

Since $\sum_{x \leq 1} P_0(|X_u - y| \geq -x) = E_0[|X_u - y|] < \infty$, we deduce that $x \mapsto \mathsf{E}_x^\xi[N^{\geq}(u,y)] \in \ell^1(-\mathbb{N})$. Therefore, $x \mapsto \mathsf{P}_x^\xi(M(u) \geq y) \in \ell^1(-\mathbb{N})$, which concludes the proof of the claim. □

Proof of Theorem 2.1. It is enough to prove tightness of $(M(t)-m(t))_{t \geq \mathcal{T}_0}$ for some $\mathbb{P}$-a.s. finite $\mathcal{T}_0 \geq 0$ (which will be chosen later). Indeed, by (4.13), for any $\mathcal{T}_0 \geq 0$ there exists $C = C(\mathcal{T}_0) \in \mathbb{N}$ such that, $\mathbb{P}$-a.s.,

$$\inf_{t \leq \mathcal{T}_0} \mathsf{P}_0^\xi(M(t) \geq -C) - \mathsf{P}_0^\xi(M(t) \geq C) \geq \inf_{t \leq \mathcal{T}_0} P_0(X_t \geq -C) - e^{\mathbf{es}\cdot t} P_0(X_t \geq C) \geq 1 - \varepsilon,$$

which implies tightness of the family $(M(t) - m(t))_{t \leq \mathcal{T}_0}$.

For a given $\varepsilon \in (0, 1/2)$, let x_t be the quenched quantile defined in (4.8). The desired tightness follows if we can find $\Delta = \Delta(\varepsilon) \in \mathbb{N}$ such that, $\mathbb{P}$-a.s.,

$$w^{x_t - \Delta}(t, 0) = \mathsf{P}_0^\xi(M(t) \geq x_t - \Delta) \geq 1 - \varepsilon, \quad \text{for } t \geq \mathcal{T}_0. \tag{4.15}$$

Indeed, $x_t - \Delta \leq m(t) \leq x_t$ by (4.15) and definition of $m(t)$, so that

$$\mathsf{P}_0^\xi(|M(t) - m(t)| > \Delta) \leq \mathsf{P}_0^\xi(M(t) > (x_t - \Delta) + \Delta) + \mathsf{P}_0^\xi(M(t) < x_t - \Delta) < 2\varepsilon,$$

by definition of x_t and (4.15).

We now prove (4.15). Since $w^{x_t}(t, 0) \geq \varepsilon$, by applying Lemma 4.3 we obtain $u = u(\varepsilon) \in (0, \infty)$ such that, $\mathbb{P}$-a.s.,

$$w^{x_t}(t + t', 0) \geq 1 - \varepsilon$$

for all $t' \geq u$ and $t \geq 0$. Hence, (4.15) follows if for any $u \in (0, \infty)$ we can find some $\Delta = \Delta(u) \in \mathbb{N}$ such that, $\mathbb{P}$-a.s.,

$$w^{x_t - \Delta}(t, 0) \geq w^{x_t}(t + u, 0), \quad \text{for } t \geq \mathcal{T}_0. \tag{4.16}$$

To prove (4.16), we employ Theorem 3.1 and Lemma 4.4, and study the zeros of

$$W(s, x) := w^{x_t - \Delta}(s, x) - w^{x_t}(s + u, x), \quad s \geq 0, \ x \in \mathbb{Z}.$$

Since $w^{x_t - \Delta}$ and w^{x_t} solve the F-KPP differential equation (4.1), it is straightforward to verify that W solves a differential equation of the form (3.1), with

$$\kappa(s, x) := \xi(x)\big(1 - w^{x_t - \Delta}(s, x) - w^{x_t}(s + u, x)\big) \in [-\mathbf{es}, \mathbf{es}],$$

and the initial condition

$$W(0, x) := \mathbf{1}_{x \geq x_t - \Delta} - \mathsf{P}_x^\xi(M(u) \geq x_t).$$

By Claim 4.7, we deduce that $W(0,\cdot)$ satisfies the assumptions of Theorem 3.1. We now show that, $\mathbb{P}$-a.s., for all $t \geq \mathcal{T}_0$ there exists

$$x^* = x^*(t) \in -\mathbb{N} \quad \text{such that} \quad W(t,x^*) \geq 0. \tag{4.17}$$

By the second part of Theorem 3.1, finding such an x^* implies, $\mathbb{P}$-a.s.,

$$W(t,0) = w^{x_t-\Delta}(t,0) - w^{x_t}(t+u,0) \geq 0, \quad \text{for } t \geq \mathcal{T}_0,$$

and therefore (4.16).

It remains to show (4.17). To this end we apply Lemma 4.4 to $y \approx x_t$. Let $v_2 > \mathbf{es} + 2$ as in the statement of Lemma 4.4 and $v := v_2 + 1$. By (4.10) and Claim 4.6, for all $\Delta \in \mathbb{N}$ there exists a $\mathbb{P}$-a.s. finite $\mathcal{T}_1 = \mathcal{T}_1(\Delta, v) \geq 0$ such that $0 \leq x_t - \Delta < \lceil vt \rceil$ for all $t \geq \mathcal{T}_1$. Therefore, picking $\Delta = \Delta(u,v) \in \mathbb{N}$ and $\mathcal{T} = \mathcal{T}(u,v) \geq 0$ as in the statement of Lemma 4.4, gives, $\mathbb{P}$-a.s.,

$$w^{x_t-\Delta}(t,x^*) \geq w^{x_t}(t+u,x^*),$$

for $t \geq \mathcal{T}_0 := \mathcal{T} \vee \mathcal{T}_1$ and $x^* := \lfloor x_t - \Delta - vt \rfloor < 0$. This establishes (4.17) and completes the proof of the theorem. □

5. Tilting and exponential change of measure

This section serves as a preparation for the proof of Lemma 4.4. We introduce the previously mentioned family of tilted measures associated to the BRWRE, and derive some of their properties.

We define $\zeta := \xi - \mathbf{es}$ and $\triangle := \mathbf{es} - \mathbf{ei}$. Note that $\zeta(x) \in [-\triangle, 0]$ for all $x \in \mathbb{Z}$. We recall that X stands for the simple random walk on $\mathbb{Z}$. We write $H_y := \inf\{t \geq 0 : X_t = y\}$ for the hitting time of $y \in \mathbb{Z}$. For $\eta \leq 0$ and $x, y \in \mathbb{Z}$ with $y \geq x$, we define a probability measure on the stopped σ-algebra $\sigma((X_{t\wedge H_y})_{t\geq 0})$ via

$$P^{\zeta,\eta}_{x,y}(A) := \frac{1}{Z^{\zeta,\eta}_{x,y}} E_x\Big[\exp\Big(\int_0^{H_y} (\zeta(X_s)+\eta)ds\Big); A\Big],$$

where the normalizing constant is given by

$$Z^{\zeta,\eta}_{x,y} := E_x\Big[\exp\Big(\int_0^{H_y} (\zeta(X_s)+\eta)ds\Big)\Big]. \tag{5.1}$$

By the Markov property of X under P_x,

$$Z^{\zeta,\eta}_{x,z} = Z^{\zeta,\eta}_{x,y} Z^{\zeta,\eta}_{y,z}, \quad x \leq y \leq z, \tag{5.2}$$

so that, for given $x \in \mathbb{Z}$, the measures $(P^{\zeta,\eta}_x)_{y \geq x}$ are consistent. In particular, by the Kolmogorov extension theorem, these measure extend to a probability measure $P^{\zeta,\eta}_x$ on $\sigma((X_t)_{t\geq 0})$.

The following result provides an explicit interpretation of the process X under the measure $P^{\zeta,\eta}_x$, as a continuous-time random walk with inhomogeneous transition probabilities and jump rates. To state the result, we first extend the definition of $Z^{\zeta,\eta}_{x,y}$ to any $x, y \in \mathbb{Z}$ by setting $Z^{\zeta,\eta}_{x,y} = 1/Z^{\zeta,\eta}_{y,x}$ whenever $y < x$. A direct computation shows that, with this definition, (5.2) extends to all $x, y, z \in \mathbb{Z}$.

Proposition 5.1. *Given $x \in \mathbb{Z}$, $\eta \le 0$ and $\zeta : \mathbb{Z} \mapsto [-\triangle, 0]$, let*

$$\lambda^{\zeta,\eta}(y) := 1 - \zeta(y) - \eta \in \big[1, 1 + \triangle + |\eta|\big], \quad y \in \mathbb{Z}. \tag{5.3}$$

Under the measure $P^{\zeta,\eta}_x$, X is a continuous-time nearest-neighbor random walk (started at x) with spatially inhomogeneous transition probabilities

$$p^{\zeta,\eta}(y, y \pm 1) := \frac{Z^{\zeta,\eta}_{y\pm1,y}}{2\lambda^{\zeta,\eta}(y)}, \quad y \in \mathbb{Z}, \tag{5.4}$$

and jump rates $(\lambda^{\zeta,\eta}(y))_{y\in\mathbb{Z}}$.

Proof. Let $Y = (Y_n)_{n\ge0}$ be a discrete-time simple random walk and $(e_i)_{i\ge1}$ a family of i.i.d. Exp(1)-distributed random variables independent of Y such that, P_x-a.s.,

$$X_t = Y_{N(t)}, \quad N(t) = 0 \vee \sup\big\{m \ge 1 : e_1 + \cdots + e_m \le t\big\}.$$

We begin by studying the law of e_1 and Y_1 under the measure $P^{\zeta,\eta}_x$. By definition of the probability measure $P^{\zeta,\eta}_x$,

$$Z^{\zeta,\eta}_{x,x+1} P^{\zeta,\eta}_x(e_1 \ge t) = E_x\left[e^{\int_0^{H_{x+1}}(\zeta(X_s)+\eta)ds}; e_1 \ge t\right].$$

By the Markov property of X under P_x, this is equal to

$$\begin{aligned}
&E_x\Big[E_{Y_1}\Big[e^{\int_0^{H_{x+1}}(\zeta(X_s)+\eta)ds}\Big]e^{\int_0^{e_1}(\zeta(X_s)+\eta)ds}; e_1 \ge t\Big] \\
&= \left(\frac{1}{2} + \frac{1}{2}E_{x-1}\Big[e^{\int_0^{H_{x+1}}(\zeta(X_s)+\eta)ds}\Big]\right)\int_t^\infty e^{-s+s(\zeta(x)+\eta)}ds \\
&= \big(1 + Z^{\zeta,\eta}_{x-1,x+1}\big)\frac{e^{-(1-\zeta(x)-\eta)t}}{2(1-\zeta(x)-\eta)} = \big(1 + Z^{\zeta,\eta}_{x-1,x+1}\big)\frac{e^{-\lambda^{\zeta,\eta}(x)t}}{2\lambda^{\zeta,\eta}(x)}.
\end{aligned}$$

In particular, with $t = 0$ this gives

$$2\lambda^{\zeta,\eta}(x) Z^{\zeta,\eta}_{x,x+1} = 1 + Z^{\zeta,\eta}_{x-1,x+1}. \tag{5.5}$$

By combining these observations we get $P^{\zeta,\eta}_x(e_1 \ge t) = e^{-\lambda^{\zeta,\eta}(x)t}$. Hence, under $P^{\zeta,\eta}_x$, the holding time e_1 is exponentially distributed with parameter $\lambda^{\zeta,\eta}(x)$. Similarly, using the Markov property of X under P_x once more,

$$\begin{aligned}
Z^{\zeta,\eta}_{x,x+1} P^{\zeta,\eta}_x(Y_1 = x \pm 1) &= E_x\Big[e^{\int_0^{H_{x+1}}(\zeta(X_s)+\eta)ds}; Y_1 = x \pm 1\Big] \\
&= \frac{1}{2}E_{x\pm1}\Big[e^{\int_0^{H_{x+1}}(\zeta(X_s)+\eta)ds}\Big]\int_0^\infty e^{-s(1-\zeta(x)-\eta)}ds.
\end{aligned}$$

Therefore, the first step of the random walk has the transition probabilities

$$P_x^{\zeta,\eta}(Y_1 = x+1) = \frac{1}{2Z_{x,x+1}^{\zeta,\eta}(1-\zeta(x)-\eta)} = \frac{Z_{x+1,x}^{\zeta,\eta}}{2\lambda^{\zeta,\eta}(x)}, \tag{5.6}$$

$$P_x^{\zeta,\eta}(Y_1 = x-1) = \frac{Z_{x-1,x+1}^{\zeta,\eta}}{2Z_{x,x+1}^{\zeta,\eta}(1-\zeta(x)-\eta)} = \frac{Z_{x-1,x}^{\zeta,\eta}}{2\lambda^{\zeta,\eta}(x)}. \tag{5.7}$$

Note that the last equality in (5.7) follows from (5.2).

To complete the proof, it remains to to show that X satisfies the Markov property under the measure $P_x^{\zeta,\eta}$. Hence, we need to show that, for all $s,t \geq 0$, $x,y,z \in \mathbb{Z}$ and $A \in \sigma((X_r)_{r\leq t})$,

$$P_x^{\zeta,\eta}(A, X_t = y, X_{t+s} = z) = P_x^{\zeta,\eta}(A, X_t = y)P_y^{\zeta,\eta}(X_s = z). \tag{5.8}$$

To this end, let $n > |x|+|y|+|z|$. In the following, for each process W and each $0 \leq u \leq r$ we let $W_{u,r}^* := \sup\{W_s : u \leq s \leq r\}$ and $W_r^* := W_{0,r}^*$.

By the Markov property of X under P_x,

$$\begin{aligned} &E_x\left[e^{\int_0^{H_n}(\zeta(X_r)+\eta)dr}; A, X_t = y, X_t^* < n, X_{t+s} = z, X_{t,t+s}^* - y < n\right] \\ &= E_x\left[e^{\int_0^{t}(\zeta(X_r)+\eta)dr}; A, X_t = y, X_t^* < n\right] E_y\left[e^{\int_0^{H_n}(\zeta(X_r)+\eta)dr}; X_s = z, X_s^* < n\right] \\ &= E_x\left[e^{\int_0^{t}(\zeta(X_r)+\eta)dr}; A, X_t = y, X_t^* < n\right] Z_{y,n}^{\zeta,\eta}P_y^{\zeta,\eta}(X_s = z, X_s^* < n). \end{aligned}$$

In the last step, $\{X_s = z, X_s^* < n\} \subset \sigma((X_{t\wedge H_n})_{t\geq 0})$ and the definition of $P_y^{\zeta,\eta}$ were used. Moreover, again by the Markov property of X under P_x,

$$\begin{aligned} &E_x\left[e^{\int_0^{t}(\zeta(X_r)+\eta)dr}; A, X_t = y, X_t^* < n\right] E_y\left[e^{\int_0^{H_n}(\zeta(X_r)+\eta)dr}\right] \\ &= E_x\left[e^{\int_0^{t}(\zeta(X_r)+\eta)dr + \int_t^{H_n}(\zeta(X_r)+\eta)dr}; A, X_t = y, X_t^* < n\right] \\ &= E_x\left[e^{\int_0^{H_n}(\zeta(X_r)+\eta)dr}; A, X_t = y, X_t^* < n\right] = Z_{x,n}^{\zeta,\eta}P_x^{\zeta,\eta}(A, X_t = y, X_t^* < n). \end{aligned}$$

By combining the two previous displays, we obtain

$$\begin{aligned} &E_x\left[e^{\int_0^{H_n}(\zeta(X_r)+\eta)dr}; A, X_t = y, X_t^* < n, X_{t+s} = z, X_{t,t+s}^* - y < n\right] \\ &= Z_{x,n}^{\zeta,\eta}P_x^{\zeta,\eta}(A, X_t = y, X_t^* < n)P_y^{\zeta,\eta}(X_s = z, X_s^* < n). \end{aligned}$$

Dividing both sides by $Z_{x,n}^{\zeta,\eta}$ and taking the limit $n \to \infty$ gives (5.8). This concludes the proof of the proposition. □

By (5.6), (5.7) and (5.2), the ratio of the transition probabilities satisfies

$$\frac{P_x^{\zeta,\eta}(Y_1 = x-1)}{P_x^{\zeta,\eta}(Y_1 = x+1)} = Z_{x-1,x+1}^{\zeta,\eta} \in (0,1]. \tag{5.9}$$

Hence, under the tilted measure, the random walk exhibits a positive drift. Since (for given η) this drift depends on ζ, we can use (5.9) to compare the law of X under the measures $P_x^{0,\eta}$, $P_x^{\zeta,\eta}$ and $P_x^{-\triangle,\eta}$. (In the following, $P_x^{\gamma,\eta}$ denotes $P_x^{\zeta,\eta}$ with $\zeta \equiv \gamma \leq 0$.) More precisely, in Corollary 5.3 below we compare the law of the hitting time H_y under this three measures. Note that this is particularly useful, as under the measures $P_x^{0,\eta}$ and $P_x^{-\triangle,\eta}$ the process X is simply a continuous-time random walk with a constant drift.

The coupling leading to Corollary 5.3 is presented in the following lemma. Before stating the result, we recall that the transition probabilities and jump rates of X under the measure $P_x^{\zeta,\eta}$ are given by, respectively, (5.4) and (5.3).

Lemma 5.2. *For any $x \in \mathbb{Z}$, $\eta \leq 0$ and $\zeta : \mathbb{Z} \mapsto [-\triangle, 0]$, there exist discrete-time nearest-neighbor random walks $Y^{0,\eta}$, $Y^{\zeta,\eta}$ and $Y^{-\triangle,\eta}$ with transition probabilities given by, respectively,*

$$p^{0,\eta}(y, y \pm 1), p^{\zeta,\eta}(y, y \pm 1) \text{ and } p^{-\triangle,\eta}(y, y \pm 1), \quad y \in \mathbb{Z},$$

such that

$$Y_0^{0,\eta} = Y_0^{\zeta,\eta} = Y_0^{-\triangle,\eta} = x \quad \text{and} \quad Y_n^{0,\eta} \leq Y_n^{\zeta,\eta} \leq Y_n^{-\triangle,\eta}, \quad n \geq 1. \tag{5.10}$$

Moreover, there exist counting processes $N^{0,\eta}, N^{\zeta,\eta}, N^{-\triangle,\eta} : [0, \infty) \to \mathbb{N}$ with jump size one, such that

$$N^{0,\eta} \leq N^{\zeta,\eta} \leq N^{-\triangle,\eta} \tag{5.11}$$

and so that the processes

$$X^{0,\eta} := \left(Y^{0,\eta}_{N^{0,\eta}(t)}\right)_{t\geq 0}, \; X^{\zeta,\eta} := \left(Y^{\zeta,\eta}_{N^{\zeta,\eta}(t)}\right)_{t\geq 0} \text{ and } X^{-\triangle,\eta} := \left(Y^{-\triangle,\eta}_{N^{-\triangle,\eta}(t)}\right)_{t\geq 0}$$

have the same law as X under $P_x^{0,\eta}$, $P_x^{\zeta,\eta}$ and $P_x^{-\triangle,\eta}$, respectively.

Before proving Lemma 5.2, we present a corollary of it, which serves as a discrete-space analog of Lemma 4.3 in [ČDO25]. Because of the differences between the discrete and the continuous-space setting, the resulting statement is slightly weaker than in [ČDO25], but it will be sufficient for our purposes.

Corollary 5.3. *For any $x \leq y$, $\eta \leq 0$, $t \geq 0$ and $\zeta : \mathbb{Z} \mapsto [-\triangle, 0]$,*

$$P_x^{0,\eta}(H_y \leq t) \leq P_x^{\zeta,\eta}(H_y \leq t) \leq P_x^{-\triangle,\eta}(H_y \leq t). \tag{5.12}$$

Proof of Corollary 5.3. We prove the first inequality, the second then follows by an analogous argument. By Lemma 5.2, $X^{0,\eta}$ and $X^{\zeta,\eta}$ have the same law as X under, respectively, $P_x^{0,\eta}$ and $P_x^{\zeta,\eta}$. In particular, it is enough to show that

$$\left\{\exists s \leq t : X_s^{0,\eta} \geq y\right\} \subset \left\{\exists s \leq t : X_s^{\zeta,\eta} \geq y\right\}$$

or, equivalently, that

$$\left\{\exists s \leq t : Y^{0,\eta}_{N^{0,\eta}(s)} \geq y\right\} \subset \left\{\exists s \leq t : Y^{\zeta,\eta}_{N^{\zeta,\eta}(s)} \geq y\right\}. \tag{5.13}$$

By (5.11) and since $N^{\zeta,\eta}$ is a counting process with jump size one, we have

$$N^{0,\eta}(s) \leq N^{\zeta,\eta}(s) \leq N^{\zeta,\eta}(t) \text{ if } s \leq t, \text{ and } N^{\zeta,\eta}([0,t]) = \{0, \dots, N^{\zeta,\eta}(t)\}. \quad (5.14)$$

If $Y^{0,\eta}_{N^{0,\eta}(s)} \geq y$ for some $s \leq t$ then, by (5.10), $Y^{\zeta,\eta}_{N^{0,\eta}(s)} \geq y$. By (5.14), this implies

$$Y^{\zeta,\eta}_{N^{\zeta,\eta}(r)} \geq y, \qquad \text{for some } r \leq t.$$

We deduce that (5.13) is satisfied, which concludes the proof. □

Proof of Lemma 5.2. Since $\zeta \mapsto Z^{\zeta,\eta}_{y-1,y+1}$ is non-decreasing, (5.9) implies that

$$\frac{p^{0,\eta}(y,y-1)}{p^{0,\eta}(y,y+1)} \geq \frac{p^{\zeta,\eta}(y,y-1)}{p^{\zeta,\eta}(y,y+1)} \geq \frac{p^{\triangle,\eta}(y,y-1)}{p^{\triangle,\eta}(y,y+1)}.$$

Hence, $p^{0,\eta}(y,y+1) \leq p^{\zeta,\eta}(y,y+1) \leq p^{-\triangle,\eta}(y,y+1)$, and the first part of the lemma follows from a standard coupling argument.

Let $\{e_i\}_{i\geq 1}$ be i.i.d. Exp(1)-distributed random variables and, for $c \in \{0, \triangle\}$,

$$N^{-c,\eta}(t) := \sup\{n \in \mathbb{N} : e_1 + \cdots + e_m \leq t(1 + c - \eta)\}. \quad (5.15)$$

It is straightforward to verify that $X^{0,\eta}$ and $X^{-\triangle,\eta}$ satisfy the desired properties. Similarly, by letting

$$N^{\zeta,\eta}(t) := \sup\left\{n \in \mathbb{N} : \frac{e_1}{1 - \zeta(Y^{\zeta,\eta}_0) - \eta} + \cdots + \frac{e_m}{1 - \zeta(Y^{\zeta,\eta}_{m-1}) - \eta} \leq t\right\} \quad (5.16)$$

we get that $\big(Y^{\zeta,\eta}_{N^{\zeta,\eta}(t)}\big)_{t\geq 0}$ has the same law as X under $P^{\zeta,\eta}_x$. We conclude the proof by noticing that, by (5.15) and (5.16), $-\triangle \leq \zeta \leq 0$ implies (5.11). □

We conclude this section with a first application of Corollary 5.3, which provides an upper bound for the critical speed v_c defined in (4.5). Before doing so, we recall an important property of v_c, which will be useful also in the next section: By Lemmas 4.2 and A.1 in [ČD20], for every $v > v_c$ there exists a unique $\overline{\eta}(v) < 0$ satisfying

$$v = \frac{1}{\mathbb{E}\Big[E^{\zeta,\overline{\eta}(v)}_0[H_1]\Big]}. \quad (5.17)$$

(Recall that $\mathbb{E}$ denotes expectation with respect to the environment.) Roughly speaking this means that for any $v > v_c$ one can choose the parameter η so that the tilted random walk has, on average, speed v. In what follows we denote by

$$v^{\gamma,\eta} := \frac{1}{E^{\gamma,\eta}_0[H_1]} \quad (5.18)$$

the speed of X under the measure $P^{\gamma,\eta}_0$, where $\gamma, \eta \leq 0$.

Lemma 5.4. *For each* $\gamma, \eta \le 0$,

$$v^{\gamma,\eta} = \sqrt{(\gamma+\eta)(\gamma+\eta-2)}. \tag{5.19}$$

Moreover, for $v_c \in (0,\infty)$ *being be the critical speed defined in* (4.5),

$$v_c \le \sqrt{\triangle(2+\triangle)} < \sqrt{\mathbf{es}(2+\mathbf{es})} < \mathbf{es}+1. \tag{5.20}$$

Proof. We first prove (5.19). By (5.18) and (5.4),

$$\begin{aligned} v^{\gamma,\eta} &= \frac{1}{E_0^{\gamma,\eta}[H_1]} = \lambda^{\gamma,\eta}(0)\big(1-2p^{\gamma,\eta}(0,-1)\big) \\ &= \lambda^{\gamma,\eta}(0)\Big(1-\frac{2Z^{\gamma,\eta}_{-1,0}}{2\lambda^{\gamma,\eta}(0)}\Big) = \lambda^{\gamma,\eta}(0) - Z^{\gamma,\eta}_{-1,0} = 1-\gamma-\eta-Z^{\gamma,\eta}_{-1,0}. \end{aligned} \tag{5.21}$$

In particular, (5.19) follows once we show that

$$Z^{\gamma,\eta}_{-1,0} = 1-\gamma-\eta-\sqrt{(\gamma+\eta)(\gamma+\eta-2)}. \tag{5.22}$$

To prove (5.22), we first notice that $Z^{\gamma,\eta}_{x,x+1} = Z^{\gamma,\eta}_{y,y+1}$ for all $x, y \in \mathbb{Z}$. In particular, by (5.2), we deduce that $Z^{\gamma,\eta}_{-1,1} = Z^{\gamma,\eta}_{-1,0}Z^{\gamma,\eta}_{0,1} = (Z^{\gamma,\eta}_{-1,0})^2$. By (5.5), this implies

$$2(1-\gamma-\eta)Z^{\gamma,\eta}_{-1,0} = 1 + (Z^{\gamma,\eta}_{-1,0})^2.$$

Solving this quadratic equation gives (5.22).

We now prove (5.20). Since $\triangle = \mathbf{es}-\mathbf{ei}$, we just need to prove the first inequality. By Lemma A.1 and Proposition A.3 in [ČD20],

$$v_c = \lim_{\eta\downarrow 0} \mathbb{E}\Big[E_0^{\zeta,\eta}[H_1]\Big]^{-1}. \tag{5.23}$$

Moreover, Corollary 5.3 implies that $\mathbb{E}[E_0^{\zeta,\eta}[H_1]] \ge E_0^{-\triangle,\eta}[H_1]$ for all $\eta \le 0$. By combining this observation with (5.23), (5.18) and (5.19), we get

$$v_c \le \lim_{\eta\downarrow 0}\frac{1}{E_0^{-\triangle,\eta}[H_1]} = \lim_{\eta\downarrow 0} v^{-\triangle,\eta} = \lim_{\eta\downarrow 0}\sqrt{(-\triangle+\eta)(-\triangle+\eta-2)} = \sqrt{\triangle(2+\triangle)},$$

which completes the proof of the lemma. □

The bound in Lemma 5.4 implies, in particular, that $\overline{\eta}(v)$ exists for every speed $v \ge \mathbf{es}+1$, regardless of the actual value of v_c. This observation will be important in the proof of Lemma 4.4.

6. Proofs of Lemmas 4.3 and 4.4

In this section, we prove Lemmas 4.3 and 4.4. We begin with Lemma 4.3, which addresses the growth of $w^y(\cdot,0) = \mathtt{P}_0^\xi(M(\cdot)\ge y)$ by providing a time $u = u(\varepsilon) > 0$ such that $w^y(t,0) \ge \varepsilon$ implies $w^y(t+u,0) \ge 1-\varepsilon$. In [ČDO25], the corresponding statement, Corollary 3.6, was proved using rather heavy PDE arguments. In the discrete setting, we provide a simple probabilistic proof.

Proof of Lemma 4.3. By (4.11), there exists $t_1 = t_1(\varepsilon) > 0$ such that

$$\mathsf{P}_0^\xi(N(t',0) \le t') \le \mathsf{P}_0^{\mathbf{ei}}(N(t',0) \le t') \le \varepsilon/2, \qquad \text{for all } t' \ge t_1,\ \mathbb{P}\text{-a.s.} \tag{6.1}$$

Moreover, for all $y \in \mathbb{Z}$,

$$\mathsf{P}_0^\xi(M(t+t') < y) \le \mathsf{P}_0^\xi(M(t+t') < y \mid N(t',0) > t') + \mathsf{P}_0^\xi(N(t',0) \le t'), \tag{6.2}$$

where, by the Markov property of the BRWRE,

$$\mathsf{P}_0^\xi(M(t+t') < y \mid N(t',0) > t') \le \left(\mathsf{P}_0^\xi(M(t) < y)\right)^{t'} = (1 - w^y(t,0))^{t'}. \tag{6.3}$$

Since $w^y(t,0) \ge \varepsilon$, there exists $t_2 = t_2(\varepsilon) > 0$ such that

$$(1 - w^y(t,0))^{t'} \le (1-\varepsilon)^{t'} \le \varepsilon/2, \qquad \text{for all } t' \ge t_2. \tag{6.4}$$

By combining (6.1), (6.2), (6.3) and (6.4), we conclude that, $\mathbb{P}$-a.s.,

$$w^y(t+t',0) = 1 - \mathsf{P}_0^\xi(M(t+t') < y) \ge 1 - \varepsilon,$$

for all $t' \ge u := t_1 \vee t_2$. □

The proof of Lemma 4.4 closely follow the arguments in Sections 5 and 6 in [ČDO25]. To avoid reproducing their essentially identical parts, we focus on the main differences and only explain the adaptations required to carry over the arguments from the continuous-space case to the discrete one.

As preparation for the proof of Lemma 4.4, the first (and most important) step is to adapt the definitions of the auxiliary velocities in [ČDO25] (see (6.1) and (6.2) therein). Specifically, we let

$$v_1 := \mathbf{es} + 2, \quad v_2 := \inf\{v > v_1 + 1 : |\overline{\eta}(v)| \ge 2v_1 + 2\}. \tag{6.5}$$

(Recall that $\overline{\eta}(v)$ was defined using (5.17).) Note that, by (5.20), we have $v_c < v_1$. Therefore, any speed satisfying $v \ge v_1$ admits some $\overline{\eta}(v) < 0$ satisfying (5.17). In particular, v_2 is well-defined and, since $v \mapsto \overline{\eta}(v)$ is a continuous decreasing function such that $\lim_{v\to\infty} \overline{\eta}(v) = -\infty$ (see Lemma 4.2 in [ČD20]), also finite.

The definition of v_2 in (6.5) is motivated by the following observation: For any $v > v_2$ one can choose $\eta < 0$ such that

$$2v_1 < |\eta| < |\overline{\eta}(v)| - 1, \tag{6.6}$$

which, by (5.19), implies in particular that η satisfies

$$v^{0,\eta} = \sqrt{\eta(\eta-2)} = \sqrt{|\eta|(2+|\eta|)} > 2v_1. \tag{6.7}$$

(This inequality will be very useful later, see statement of Lemma 6.1 below.)

The second step in the preparation for the proof of the Lemma 4.4 consists in observing that each of the statements in Section 5 in [ČDO25] holds also in the discrete setting, with obvious modifications. In particular, the perturbation results of Proposition 5.1 in [ČDO25] remain valid in the discrete framework considered here. In fact, the arguments in Section 5 in [ČDO25] adapt the results in

[DS22]. As these results are an extension of those obtained in [ČD20], they remain applicable in our setting.

To explain how the proof of Lemma 4.4 is structured, we briefly recall the main steps in the proof of its continuous-space analog Lemma 6.1 in [ČDO25]. In [ČDO25], the authors first make use of Proposition 5.1 to show that Lemma 6.1 follows from the (more technical) result Lemma 6.2. The latter is then proved via two auxiliary results, Lemmas 6.3 and 6.4.

We proceed in reverse order and first establish discrete-space analogs of Lemmas 6.3 and 6.4 in [ČDO25]. By virtue of the results in Section 4.3 in [ČD20], the proof of Lemma 6.4 in [ČDO25] carries over to the discrete case without substantial modification. It therefore suffices to address Lemma 6.3 in [ČDO25].

To this end, we define, for $K \geq 0$, $t \geq 0$ and $y \in \mathbb{Z}$, the hitting time

$$\mathcal{T}_{y,t} := \inf\{s \geq 0 : X_s \geq \lceil \beta_{y,t}(s) \rceil\}, \quad \text{where } \beta_{y,t}(s) := y - v_1(t-s),$$

and the event $\mathcal{G}_K := \{\mathcal{T}_{y,t} \in [t-K, t]\}$ (cf. (6.9) and (6.10) in [ČDO25]).

By the nature of the differences between the discrete-space setting considered here and that of [ČDO25], both the statement and the proof of Lemma 6.1 below differ slightly from those of its continuous-space analog Lemma 6.3 in [ČDO25].

Lemma 6.1. *Let $\eta < 0$ be such that $v^{0,\eta} > 2v_1$. Then there exists $K_0 = K_0(\eta) \in (0, \infty)$ such that, $\mathbb{P}$-a.s., for all $v > v_1$, $y \in \mathbb{Z}$, $K \geq K_0$, $t \geq K$ and $L \in (0, K/3]$,*

$$P^{\zeta,\eta}_{\lfloor y - vt \rfloor}(H_y \leq t,\ \mathcal{T}_{y,t} \leq t - K) \leq 2 P^{\zeta,\eta}_{\lfloor y - vt \rfloor}(H_y < t - L).$$

Proof. By a straightforward adaptation of the proof of Lemma 6.3 in [ČDO25], it is sufficient to show that (cf. (6.24) therein)

$$P^{\zeta,\eta}_{\lceil y - v_1(t-u) \rceil}(H_y \leq t - u - L) \geq 1/2, \quad \text{whenever } 0 \leq u \leq t - K. \tag{6.8}$$

In [ČDO25], (6.8) is proved via a lower bound on the expectation of X_{t-u-L} under the measure $P^{0,\eta}_{\lceil y - v_1(t-u) \rceil}$. Our setting requires a different approach, as (6.8) does not follow directly from a lower bound of this form.

By Corollary 5.3 and translation invariance of X under $P^{0,\eta}_0$,

$$P^{\zeta,\eta}_{\lceil y - v_1(t-u) \rceil}(H_y \leq t - u - L) \geq P^{0,\eta}_0(H_{\lfloor v_1(t-u) \rfloor} \leq t - u - L). \tag{6.9}$$

Moreover, since $2L/K \leq 2/3$, $t - u \geq K$ and $K - L \geq 2K/3$,

$$\begin{aligned}
\frac{5}{3} v_1(t-u-L) &\geq v_1(t-u-L) + v_1 \frac{2L}{K}(t-u-L) \\
&\geq v_1(t-u) - v_1 L + v_1 \frac{2L}{K}(K-L) \\
&\geq v_1(t-u) - v_1 L + v_1 \frac{4L}{3} \geq v_1(t-u).
\end{aligned}$$

Hence,

$$v^{0,\eta}(t-u-L) \geq 2v_1(t-u-L) \geq v_1(t-u) + \frac{v_1}{3}(t-u-L),$$

so that

$$t-u-L \geq \frac{v_1(t-u)}{v^{0,\eta}-v_1/3}. \tag{6.10}$$

Since, by (5.18), X has speed $v^{0,\eta}$ under the measure $P_0^{0,\eta}$, the law of large numbers for the continuous-time random walk gives, by (6.10),

$$P_0^{0,\eta}\left(H_{\lfloor v_1(t-u)\rfloor} \leq t-u-L\right) \geq P_0^{0,\eta}\left(H_{\lfloor v_1(t-u)\rfloor} \leq \frac{v_1(t-u)}{v^{0,\eta}-v_1/3}\right) \geq 1/2$$

whenever $t-u \geq K \geq K_0$ and $K_0 = K_0(\eta)$ is sufficiently large. By (6.9), this gives the desired relation (6.8), and completes the proof. □

We can now state and prove a discrete analog of Lemma 6.2 in [ČDO25].

Lemma 6.2. *For every $v > v_2$ there exist constants $K = K(v) \in (0,\infty)$ and $C = C(v) \in (0,\infty)$ such that, $\mathbb{P}$-a.s.,*

$$E_{\lfloor y-vt\rfloor}\left[e^{\int_0^t \xi(X_s)ds}; X_t \geq y\right] \leq CE_{\lfloor y-vt\rfloor}\left[e^{\int_0^t \xi(X_s)ds}; X_t \geq y, \mathcal{G}_K\right]$$

for all $t \geq 0$ large enough and all $y \in \{0,\dots,\lceil vt\rceil\}$.

Proof. By a straightforward adaptation of the proof of Lemma 6.2 in [ČDO25], the desired result follows once we establish, for suitably chosen $\eta \leq 0$, $L \geq 0$, $K \geq 0$ and $\delta = \delta(K,\eta) > 0$, $C = C(\eta,K) < \infty$, that, $\mathbb{P}$-a.s.,

$$p_y^{\zeta,\eta}(s) \coloneqq P_y^{\zeta,\eta}(X_s \geq y) \geq \delta, \quad \text{for all } y \in \mathbb{Z} \text{ and } s \leq K, \tag{6.11}$$

and

$$P_{\lfloor y-vt\rfloor}^{\zeta,\eta}(H_y \in [t-L,t]) \leq CP_{\lfloor y-vt\rfloor}^{\zeta,\eta}(H_y \in [t-K,t], \mathcal{T}_{y,t} \geq t-K), \quad \mathbb{P}\text{-a.s.}, \tag{6.12}$$

for all t large enough and $y \in \{0,\dots,\lceil vt\rceil\}$ (cf. (6.17) and (6.18) in [ČDO25]).

However, (6.11) follows directly from (5.3), as

$$P_y^{\zeta,\eta}(X_s \geq y) \geq P_y^{\zeta,\eta}(X_r = y,\ \forall\, r \leq K) \geq e^{-(1+\triangle+|\eta|)K} =: \delta > 0,$$

for all $y \in \mathbb{Z}$ and $s \leq K$. Moreover, by (6.6) and (6.7), we can choose η, L and K so that both Lemma 6.1 and Lemma 6.4 in [ČDO25] (which, we recall, extend to the discrete setting) are applicable. By the exact same arguments as in the proof of Lemma 6.2 in [ČDO25], this gives the desired inequality (6.12). □

Proof of Lemma 4.4. In [ČDO25], Lemma 6.1 is established via two intermediate inequalities, see (6.5) and (6.14) therein. The arguments leading to the first one extend directly to the discrete setting, since (as explained before) Proposition 5.1 in [ČDO25] continues to hold in our framework. On the other hand, the arguments leading to the second intermediate inequality require a small adaptation, as it is a priori not clear whether, in our setting,

$$\int_0^{t-K} e^{\mathbf{es}(t-s)} P_{X_s}(X_{t-s} \geq y) ds \leq 1, \quad \text{for all } K \geq 1, \text{ on the event } \mathcal{G}_K. \tag{6.13}$$

(cf. (6.11) in [ČDO25] and the display following that equation.) To solve this issue, we first notice that, on the event $\mathcal{G}_K$,

$$X_s \leq \lfloor y - v_1(t-s) \rfloor = y - \lceil v_1(t-s) \rceil, \quad \text{for all } s \in [0, t-K).$$

In particular, on the event $\mathcal{G}_K$, every $s \in [0, t-K)$ satisfies

$$P_{X_s}(X_{t-s} \geq y) \leq P_{y - \lceil v_1(t-s) \rceil}(X_{t-s} \geq y) \leq P_0(X_{t-s} \geq v_1(t-s)). \tag{6.14}$$

Moreover, by (4.12) and (6.5),

$$\begin{aligned} \int_0^{t-K} e^{\mathbf{es}(t-s)} P_0(X_{t-s} \geq v_1(t-s)) ds &\leq \int_0^{t-K} e^{\mathbf{es}(t-s)} e^{-(t-s)(v_1 - 1)} ds \\ &\leq \int_K^t e^{-(v_1 - 1 - \mathbf{es})s} ds \leq e^{-K}. \end{aligned} \tag{6.15}$$

By combining (6.14) and (6.15), we obtain (6.13).

Using (6.13), we can replicate the arguments in [ČDO25] and deduce that (6.14) therein extend to our setting. With a discrete counterpart to equation (6.14) and Lemma 6.2 at our disposal, the last part of the proof of Lemma 6.1 in [ČDO25] carries over verbatim to our framework, which concludes the proof. □

7. Proof of Theorem 3.1

We now prove Theorem 3.1. The proof consists of three parts and, as anticipated in the introduction, follows the approach proposed by Evans and Williams in [EW99] for the real-line setting. First, we prepare for the proof by reducing the statement to a simpler form. Specifically, we express the solution to (3.1) in terms of a killed random walk by recalling its Feynman–Kac representation. Then, we construct a sequence of processes converging, in a suitable sense, to $t \mapsto u(t, \cdot)$. Finally, by exploiting the properties of these processes, we conclude the proof of the theorem.

7.1. Killed random walk and the Feynman–Kac formula

To prepare for the proof of Theorem 3.1, we begin by showing that the solution to (3.1) admits a representation in terms of a killed random walk. To do so, note first that if $\|\kappa\|_\infty < C$ and u solves (3.1), then $\tilde{u}(t, y) = u(t, y) e^{-2Ct}$ is a solution to (3.1) with $\kappa + 2C \in (C, 3C)$ in place of κ. Because u and $\tilde{u}$ share the same initial condition, zeros and sign properties, we may, without loss of generality, assume from now on that κ is bounded above and below by positive constants.

We now consider a one-dimensional continuous-time simple random walk with jump rate one, which is killed with rate $\kappa(t,y) > 0$ when located at position $y \in \mathbb{Z}$ at time $t \geq 0$. This process can be constructed as follows. Let, as previously, X be a one-dimensional continuous-time simple random walk with rate one, started at $x \in \mathbb{Z}$ under P_x. We denote by † the point at infinity in the one-point compactification $\overline{\mathbb{Z}}$ of $\mathbb{Z}$. The killed random walk (KRW) is the $\overline{\mathbb{Z}}$-valued process $(\tilde{X}_t)_{t\geq 0}$ defined by

$$\tilde{X}_t = \begin{cases} X_t, & t < \zeta \\ \dagger, & t \geq \zeta \end{cases} \quad \text{with} \quad P_x(\zeta > t) = E_x\left[\exp\left(- \int_0^t \kappa(s, X_s) ds\right)\right].$$

As the killing rate is bounded away from zero, the life-time ζ of $\tilde{X}$ is finite a.s.

To write the unique solution to (3.1) it terms of the KRW, we use the Feynman–Kac formula for time-dependent potentials (see Section II.1 in [CM94]). Since u_0 is summable, the formula applies even if the initial condition is not non-negative, so that for all $t \geq 0$ and $y \in \mathbb{Z}$

$$\begin{aligned} u(t,y) = \ & E_y\left[\exp\left(-\int_0^t \kappa(t-s, X_s) ds\right) u_0(X_t)\right] \\ = & \sum_{x\in\mathbb{Z}} u_0(x)\ E_x\left[\exp\left(- \int_0^t \kappa(s, X_s) ds\right); X_t = y\right] = \sum_{x\in\mathbb{Z}} u_0(x)\ P_x(\tilde{X}_t = y). \end{aligned}$$

At this point, one could deduce Theorem 3.1 from the previous display by combining the Karlin-McGregor determinant formula of coincidence probabilities in [Kar88] with well-known variation diminishing properties of strictly totally positive matrices (see, for example, Chapter 3 in [Pin10]). Nevertheless, we believe that the arguments we present in the following might be of independent interest, as they do not involve determinants, but rather a purely probabilistic study of particle systems.

We conclude this section with a few reductions of the statement of the theorem. First, by rescaling $u_0 \in \ell^1(\mathbb{Z})$ by its ℓ^1-norm, we may assume from now on that $|u_0|$ defines a probability measure on $\mathbb{Z}$. Second, we claim that it is enough to proof the monotonicity of the number of zero-crossings in the case $s = 0$. Indeed, by the above representation of u and the Markov property,

$$u(t,\cdot) = \sum_{x\in\mathbb{Z}} u(s,x) P_0(\tilde{X}_t = \cdot\ | \tilde{X}_s = x),$$

for each $0 \leq s \leq t$. But $u(s,\cdot) \in \ell^1(\mathbb{Z})$, and $(\tilde{X}_{s+t})_{t\geq 0}$ conditional on $\tilde{X}_s = x$ is again a KRW, now started at x.

7.2. Annihilating particles

The first part of this section provides the main tool for the proof of Theorem 3.1, namely a sequence of measure-valued processes that converge, in a suitable sense, to the solution of (3.1).

In the following, we let $\mathcal{M}$, $\overline{\mathcal{M}}$ be, respectively, the spaces of finite signed measures on $\mathbb{Z}$, $\overline{\mathbb{Z}}$, and $\mathcal{N} \subset \mathcal{M}$, $\overline{\mathcal{N}} \subset \overline{\mathcal{M}}$ the corresponding subspaces containing the integer-valued ones. All spaces are equipped with the weak topology.

For every $\nu \in \mathcal{N}$, there exists a unique choice of integers $x_1, \dots, x_n \in \mathbb{Z}$ and signs $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ such that

$$\nu = \sum_{i=1}^{n} \varepsilon_i \delta_{x_i}, \ x_1 \leq \ \cdots \leq x_n \text{ and } x_i < x_{i+1} \text{ whenever } \varepsilon_i \neq \varepsilon_{i+1}.$$

We denote this unique sequence of signs with $\mathcal{S}(\nu) := (\varepsilon_1, \dots, \varepsilon_n)$. For $\tilde{\nu} \in \overline{\mathcal{N}}$ we define $\mathcal{S}(\tilde{\nu}) = \mathcal{S}(\tilde{\nu}|_{\mathbb{Z}})$.

Our first goal is to construct, for every $\nu \in \mathcal{N}$, a $\overline{\mathcal{N}} \times \overline{\mathcal{N}}$-valued process (Y, Z) that describes the behavior of n interacting signed particles on the integer line. More specifically, Z records the positions of the particles—which we start at $x_1, \dots, x_n$—when considered as unsigned and non-interacting, while Y incorporates the sign information and the corresponding interaction structure. Later, the number of particles will be sent to infinity.

The explicit construction of the process goes as follows. For a given $\nu \in \mathcal{N}$, let $\tilde{X}^1, \dots, \tilde{X}^n$ be independent copies of the KRW from the previous section, starting at $x_1, \dots, x_n$. We define Z via $Z_t := \sum_{i=1}^n \delta_{\tilde{X}^i_t}$. The process Y is constructed from Z by first assigning to each particle the sign ε_i, so that $Y_0 = \sum_{i=1}^n \varepsilon_i \delta_{\tilde{X}^i_0}$, and then annihilating any two particles with opposite sign upon meeting at any site other than the cemetery $\dagger$. As annihilating two such signed particles has the same effect as freezing them, the process can be written as $Y_t = \sum_{i=1}^n \varepsilon_i \delta_{\tilde{X}^i_{t\wedge\tau_i}}$ for suitable stopping times $\tau_1, \dots, \tau_n \in (0, \infty]$.

We now explain how these stopping times are constructed. The annihilation time τ_i of the i-th particle is defined as the first time it encounters another particle, not yet annihilated, with opposite sign. If an alive particle $\delta_{\tilde{X}^i}$ (not yet annihilated nor sent to the cemetery) with sign ε_i jumps to a site $x \in \mathbb{Z}$ where possibly multiple (not yet annihilated) particles, say $\delta_{\tilde{X}^{j_1}}, \dots, \delta_{\tilde{X}^{j_m}}$, having sign $-\varepsilon_i$ are located, then only one of them (say the one with the smallest index) will annihilate with $\delta_{\tilde{X}^i}$, which, for the corresponding stopping times, means $\tau_i = \tau_{j_1} < \tau_{j_k}$ for $2 \leq k \leq m$. The independence of the random walks implies that their jumps—and hence the annihilations—almost surely occur at distinct times. Consequently, if $\tau_i = \tau_j = \tau_k$ and $i \neq j \neq k$, then $\tau_i = \tau_j = \tau_k = \infty$ almost surely, which is possible as no annihilation occurs at the cemetery.

Our next focus is the study of the zero-crossings of the constructed process. To this end, we extend the notion of zero-crossing to sign sequences and finite signed measures on $\mathbb{Z}$ by identifying them with sequences in $\ell^1(\mathbb{Z})$. The definition can be extended to any finite signed measure μ on $\overline{\mathbb{Z}}$ via $\Sigma(\mu) := \Sigma(\mu|_{\mathbb{Z}})$.

By considering the process $(\mathcal{S}(Y_t))_{t\geq 0}$, which records the signs of the particles together with their ordering, we notice that $\Sigma(\mu) = \Sigma(\mathcal{S}(\mu))$ for any $\mu \in \overline{\mathcal{N}}$. In particular, it is sufficient to describe the time evolution of the zero-crossings of $\mathcal{S}(Y_t)$. To do so, we introduce the notion of a substring. For $\mu, \tilde{\mu} \in \overline{\mathcal{N}}$, we say that $\mathcal{S}(\mu)$ is a substring of the sign sequence $\mathcal{S}(\tilde{\mu})$, denoted $\mathcal{S}(\mu) \preceq \mathcal{S}(\tilde{\mu})$, if $\mathcal{S}(\mu)$ is either identical to $\mathcal{S}(\tilde{\mu})$ or can be obtained from it by removing finitely many signs.

The following result, which will be used in the next section, serves as our counterpart to part *(iv)* of Lemma 3.1 in [EW99]. The proof can be adapted to our framework because particles move via nearest-neighbor jumps, a property that plays the role of path continuity in the original argument.

Lemma 7.1. *Almost surely, for each $s \leq t$*

$$\mathcal{S}(Y_t) \text{ is a substring of } \mathcal{S}(Y_s) \quad \text{and} \quad \Sigma(Y_t) \leq \Sigma(Y_s) \leq \Sigma(\nu).$$

Proof. Almost surely, each particle has a finite life-time and therefore jumps finitely many times. Therefore, there are finitely many collisions, and the right-continuous $\overline{\mathcal{N}}$-valued process $(Y_t)_{t\geq 0}$ is almost surely piecewise constant with finitely many jumps. The same holds for the process $(\mathcal{S}(Y_t))_{t\geq 0}$. Moreover, if s and t are such that the process has exactly one jump in the time interval $(s, t]$, then $\mathcal{S}(Y_t)$ is obtained from $\mathcal{S}(Y_s)$ by the removal of either one sign, in the case where a particle is sent to the cemetery, or two consecutive signs, in the case of a particle pair undergoing annihilation. In fact, since particles do not jump simultaneously, $\mathcal{S}(Y_t)$ cannot be obtained by permuting two signs in $\mathcal{S}(Y_s)$. In particular, one has $\mathcal{S}(Y_t) \preceq \mathcal{S}(Y_s)$. By transitivity, we conclude that $\mathcal{S}(Y_t) \preceq \mathcal{S}(Y_s)$ for all $0 \leq s \leq t$. The second part of the lemma is a direct consequence of the first one. □

We now construct a sequence of $\overline{\mathcal{N}}$-valued processes $(Y^n)_{n\in\mathbb{N}}$, each representing the dynamics of n particles evolving under the interaction and annihilation mechanism of the process Y above. We add an additional layer of randomness by choosing random initial configurations $(Y_0^n)_{n\in\mathbb{N}}$.

Let $(\mathcal{X}_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of $|u_0|$-distributed starting positions, with corresponding signs $\mathcal{E}_i := \operatorname{sgn}(u_0(\mathcal{X}_i))$. We recall that we assume $|u_0|$ is a probability measure. Then, defining

$$Y_0^n := \sum_{i=0}^{n} \mathcal{E}_i \delta_{\mathcal{X}_i}$$

yields, by repeating the previous construction with Y_0^n in place of ν, a sequence of $\overline{\mathcal{N}}$-valued processes $(Y^n)_{n\in\mathbb{N}}$.

Once normalized, these processes converge (in a suitable sense) to the deterministic function $t \in [0, \infty) \to u(t, \cdot)$, which can be extended to $\overline{\mathbb{Z}}$ by setting $u(t, \dagger) := \sum_{x\in\mathbb{Z}} u_0(x) P_x(\tilde{X}_t = \dagger)$ and thus can be viewed as an element of $\overline{\mathcal{M}}$.

Lemma 7.2. *The sequence* $(\frac{1}{n}Y^n)_{n\in\mathbb{N}}$ *of càdlàg* $\overline{\mathcal{M}}$*-valued processes converges in probability in the Skorokhod topology to the continuous deterministic function* $t \mapsto u(t,\cdot) \in \overline{\mathcal{M}}$.

The proof of Lemma 7.2 is a straightforward adaptation of the arguments in the proof of Lemma 4.1 in [EW99]. The crucial observation, which makes the adaptation to our setting straightforward, is that our KRW induces a Feller semigroup $(\mathcal{P}_t)_{t\geq 0}$ of linear operators on the space or continuous functions on $[0,\infty)\times\mathbb{Z}$. More precisely, the family $(P_{(s,x),t})_{s,t\geq 0,x\in\mathbb{Z}}$ of subprobability measures on $[0,\infty)\times\mathbb{Z}$, given by

$$P_{(s,x),t}(A\times B) := \delta_{s+t}(A)P_x(\tilde{X}_{s+t}\in B|\tilde{X}_s = x),$$

induces a semigroup with the desired properties via

$$\mathcal{P}_t(f)(s,x) := \int P_{(s,x),t}(dz)f(z) = \sum_{y\in\mathbb{Z}} f(s+t,y)P_x(\tilde{X}_{s+t} = y|\tilde{X}_s = x).$$

Proof of Theorem 3.1. Building on the above results, we conclude the proof of Theorem 3.1 via a minor modification of the arguments in Section 4 in [EW99].

We begin by establishing (3.2). By Lemma 7.2, we can pick a subsequence $(\frac{1}{n_k}Y^{n_k})_{k\in\mathbb{N}}$ converging almost surely to $(u(t,\cdot))_{t\geq 0}$ and hence, by continuity of the function $t\mapsto u(t,\cdot)\in\overline{\mathcal{M}}$ with respect to the weak topology, such that for all $t\geq 0$ the random measures $(\frac{1}{n_k}Y_t^{n_k})_{k\in\mathbb{N}}$ converge almost surely to $u(t,\cdot)$. In particular, if $\Sigma(u(t,\cdot)) = m$ then, almost surely, there exist $x_1 < \cdots < x_{m+1}\in\mathbb{Z}$ and a (random) $\mathcal{K}_0\in\mathbb{N}$ such that

$$\frac{Y_t^{n_k}(x_i)}{n_k}\frac{Y_t^{n_k}(x_{i+1})}{n_k} < 0, \qquad \text{for } k\geq\mathcal{K}_0 \text{ and } 1\leq i\leq m,$$

which implies that $\liminf_{k\to\infty}\Sigma(Y_t^{n_k})\geq m$. Therefore, almost surely,

$$\Sigma(u(t,\cdot)) \leq \liminf_{k\to\infty}\Sigma(Y_t^{n_k}).$$

By Lemma 7.1, the right-hand side is almost surely bounded by $\liminf_{k\to\infty}\Sigma(Y_0^{n_k})$, which, by construction, coincides with $\Sigma(u_0)$.

To prove the second part of the theorem, we first show the following result.

Claim 7.3. *Under the assumptions of Theorem 3.1, if* u_0 *is such that* $u_0(x) < 0$ *implies* $u_0(y)\leq 0$ *for each* $y < x$ *and* $u_0(x) > 0$ *implies* $u_0(y)\geq 0$ *for each* $y > x$*, then* $u(t,\cdot)$ *satisfies the same property for every* $t\geq 0$.

Even though this property seems to be a direct consequence of (3.2), we need to rule of the possibility that the two signs in $\mathcal{S}(\mathrm{sgn}(u(t,\cdot)))$ change order in time.

Proof of Claim 7.3. We argue by contradiction, and assume that $u(t,x) > 0 > u(t,y)$ for some $y > x$. Then, by repeating the above arguments, $Y_t^n(x) > 0 > Y_t^n(y)$ for sufficiently large $n\in\mathbb{N}$, almost surely. In particular, $(+1,-1)\preceq\mathcal{S}(Y_t^n)$.

Since $\mathcal{S}(Y_t^n) \preceq \mathcal{S}(Y_0^n)$ by Lemma 7.1, one can find $y_0 > x_0$ with $Y_0^n(x_0) > 0 > Y_0^n(y_0)$. By construction of Y_0^n, this contradicts the assumptions on u_0. □

We now use this claim to derive a stronger result, namely that $u(t,y) > 0$ implies $u(t,x) > 0$ for all $x > y$. Assume by contradiction that there exists $x > y$ with $u(t,x) = 0 < u(t,y)$ (by Claim 7.3, $u(t,x)$ must be non-negative). If we set

$$0 < \varepsilon \coloneqq \begin{cases} \frac{1}{2}u(t,y) \wedge u_0(x), & \text{if } u_0(x) > 0 \\ \frac{1}{2}u(t,y), & \text{if } u_0(x) \leq 0 \end{cases}$$

and consider the solution

$$u^\varepsilon(t,\cdot) = -\varepsilon P_x(\tilde{X}_t = \cdot\,) \in (-\varepsilon, 0), \quad t > 0,$$

of (3.1) with initial condition $u_0^\varepsilon \coloneqq -\varepsilon \mathbf{1}_x$, then $u(t,\cdot) + u^\varepsilon(t,\cdot)$ describes a solution of (3.1) with initial condition $u_0 + u_0^\varepsilon \in \ell^1(\mathbb{Z})$. By the construction of ε and the assumptions on u_0, $u_0 + u_0^\varepsilon \in \ell^1(\mathbb{Z})$ satisfies the assumptions of Claim 7.3. However, since $u(t,y) + u^\varepsilon(t,y) \geq \frac{1}{2}u(t,y) > 0$ and $u(t,x) + u^\varepsilon(t,x) < 0$, the claim yields a contradiction. Therefore, $\{x : u(t,x) > 0\}$ is necessarily a (possibly empty) set of the form $\{b_t, b_t + 1, \ldots,\}$. A similar argument can be carried out for the negative case. This concludes the proof of Theorem 3.1. □

We conclude with a few remarks on possible generalizations of Theorem 3.1. Replacing $\tilde{X}$ with a killed simple random walk with non-zero drift and arbitrary jump rate does not affect the arguments presented above. In particular, Theorem 3.1 can be extended to a broader class of discrete differential equations. More precisely, if $\alpha > 0$ and $\beta \in (-\alpha, \alpha)$ then one can replace equation (3.1) in the theorem with

$$\partial_t u(t,x) = \alpha \Delta_d u(t,x) + \beta \partial_x u(t,x) - \kappa(t,x) u(t,x),$$

where $\partial_x u(t,x) = u(t,x+1) - u(t,x-1)$. The nearest-neighbor killed random walk corresponding to this differential equation is the one which jumps with rate 2α, has drift $-\beta/\alpha$, and is killed with rate $\kappa(t,x)$.

On the other hand, the restriction to nearest-neighbor jumps is crucial for the theorem to hold without additional assumptions on $u_0 \in \ell^1(\mathbb{Z})$. Dropping this condition allows simple counterexamples in which the number of zero-crossings increases over time. For instance, if $u_0 = \mathbf{1}_{\{0\}} - \mathbf{1}_{\{1\}}$ and the random walk satisfies $\{x\} \subsetneq \operatorname{supp}(P_x(\tilde{X}_t = \cdot\,)) \subset x + 2\mathbb{Z}$ for $x \in \{0,1\}$, then $\Sigma(u(t,\cdot)) > 1 = \Sigma(u_0)$.

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