ON THE TIGHTNESS OF THE MAXIMUM OF BRANCHING BROWNIAN MOTION IN RANDOM ENVIRONMENT

JIŘÍ ČERNÝ, ALEXANDER DREWITZ, AND PASCAL OSWALD

Abstract. We consider one-dimensional branching Brownian motion in spatially random branching environment (BBMRE) and show that for almost every realisation of the environment, the distribution of the maximal particle of the BBMRE re-centred around its median is tight. This result is in stark contrast to the fact that the transition fronts in the solution to the randomised F-KPP equation are, in general, not bounded uniformly in time. In particular, this highlights that—when compared to the setting of homogeneous branching—the introduction of a random environment leads to a much more intricate behaviour.

1. Introduction

The behaviour of the position of the maximally—or, equivalently, minimally—displaced particle in various variants of branching random walk (BRW) and branching Brownian motion (BBM) has been the subject of intensive research over the last couple of decades [Bra78, Bra83, BZ07, ABR09, HS09, Aïd13]. While initially most of the work focused on branching systems with homogeneous branching rates, there has recently been an increased activity in the investigation of branching random walks with non-homogeneous branching rates that depend on either time or space mostly in special *deterministic* ways, see [LS88, LS89, FZ12a, FZ12b, BBH+15, MZ16, Mal15, BH14, BH15, ČD20, Kri21, HRS22, Kri22].

In this article we continue the study of the maximally displaced particle in the model of branching Brownian motion with spatially random branching environment (BBMRE) which was initiated in [DS22], building on the previous work [ČD20] on a discrete-space analogue, the branching random walk in i.i.d. random environment (BRWRE). The techniques developed in [ČD20, DS22] also lent themselves to obtain refined information on the front of the solution of the randomised Fisher-Kolmogorov-Petrovskii-Piskunov (F-KPP) equation [ČDS22]. Subsequently, the techniques and results of [ČD20] have been extended to the continuum space setting of BBMRE in [HRS22].

We complement the above body of findings by addressing a seemingly simple, but subtle problem that arises naturally, and which has also been formulated as an open question in [ČD20]. More precisely, we show that the distributions of the position of the maximally displaced particle of the BBMRE, when re-centred around its median, form a tight family of distributions as time evolves. While establishing tightness might a priori not look like an overly intricate problem, we take the opportunity to emphasise that such a preconception is erroneous, see also [BZ09, BZ07]. Our result is particularly interesting as it sharply contrasts the result established in [ČDS22] that the transition fronts of the solution to the randomised F-KPP equation are, in general, unbounded in time. In the homogeneous setting, such a dichotomy cannot be observed since, a fortiori, there is a duality between these two objects in that setting in that tightness of the re-centred maximum of

Date: December 23, 2022.

BBM is equivalent to the uniform boundedness in time of the transition fronts of the solution to F-KPP.

1.1. **Homogeneous BBM and F-KPP equation.** To explain this duality more in detail, we start with recalling the model in the homogeneous situation, which will also serve as a point of reference throughout the article. For a (binary) branching Brownian motion with homogeneous branching rate equal to one, started from a single particle located at the origin, we denote its maximal displacement at time t by $M(t)$, and write

$$w(t,x) = P(M(t) \geq x), \tag{1.1}$$

for the probability that this displacement exceeds $x \in \mathbb{R}$. Then, the function $w(t,x)$ solves a non-linear PDE, known as the Fisher-Kolmogorov-Petrovskii-Piskunov (F-KPP) equation,

$$\partial_t w(t,x) = \frac{1}{2}\partial_x^2 w(t,x) + w(t,x)(1-w(t,x)), \qquad t>0, x\in\mathbb{R}, \tag{1.2}$$

with the initial datum $w(0,\cdot) = \mathbf{1}_{(-\infty,0]}$ of Heaviside type, see [INW68, McK75]. Moreover, it is well known that as $t\to\infty$, the solution to (1.2) approaches a *travelling wave* g in the following sense: for an appropriate function $m:(0,\infty)\to[0,\infty)$ one has that

$$w(t, m(t)+\cdot) \to g \quad \text{uniformly as } t\to\infty \tag{1.3}$$

for a decreasing function g satisfying $\lim_{x\to\infty} g(x) = 0$ and $\lim_{x\to-\infty} g(x) = 1$. A critical ingredient in the proof of this convergence is that, again for $m(t)$ being chosen appropriately, one has

$$\begin{aligned} &w(t, x+m(t)) \text{ is increasing in } t \text{ for } x<0, \text{ and} \\ &w(t, x+m(t)) \text{ is decreasing in } t \text{ for } x>0. \end{aligned} \tag{1.4}$$

Property (1.3) immediately yields for every $\varepsilon>0$ the existence of some $r_\varepsilon\in(0,\infty)$ such that

$$w(t, m(t)+r_\varepsilon) - w(t, m(t)-r_\varepsilon) > 1-\varepsilon \qquad \text{for all } t\geq 0. \tag{1.5}$$

In other words, the family $(M(t)-m(t))_{t\geq 0}$ is tight. Another, essentially trivial, consequence of (1.3) is the uniform boundedness of the width of the transition front of the solution to (1.2), namely that for every $\varepsilon\in(0,1/2)$,

$$\limsup_{t\to\infty} \operatorname{diam}\big(\{x\in\mathbb{R} : w(t,x)\in[\varepsilon, 1-\varepsilon]\}\big) < \infty. \tag{1.6}$$

In this context, it is worth pointing out that the above line of reasoning implicitly uses the reflection symmetry of Brownian motion and the homogeneity of the branching environment. As a consequence, it breaks down in the presence of an inhomogeneous environment, and the relationship between the solutions of the F-KPP equation and the maximum of BBMRE becomes more intricate than that given in (1.1) and (1.2), cf. Section 3.1.

1.2. **Randomised F-KPP equation.** In the inhomogeneous setting of a random potential, as considered in the current paper, the respective randomised F-KPP equation has been investigated in [ČDS22]. In that source it has been established that for a canonical choice of random potentials ξ, the transition front of the solution to the inhomogeneous F-KPP equation (which is discussed in more detail in Section 3.1)

$$\partial_t w^\xi(t,x) = \frac{1}{2}\partial_x^2 w^\xi(t,x) + \xi(x) w^\xi(t,x)(1-w^\xi(t,x)), \qquad t>0, x\in\mathbb{R}, \tag{1.7}$$

with the initial condition $w^\xi(0,\cdot) = \mathbf{1}_{(-\infty,0]}$ does not need to be uniformly bounded in time, in the sense that the width of their transition fronts can be unbounded. More precisely, cf. (1.6), it follows from [ČDS22, Theorem 2.3] that there are random potentials ξ within the class of inhomogeneities considered in the current paper, such that $\mathbb{P}$-a.s., for all $\varepsilon \in (0,1/2)$,

$$\limsup_{t\to\infty} \operatorname{diam}\big(\{x \in \mathbb{R} : w^\xi(t,x) \in [\varepsilon, 1-\varepsilon]\}\big) = +\infty. \tag{1.8}$$

It is hence non-trivial and might be surprising that for BBMRE in the random potential ξ we obtain tightness for the re-centred family of maxima, and a novel approach is required in order to address this situation adequately.

It is worthwhile to note that the PDE results of [ČDS22] have been obtained by taking advantage of almost exclusively probabilistic techniques. In the current article, however, the probabilistic main result is proven via a symbiosis of analytic and probabilistic techniques.

2. Definition of the model and the main result

We work with a model of *branching Brownian motion in random branching environment* (BBMRE) introduced in [ČDS22, DS22] as a continuous space version of the branching random walk in random environment model studied in [ČD20]. The random environment is given by a stochastic process $\xi = (\xi(x))_{x\in\mathbb{R}}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which fulfils the following assumptions.

Assumption 1.
- The sample paths of ξ are $\mathbb{P}$-a.s. locally Hölder continuous, that is, for almost every ξ there exists $\alpha = \alpha(\xi) \in (0,1)$ and for every compact $K \subseteq \mathbb{R}$ a constant $C = C(K,\xi) > 0$ such that
$$|\xi(x) - \xi(y)| \le C|x-y|^\alpha, \qquad \text{for all } x, y \in K. \tag{2.1}$$
- ξ is uniformly elliptic in the sense that
$$0 < \texttt{ei} := \operatorname{ess\,inf} \xi(0) < \operatorname{ess\,sup} \xi(0) =: \texttt{es} < \infty. \tag{2.2}$$
- ξ is stationary, that is, for every $h \in \mathbb{R}$
$$(\xi(x))_{x\in\mathbb{R}} \overset{(d)}{=} (\xi(x+h))_{x\in\mathbb{R}}. \tag{2.3}$$
- ξ fulfils a ψ*-mixing* condition: There exists a continuous non-increasing function $\psi : [0,\infty) \to [0,\infty)$ satisfying $\sum_{k=1}^\infty \psi(k) < \infty$ such that (using the notation $\mathcal{F}_A = \sigma(\xi(x) : x \in A)$ for $A \subset \mathbb{R}$) for all $Y \in L^1(\Omega, \mathcal{F}_{(-\infty,j]}, \mathbb{P})$, and all $Z \in L^1(\Omega, \mathcal{F}_{[k,\infty)}, \mathbb{P})$ we have
$$\begin{aligned} \big|\mathbb{E}\big[Y - \mathbb{E}[Y] \mid \mathcal{F}_{[k,\infty)}\big]\big| &\le \mathbb{E}[|Y|]\,\psi(k-j), \\ \big|\mathbb{E}\big[Z - \mathbb{E}[Z] \mid \mathcal{F}_{(-\infty,j]}\big]\big| &\le \mathbb{E}[|Z|]\,\psi(k-j). \end{aligned} \tag{2.4}$$
(Note that this conditions implies the ergodicity of ξ with respect to the usual shift operator.)

In the current article we do not explicitly make use of the mixing condition. However, in particular in the Appendix, we will employ some of the results developed in [ČDS22, DS22] which depend on this mixing assumption.

The dynamics of BBMRE started at a position $x \in \mathbb{R}$ is as follows. Given a realisation of the environment ξ, we place one particle at x at time $t = 0$. As time evolves, the particle follows the trajectory of a standard Brownian motion $(X_t)_{t\ge0}$. Additionally and independently of everything else, while at position y, the particle gets killed with rate $\xi(y)$. Immediately after its death, the particle is replaced by k independent copies at the site of death, according to some fixed offspring distribution $(p_k)_{k\in\mathbb{N}}$. All k descendants evolve independently of each other according to the same stochastic diffusion-branching dynamics.

We denote by $\mathtt{P}_x^\xi$ the quenched law of a BBMRE, started at x and write $\mathtt{E}_x^\xi$ for the corresponding expectation. Moreover, we denote by $N(t)$ the set of particles alive at time t. For any particle $\nu \in N(t)$ we denote by $(X_s^\nu)_{s\in[0,t]}$ the spatial trajectory of the genealogy of ancestral particles of ν up to time t. Our main focus of interest lies in the maximally displaced particle of the BBMRE at time t,

$$M(t) := \sup\{X_t^\nu : \nu \in N(t)\}.$$

Throughout this article we deal with supercritical branching such that the offspring distribution has second moments and particles always have at least one offspring.

Assumption 2. The offspring distribution $(p_k)_{k\geq 1}$ satisfies

$$\sum_{k=1}^{\infty} kp_k =: \mu > 1, \qquad \text{and} \qquad \sum_{k=1}^{\infty} k^2 p_k =: \mu_2 < \infty. \tag{2.5}$$

It is well known that under these assumptions the maximally displaced particle $M(t)$ satisfies a law or large numbers for some non-random asymptotic velocity $v_0 \in (0,\infty)$. The asymptotic velocity can be characterised as the unique positive root of the *Lyapunov exponent* λ, which is a deterministic function $\lambda : \mathbb{R} \to \mathbb{R}$ that admits the representation

$$\lambda(v) = \lim_{t\to\infty} \frac{1}{t} \ln \mathtt{E}_0^\xi\big[\big|\{\nu \in N(t) : X_t^\nu \geq vt\}\big|\big], \quad \mathbb{P}\text{-a.s.} \tag{2.6}$$

Under Assumptions 1 and 2, the function λ is non-increasing, concave, and there exists a critical value $v_c \geq 0$ pertaining to a linear facet in the graph of λ such that λ is strictly concave on $[v_c,\infty)$, see e.g. [DS22, Proposition A.3]. As in [ČD20, ČDS22, DS22] we make the following technical assumption.

Assumption 3. We only consider BBMREs whose asymptotic speed satisfies

$$v_0 > v_c. \tag{2.7}$$

Essentially, this condition allows the introduction of a *tilted* probability measure, in the *ballistic phase*, under which a Brownian particle $(X_t)_{t\geq 0}$ moves *on average* with speed v_0 up to time t, cf. Section 4. We refer also to [DS22, Section 4.4] for a detailed discussion on the condition (2.7), as well as for examples of potentials ξ which do and do not satisfy (2.7).

Finally, we also define for $\varepsilon \in (0,1)$ the quenched quantiles for the distribution of $M(t)$ where the process is started at the origin,

$$m_\varepsilon^\xi(t) := \inf\big\{y \in \mathbb{R} : \mathtt{P}_0^\xi(M(t) \leq y) \geq \varepsilon\big\}. \tag{2.8}$$

For notational convenience, we drop the subscript when $\varepsilon = 1/2$ and write $m^\xi(t)$ for the median of the distribution.

With this, we can state our main result.

Theorem 2.1. *Under Assumptions 1–3, for almost every realisation of the environment ξ, the family $\big(M(t) - m^\xi(t)\big)_{t\geq 0}$ is tight under $\mathtt{P}_0^\xi$.*

This result should be contrasted with the behaviour (1.8) of transition fronts of solutions to the inhomogeneous F-KPP equation (1.7) discussed in the introduction. In [ČDS22, Theorem 2.3, Theorem 2.4] environments ξ were constructed, under the additional requirement that $\texttt{es}/\texttt{ei} > 2$, and which are within the framework being discussed here, for which solutions w^ξ of the F-KPP equation are not tight, in the sense that they have transition fronts that grow logarithmically in time, along a subsequence. More precisely, it is shown that environments satisfying Assumption 1 exist such that for small enough $\varepsilon > 0$, there exist times and positions $(t_n)_n, (x_n)_n \in \Theta(n)$ and a function $\varphi \in \Theta(\ln n)$ such that

$$w^\xi(t_n, x_n) \geq w^\xi(t_n, x_n + \varphi(n)) + \varepsilon. \tag{2.9}$$

This also implies spatial non-monotonicity of the functions $w^\xi(t, \cdot)$.

The existence of environments for which (1.8) holds and the non-monotonicity of (2.9) sharply contrast the homogeneous case, as indicated by (1.4) and (1.5), where the usual argument for tightness of BBM is by the uniform boundedness in time of transition fronts for the corresponding homogeneous F-KPP solutions.

Questions of tightness also arise naturally and have been addressed in many other classes of models. In [BZ09] analytic tools have been developed in order to establish tightness for a class of discrete time models whose distribution function satisfy certain recursive equations, analogous to the F-KPP equation in the case of BBM. These tools are powerful and were applied and adapted to show tightness for several models, e.g. [ABR09, BDZ11, DRZ21, FZ12a, HS09, NZ21] to name a few. For BBM in a periodic environment [LTZ22] used an analytic result on the F-KPP front in periodic environment [HNRR16] which directly implies tightness.

In the context of the discrete space model of [ČD20], sub-sequential tightness along a deterministic sequence is shown for the quenched and annealed law of the maximally displaced particle in [Kri21] using a Dekking-Host type argument. Our method relies crucially on analytic properties of solutions to the F-KPP equation, and differs from the approaches in the above mentioned articles.

The tightness result of Theorem 2.1 naturally suggests the question whether the random variables $M_t - m^\xi(t)$ converge in distribution as $t \to \infty$. Supported by the numerical simulations presented in Figure 1, we conjecture that the answer to this question is negative.

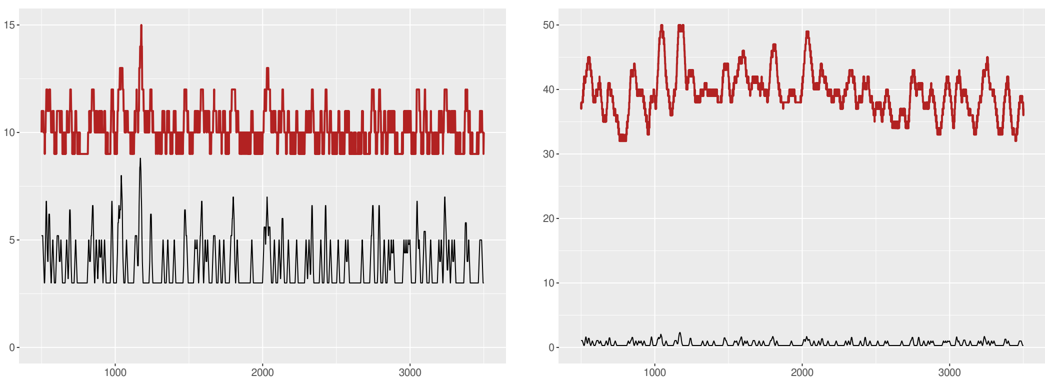

Figure 1. Numerical simulations suggesting that the distributions of $M(t) - m^\xi(t)$ do not converge as $t \to \infty$. The red line shows the dependence of the "spread" of this distribution, that is of $m^\xi_{0.99}(t) - m^\xi_{0.01}(t)$, on the median $m^\xi(t)$. The black line shows the corresponding potential $\xi(x)$ as function of x. The simulations were performed for a discrete-space model, for realisations of ξ from two different distributions (left and right panel).

2.1. **Strategy of the proof.** One of the key ideas in the proof is making use of a powerful analytic technique from the theory of parabolic equations, which can be called a "Sturmian principle", see Section 3.3 for details. In virtue of the duality between BBMRE and the F-KPP equation, the Sturmian principle will let us translate certain comparisons of the typical behaviour of the maximally displaced particle to comparisons of the behaviour in regions governed by large deviation effects. In order to deal with these large deviation effects we employ a strategy bearing similarities to the ones in [ČD20, DS22, ČDS22]. In virtue of a "many-to-one" lemma (Feynman-Kac formula, cf. Proposition 3.3) we introduce a family of "tilted" probability measures with appropriate tilting parameters, which are amenable to standard techniques and under which the large deviation events of interest become typical.

Organisation of the article. In Section 3 we state the exact variant of randomised F-KPP equation, which is connected to our model and recall the well-known Feynman-Kac formula for it and its linearisation, the parabolic Anderson model. Moreover we give a spatial and temporal perturbation result for solutions of the parabolic Anderson model and discuss a first application of the Sturmian principle to our setting. Section 4 reviews tilted measures, which, on a technical level, will play the role of a suitable "gauging-measure" when comparing probabilities in the subsequent sections. Finally, Sections 5 and 6 deal with the proof of the main theorem. Section 5 provides the main argument and Section 6 deals with a technical lemma which is the driving force behind the proof.

Notational conventions: We often use positive finite constants c_1, c_2, etc. in the proofs. This numbering is consistent within every proof and is reset at its end. We use c, C, c' etc. to denote positive finite constants whose value may change during computations.

3. Preliminaries

This section recalls two important and well known probabilistic tools which will feature heavily in the proof of our main theorem. Furthermore, we make precise the Sturmian principle alluded to above.

3.1. **The randomised F-KPP equation and its linearisation.** As already mentioned in the introduction, there is a fundamental link between branching Brownian motion and solutions to the homogeneous F-KPP equation. It is often attributed to McKean [McK75], but can already be found in Skorohod [Sko64] and Ikeda, Nagasawa and Watanabe [INW68]. Such a connection can also be extended to the setting of random branching rates, as we now detail. For this purpose, assume given an offspring distribution (p_k) as in (2.5). We then consider the random semilinear heat equation

$$\begin{aligned} \partial_t w(t,x) &= \frac{1}{2}\partial_x^2 w(t,x) + \xi(x)F(w(t,x)), && t>0, x\in\mathbb{R}, \\ w(0,x) &= w_0(x), && x\in\mathbb{R}, \end{aligned} \tag{F-KPP}$$

where the non-linearity $F:[0,1]\to[0,1]$ is given by

$$F(w) = (1-w) - \sum_{k=1}^{\infty} p_k(1-w)^k, \qquad w\in[0,1]. \tag{3.1}$$

Then the adaptation of McKean's representation of solutions to (F-KPP) takes the following form.

Proposition 3.1. *For any function $w_0 : \mathbb{R} \to [0,1]$ which is the pointwise limit of an increasing sequence of continuous functions, and for any bounded, locally Hölder continuous function $\xi : \mathbb{R} \to (0,\infty)$, there exists a solution to* (F-KPP) *which is continuous on $(0,\infty) \times \mathbb{R}$ and which, for $t \in [0,\infty)$ and $x \in \mathbb{R}$, can be represented as*

$$w(t,x) = 1 - \mathtt{E}_x^\xi\Big[\prod_{\nu \in N(t)} \big(1 - w_0(X_t^\nu)\big)\Big]. \tag{3.2}$$

A proof of this proposition can be found e.g. in [DS22, Proposition 2.1]; the formulation in that source is under slightly more restrictive conditions, but it transfers verbatim to the assumptions we impose above.

A crucial consequence of Proposition 3.1 is that the solution w^y of (F-KPP) with Heaviside-like initial condition $w_0^y = \mathbf{1}_{[y,\infty)}$, for $y \in \mathbb{R}$, is linked to the distribution function of $M(t)$ via the identity

$$w^y(t,x) = \mathtt{P}_x^\xi(M(t) \geq y). \tag{3.3}$$

Remark 3.2. It is common practice in the F-KPP literature to normalise the nonlinearity F in such a way that its derivative at the origin is one. Using (2.5) it is easy to check that in our case, $F'(0) = \mu - 1$. In other words, the standard normalisation of equation (F-KPP) corresponds to a branching processes for which the offspring distribution has mean $\mu = 2$, as is also assumed in [DS22]. In (2.5), we assume only that $\mu > 1$ and do not a priori work under the usual F-KPP normalisation. Nevertheless, given any such offspring distribution $(p_k)_{k\in\mathbb{N}}$ with mean $\mu \neq 2$ and a corresponding BBMRE in environment ξ, one can always transform it into another BBMRE in a rescaled environment, so that the transformed process is in the usual normalisation and has the same distribution as the original process. Indeed, the transformation defined by

$$\xi \to (\mu-1)\xi, \quad p_1 \to \frac{\mu + p_1 - 2}{\mu - 1}, \quad \text{and} \quad p_k \to \frac{p_k}{\mu-1} \text{ for } k \geq 2,$$

yields a new offspring distribution with mean two. Moreover, rescaling the environment guarantees that (F-KPP), and the law $\mathtt{P}_x^\xi$ are invariant under the transformation. After rescaling, it holds $F'(0) = 1$ and $\mu_2 > 2$; hence, in light of this reasoning, we will from now on always assume that

$$\mu = 2, \qquad F'(0) = 1, \quad \text{and} \quad \mu_2 > 2. \tag{3.4}$$

Observe also, that by (3.1) this implies that

$$F'(w) \leq 1, \quad \text{and} \quad F''(w) \geq -\mu_2 + 2 \qquad \text{for all } w \in [0,1]. \tag{3.5}$$

Another PDE related to BBMRE, which we make use of later on, is the linearisation of (F-KPP), known as the *parabolic Anderson model* (PAM),

$$\begin{aligned} \partial_t u(t,x) &= \frac{1}{2}\partial_x^2 u(t,x) + \xi(x)u(t,x), && t > 0, x \in \mathbb{R} \\ u(0,x) &= u_0(x), && x \in \mathbb{R}. \end{aligned} \tag{PAM}$$

The PAM has been the subject of intense investigation in its own right, see e.g. [Kön16] and reference therein for a comprehensive overview; our main interest, however, lies in space and time perturbation results that have been developed for its solution in [ČDS22, DS22]. These will be considered in more detail in Section 3.2.

An important strategy for probabilistically investigating the solutions to the equations (F-KPP) and (PAM) is via analysing their Feynman-Kac representations. In what comes below we denote, for arbitrary $x \in \mathbb{R}$, by P_x the probability measure under which the process denoted by $(X_t)_{t\geq 0}$ is a standard Brownian motion started at x. The corresponding expectation operator is denoted by E_x. We also make repeated use of the abbreviation $E_x[f; A]$ for $E_x[f\mathbf{1}_A]$.

Proposition 3.3. *Under Assumptions 1 and 2, the unique non-negative solution u of (PAM) is given by*

$$u(t,x) = E_x\Big[\exp\Big\{\int_0^t \xi(X_r)\,\mathrm{d}r\Big\}\,u_0(X_t)\Big], \qquad t \geq 0, x \in \mathbb{R}, \tag{3.6}$$

and the unique non-negative solution w of (F-KPP) fulfils

$$w(t,x) = E_x\Big[\exp\Big\{\int_0^t \xi(X_r)\widetilde{F}(w(t-r, X_r))\,\mathrm{d}r\Big\}\,w_0(X_t)\Big], \qquad t \geq 0, x \in \mathbb{R}, \tag{3.7}$$

where $\widetilde{F}(w) = F(w)/w$ for $w \in (0,1]$, which can be continuously extended to $\widetilde{F}(0) = \lim_{w\to 0+}\widetilde{F}(w) = \sup_{w\in(0,1]}\widetilde{F}(w) = 1$.

See e.g. [Bra83, (1.32), (1.33)] for references to the former. Note that the Feynman-Kac representation (3.7) for the solution of the F-KPP equation is an implicit expression, whereas the expression in (3.6) is explicit.

Taking advantage of the above, the link between the PAM and BBMRE can be derived by combining the Feynman-Kac representation (3.6) of the solution to (PAM) with a *many-to-one formula*, see e.g. [DS22, Proposition 2.3], in order to arrive at the representation

$$u(t,x) = \mathtt{E}_x^{\xi}\Big[\sum_{\nu\in N(t)} u_0(X_t^{\nu})\Big]$$

of solutions to (PAM).

3.2. **Perturbation results for the PAM.** On a technical level, our primary interest in the PAM comes from results on the sensitivity of its solutions regarding respective disruptions in space and in time. A variant of these results was developed in [ČDS22, DS22] (cf. Lemmas 3.11 and 3.13 from [DS22], or Lemma 4.1 of [ČDS22]) for the study of the fronts of (F-KPP) and (PAM). These perturbation results will be used together with (3.3) and the Feynman-Kac representation, Proposition 3.3, in order to get bounds on the distribution function of the maximally displaced particle in Section 6.

To avoid the dependence of various constants appearing in these perturbation results on the speed, we assume for the rest of the article that the speeds we allow are contained in some arbitrary but fixed compact interval $V \subset (v_c, \infty)$ which has v_0 in its interior (in particular, we require (2.7) to hold). As we can otherwise choose V arbitrarily large, this does not pose any further restrictions for what follows in the subsequent sections.

Lemma 3.4. *(a) For every $\delta > 0$ and $A > 0$, there exist a constant $c_1 \in (1,\infty)$ and a $\mathbb{P}$-a.s. finite random variable $\mathcal{T}_1$ such that for all $t \geq \mathcal{T}_1$ uniformly in $0 \leq h \leq t^{1-\delta}$, and $x, y \in [-At, At]$ with $x < y$, $\frac{y-x}{t} \in V$ and $\frac{y-x}{t+h} \in V$,*

$$E_x\Big[e^{\int_0^{t+h}\xi(X_s)\,\mathrm{d}s}; X_{t+h} \geq y\Big] \leq c_1 e^{c_1 h} E_x\Big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \geq y\Big].$$

(b) *Let $\delta : (0,\infty) \to (0,\infty)$ be a function tending to 0 as $t \to \infty$, and let $A > 0$. Then there exists a constant $c_2 \in (1,\infty)$ and a $\mathbb{P}$-a.s. finite random variable $\mathcal{T}_2$ such that for all $t \geq \mathcal{T}_2$, uniformly in $0 \leq h \leq t\delta(t)$ and $x, y \in [-At, At]$ with $x < y$, $\frac{y-x}{t} \in V$ and $\frac{y+h-x}{t} \in V$,*

$$E_x\Big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \geq y+h\Big] \leq c_2 e^{-h/c_2} E_x\Big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \geq y\Big].$$

The proof of this lemma is a rather straightforward, but lengthy adaptation of the proofs given in [DS22, ČDS22]; we will provide it in Appendix A. It involves comparing the Feynman-Kac representation (3.6) to functionals with respect to the same family of tilted probability measures that are discussed in Section 4 below.

3.3. **Sturmian principle.** In this section we present the analytic ingredient of our proof of Theorem 2.1, which can be motivated as follows: we will later on be interested in differences of the type $W(\cdot,\cdot) = w^{y_1}(\cdot,\cdot) - w^{y_2}(\cdot + T, \cdot)$ for some $T > 0$, and $y_2 > y_1$, where we recall that for any $y \in \mathbb{R}$, we denote by w^y the solution of (F-KPP) with initial condition $w_0 = \mathbf{1}_{[y,\infty)}$. It is immediate that for a given $T > 0$, the function W satisfies the linear parabolic equation

$$\begin{aligned} \partial_t W(t,x) &= \frac{1}{2}\partial_x^2 W(t,x) + G(t,x)W(x,t), && t>0, x \in \mathbb{R}, \\ W(0,x) &= \mathbf{1}_{[y_1,\infty)}(x) - w^{y_2}(T,x), && x \in \mathbb{R}, \end{aligned} \tag{3.8}$$

where G is the bounded measurable function defined by (using the convention $F'(0) = 1$, cf. Remark 3.2)

$$G(t,x) = \begin{cases} \xi(x)\,\frac{F(w^{y_1}(t,x)) - F(w^{y_2}(t+T,x))}{w^{y_1}(t,x) - w^{y_2}(t+T,x)}, & \text{if } w^{y_1}(t,x) \neq w^{y_2}(t+T,x), \\ \xi(x), & \text{if } w^{y_1}(t,x) = w^{y_2}(t+T,x). \end{cases} \tag{3.9}$$

Let us state the following simple observation, which will be used at various stages in the following: By Proposition 3.1 it follows that

$$0 < w^{y_2}(T,x) < 1 \qquad \text{for all } x \in \mathbb{R}. \tag{3.10}$$

As a consequence, the initial condition of (3.8) has exactly one zero-crossing, and it is located at y_1.

In the analysis literature, it has been known for a long time that the cardinality of the set of zero-crossings of solutions to linear parabolic equations is monotonically non-increasing in time, with the earliest reference dating back to at least an article by Charles Sturm in 1836, cf. [Stu36]. Nevertheless, despite this result having been known for almost two centuries by now, it was not until the eighties of the last century that Sturm's ideas really revived in the theory of linear and non-linear parabolic equations, see, e.g., [Ang88, Ang91, DGM14, EW99, Nad15] for a non-exhaustive list. In this list, the ideas in [EW99] stand out, as they involve a simple and purely probabilistic proof, by interpreting the linear parabolic partial differential equations as generators of Markov processes and reducing the study of the zero-crossings to the study of Markovian transition operators acting on signed measure spaces. A more complete history and a detailed discussion of the Sturmian principle and its applications can be found in [Gal04].

Remark 3.5. In this context, it is interesting to note that already in their seminal article on the F-KPP equation, Kolmogorov, Petrovskii and Piskunov also make use of a Sturmian principle for equations of the form (3.8), see [KPP37, Theorem 11], which is proved using a parabolic maximum principle.

We include a version of such results which is formulated to fit our purpose; a more general version of this result can be found in [Nad15]. Note that the assumptions in particular fit the setting of a single zero-crossing in the initial value.

Lemma 3.6 ([Nad15, Proposition 7.1])**.** *For any $t_0 \in \mathbb{R}$, let $G \in L^\infty((t_0,\infty)\times\mathbb{R})$ and assume $W \in C((t_0,\infty)\times\mathbb{R}) \cap L^\infty((t_0,\infty)\times\mathbb{R})$ to be a weak solution of*

$$\begin{aligned} \partial_t W(t,x) &= \frac{1}{2}\partial_x^2 W(t,x) + G(t,x)W(x,t), && t>t_0, x\in\mathbb{R},\\ W(t_0,x) &= W_{t_0}(x), && x\in\mathbb{R}, \end{aligned}$$

where $W_{t_0} \not\equiv 0$ is piecewise continuous and bounded in $\mathbb{R}$, such that for some $z_{t_0} \in \mathbb{R}$ one has

$$W_{t_0}(x) \le 0, \text{ if } x < z_{t_0}, \qquad \text{and} \qquad W_{t_0}(x) \ge 0, \text{ if } x > z_{t_0}.$$

Then, for all $t > t_0$ there exists a unique point $z(t) \in [-\infty,\infty]$ such that

$$W(t,x) < 0, \text{ if } x < z(t), \qquad \text{and} \qquad W(t,x) > 0, \text{ if } x > z(t).$$

As a first application of Lemma 3.6, let us consider the effect on the solution of (F-KPP) when the discontinuity of the Heaviside-type initial condition tends to infinity. For this purpose, in order to obtain a non-trivial limit, we perform an appropriate temporal shift. More precisely, we introduce for a given realisation of the environment ξ, any $y \in \mathbb{R}$ and any $\varepsilon > 0$ the "temporal quantile at the origin" as

$$\tau_y^\varepsilon := \inf\{t \ge 0 : w^y(t,0) \ge \varepsilon\}. \tag{3.11}$$

Since $\mathbb{P}$-a.s. we have $\lim_{t\to\infty} w^y(t,0) = 1$ (due to, e.g., [Fre85, Theorem 7.6.1]), τ_y^ε is finite. By the continuity of w^y on $(0,\infty)\times\mathbb{R}$, cf. Proposition 3.1, the quantity τ_y^ε satisfies $w^y(\tau_y^\varepsilon, 0) = \varepsilon$. Moreover, cf. (3.3), as $w^y(t,0) = \mathtt{P}_0^\xi(M(t) \ge y)$ is increasing in y, so is τ_y^ε, and by the law of large numbers for the maximal displacement (cf. (2.6) and the definition of v_0), it follows readily that $\lim_{y\to\infty} \tau_y^\varepsilon = \infty$.

The shift by τ_y^ε allows to establish the following result, which follows already from [Nad15, Lemma 7.3]. Nevertheless, we provide its short proof here for the sake of completeness and as an illustration of how Lemma 3.6 can be used in this context.

Proposition 3.7. *For every $\varepsilon \in (0,1)$ and for $\mathbb{P}$-a.a. ξ, the limit*

$$w_\varepsilon^\infty(t,x) := \lim_{y\to\infty} w^y(\tau_y^\varepsilon + t, x) \tag{3.12}$$

exists locally uniformly in $(t,x) \in \mathbb{R}^2$, and is a global-in-time (that is, for all $t \in \mathbb{R}$) solution to (F-KPP).

The limiting function w_ε^∞ plays a role comparable to that of a travelling wave solution of the homogeneous F-KPP equation, cf. (1.3). However, unlike in the homogeneous situation outlined in the introduction, w_ε^∞ does not directly provide an argument for tightness because we lack a suitable quantitative control of the random variables τ_y^ε as y varies. Nonetheless, the result of Proposition 3.7 plays a vital role in our proof of tightness. We restrict ourselves to providing a proof of the convergence for $t > 0$ only, as this is sufficient for our purposes in what follows.

Proof of Proposition 3.7. Fix $y_1 < y_2$ and for $t \ge -\tau_{y_1}^\varepsilon = -\tau_{y_1}^\varepsilon \vee -\tau_{y_2}^\varepsilon$ (recall that the latter identity follows from the monotonicity of $y \mapsto \tau_y^\varepsilon$ observed below (3.11)) define the function $W(t,x) := w^{y_1}(t+\tau^\varepsilon_{y_1},x) - w^{y_2}(t+\tau^\varepsilon_{y_2},x)$. Then, similarly as for (3.8) and (3.9), it follows that

$$\partial_t W(t,x) = \frac{1}{2}\partial_x^2 W(t,x) + G(t,x)W(t,x), \qquad t > -\tau^\varepsilon_{y_1}, x \in \mathbb{R}, \tag{3.13}$$

where G is given by

$$G(t,x) = \begin{cases} \xi(x)\,\frac{F(w^{y_1}(t+\tau^\varepsilon_{y_1},x)) - F(w^{y_2}(t+\tau^\varepsilon_{y_2},x))}{w^{y_1}(t+\tau^\varepsilon_{y_1},x) - w^{y_2}(t+\tau^\varepsilon_{y_2},x)}, & \text{if } w^{y_1}(t+\tau^\varepsilon_{y_1},x) \neq w^{y_2}(t+\tau^\varepsilon_{y_2},x), \\ \xi(x), & \text{if } w^{y_1}(t+\tau^\varepsilon_{y_1},x) = w^{y_2}(t+\tau^\varepsilon_{y_2},x). \end{cases}$$

From the assumptions, it follows directly that G is a bounded measurable function. Due to [Fre85, Theorem 7.4.1], there exists for $\mathbb{P}$-a.a. ξ a unique classical solution to (3.13). Moreover, since $w^{y_1}(0,x) = \mathbf{1}_{[y_1,\infty)}(x)$, it holds that

$$W(-\tau^\varepsilon_{y_1},x) = w^{y_1}(0,x) - w^{y_2}(\tau^\varepsilon_{y_2} - \tau^\varepsilon_{y_1},x) = \mathbf{1}_{[y_1,\infty)}(x) - w^{y_2}(\tau^\varepsilon_{y_2} - \tau^\varepsilon_{y_1},x). \tag{3.14}$$

Together with the fact that $0 < w^{y_i}(t,x) < 1$ for $i = 1,2$ and for all $t > 0$ and $x \in \mathbb{R}$ (cf. (3.10)), display (3.14) implies that $W(-\tau^\varepsilon_{y_1},x) < 0$ if $x < y_1$ and $W(-\tau^\varepsilon_{y_1},x) > 0$ if $x > y_1$. By Lemma 3.6, for all $t > -\tau^\varepsilon_{y_1}$, the sets $\{x \in \mathbb{R} : W(t,x) > 0\}$ and $\{x \in \mathbb{R} : W(t,x) < 0\}$ are intervals. But due to the continuity of w^{y_1} and w^{y_2}, we also know that $W(0,0) = w^{y_1}(\tau^\varepsilon_{y_1},0) - w^{y_2}(\tau^\varepsilon_{y_2},0) = \varepsilon - \varepsilon = 0$. Therefore, the above reasoning supplies us with

$$\begin{aligned} w^{y_1}(\tau^\varepsilon_{y_1},x) &\le w^{y_2}(\tau^\varepsilon_{y_2},x), && \text{if } x < 0, \\ w^{y_1}(\tau^\varepsilon_{y_1},x) &\ge w^{y_2}(\tau^\varepsilon_{y_2},x), && \text{if } x > 0. \end{aligned} \tag{3.15}$$

That is, the function $y \mapsto w^y(\tau^\varepsilon_y,x)$ is non-decreasing if $x < 0$ and non-increasing on $x > 0$. As a consequence, the limit $w^\infty_\varepsilon(0,x) := \lim_{y\to\infty} w^y(\tau^\varepsilon_y,x)$ exists pointwise, and thus locally uniformly, for all $x \in \mathbb{R}$, and also implies $0 \le w^\infty_\varepsilon(0,\cdot) \le 1$. As a consequence, the right-hand side of (3.12) converges locally uniformly for $t = 0$. (This should be compared to (1.4) in the introduction, which describes the "spatial stretching" of re-centred solutions to the homogeneous F-KPP equation.)

To prove that the local uniform convergence postulated in (3.12) holds true for $t > 0$ also, one uses standard estimates on solutions of quasilinear parabolic equations (see, e.g., [LSU68], Chapter V). As a consequence of these estimates, the solutions $w^y(t,x)$ together with their derivatives are bounded locally uniformly in (t,x), uniformly for all y sufficiently large. Hence the set $\{w^y : y \ge 0\}$ is precompact in $C^{1,2}_{loc}(\mathbb{R}_+ \times \mathbb{R})$. It therefore contains converging sub-sequences, and every limit point of such sub-sequence is a solution to (F-KPP) with the initial condition $w^\infty(0,\cdot)$. As the solution to (F-KPP) with that given initial condition is unique, this implies that all subsequential limits must agree and thus (3.12) holds for all $t > 0$, as well as the fact that w^∞ solves (F-KPP) for $t \ge 0$. We omit here the proof for $t < 0$, as it will not be needed later on. □

A direct consequence of Proposition 3.7 that is going to be relevant later on, is the following: for almost all realisations of the environment ξ, and given that we know the value of the solution w^y of (F-KPP) at the origin at a certain time, we can find a finite time period after which we can deduce a lower bound for the value of w^y at the origin, at least for y large enough. More precisely, we obtain the following corollary.

Corollary 3.8. *For every $\varepsilon \in (0,1/2)$ there exists a $\mathbb{P}$-a.s. finite random variable $T = T(\xi)$ such that for all $y \in \mathbb{R}$ large enough, and any t for which $w^y(t,0) = \varepsilon$, it holds that*

$$w^y(t+t',0) \ge 1 - \varepsilon/2 \quad \text{for all } t' \in [T, T+1].$$

Proof. Let $y \in \mathbb{R}$ and $t \geq 0$ be such that $w^y(t,0) = \varepsilon$. By (3.11) and the finiteness of τ_y^ε deduced below that display, there exists some $s_0 = s_0(y) \geq 0$ such that $t = \tau_y^\varepsilon + s_0$.

Consider w_ε^∞ from Proposition 3.7 and let

$$s_1 = \inf\{s > s_0 : w_\varepsilon^\infty(s',0) \geq 1 - \varepsilon/4 \text{ for all } s' > s\};$$

note that as w_ε^∞ solves (F-KPP), it follows by [Fre85, Theorem 7.6.1] that for $\mathbb{P}$-a.a. realisations of the environment, $\lim_{s\to\infty} w_\varepsilon^\infty(s,x) = 1$, and hence s_1 is $\mathbb{P}$-a.s. finite. Next, taking advantage of the fact that the convergence in Proposition 3.7 is locally uniform in t, due to the continuity of the functions involved and using the compactness of $[s_1, s_1+1]$, it holds for large enough $y \in \mathbb{R}$ that

$$\sup_{s' \in [s_1, s_1+1]} |w^y(\tau_y^\varepsilon + s', 0) - w_\varepsilon^\infty(s',0)| < \varepsilon/4.$$

Setting $T = s_1 - s_0$, we thus obtain for all y large enough and for all $t' \in [T, T+1]$ (with $s' = s_0 + t' \in [s_1, s_1+1]$) that

$$w^y(t+t',0) = w^y(\tau_y^\varepsilon + s', 0) \geq w_\varepsilon^\infty(s',0) - \varepsilon/4 \geq 1 - \varepsilon/2.$$

This completes the proof. □

This result concludes our analytic preparations on how the set of zero-crossings of solutions to linear parabolic equations evolves, and of how it can be applied to the difference of temporally shifted solutions of (F-KPP).

4. Tilting and exponential change of measure

The last tool that we introduce is a change of measure for Brownian paths in the Feynman-Kac representation, which makes certain large deviation events typical. These measures have been featured heavily in [ČD20, ČDS22, DS22] already, including in the proof of Lemma 3.4. In the aforementioned articles this change of measure has been employed so as to make solutions to (PAM) amenable to the investigation by more standard probabilistic tools. Here we go a step further and consider the stochastic processes driving the tilted path measures. This in turn gives us even more precise control on the tilted measures and allows for comparisons with Brownian motion with constant drift, see Proposition 4.3 below.

To define the tilted measures we set

$$\zeta := \xi - \mathtt{es}. \tag{4.1}$$

Due to the uniform ellipticity (2.2) it follows that $\mathbb{P}$-a.s. for all $x \in \mathbb{R}$,

$$\zeta(x) \in [\mathtt{ei} - \mathtt{es}, 0], \tag{4.2}$$

and ζ is $\mathbb{P}$-a.s. locally Hölder continuous with the same exponent as ξ. Moreover, ζ also inherits the stationarity as well as the mixing property from ξ. For the Brownian motion $(X_t)_{t\geq 0}$ under the measure P_x, as used in the Feynman-Kac representations of Proposition 3.3, we introduce the first hitting times as

$$H_y := \inf\{t \geq 0 : X_t = y\} \quad \text{for } y \in \mathbb{R}.$$

Analogously to [ČD20, ČDS22, DS22], we define for $x, y \in \mathbb{R}$ with $y \geq x$, as well as $\eta < 0$, the *tilted* path measures characterised through events $A \in \sigma(X_{t\wedge H_y}, t \geq 0)$ via

$$P_{x,y}^{\zeta,\eta}(A) := \frac{1}{Z_{x,y}^{\zeta,\eta}} E_x\Big[e^{\int_0^{H_y} (\zeta(X_s)+\eta)\,\mathrm{d}s}; A\Big], \tag{4.3}$$

with normalising constant

$$Z_{x,y}^{\zeta,\eta} := E_x\Big[e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}\Big] \in (0,1]. \tag{4.4}$$

By the strong Markov property, it follows easily that the measures are consistent in the sense that $P_{x,y'}^{\zeta,\eta}(A) = P_{x,y}^{\zeta,\eta}(A)$ for $x \le y \le y'$ and $A \in \sigma(X_{t\wedge H_y}, t \ge 0)$. Hence, for any $x \in \mathbb{R}$, we can extend $P_{x,y}^{\zeta,\eta}$ to a probability measure $P_x^{\zeta,\eta}$ on $\sigma(X_t, t \ge 0)$ with the help of Kolmogorov's extension theorem. We write $E_x^{\zeta,\eta}$ for the expectation with respect to the probability measure $P_x^{\zeta,\eta}$.

Finally, as in [DS22, (2.8)], we introduce the annealed logarithmic moment generating function

$$L(\eta) := \mathbb{E}[\ln Z_{0,1}^{\zeta,\eta}], \tag{4.5}$$

and denote by $\overline{\eta}(v) < 0$ the unique solution of the equation $L'(\overline{\eta}(v)) = \frac{1}{v}$ for any $v > v_c$; observe that the former is well-defined as by [DS22, Lemma 2.4],

$$\overline{\eta}(v) \text{ exists for every } v > v_c;\ v \mapsto \overline{\eta}(v) \text{ is a continuous decreasing function and } \lim_{v\to\infty}\overline{\eta}(v) = -\infty. \tag{4.6}$$

The strong Markov property furthermore entails that, for a fixed realization ζ and any $\eta < 0$, the normalising constants (4.4) are multiplicative in the sense that for any $x < y < z$ in $\mathbb{R}$,

$$Z_{x,z}^{\zeta,\eta} = Z_{x,y}^{\zeta,\eta} Z_{y,z}^{\zeta,\eta}. \tag{4.7}$$

Defining, for some arbitrary but fixed $x_0 \in \mathbb{R}$, the function

$$Z^{\zeta,\eta}(x) := \begin{cases} (Z_{x_0,x}^{\zeta,\eta})^{-1}, & \text{if } x \ge x_0, \\ Z_{x,x_0}^{\zeta,\eta}, & \text{if } x < x_0, \end{cases} \tag{4.8}$$

the identity (4.7) thus implies that for all $x < y$ we have

$$Z_{x,y}^{\zeta,\eta} = \frac{Z^{\zeta,\eta}(x)}{Z^{\zeta,\eta}(y)}. \tag{4.9}$$

The following lemma states some useful properties of the function $Z^{\zeta,\eta}$.

Lemma 4.1. *For every bounded Hölder continuous function $\zeta : \mathbb{R} \to (-\infty, 0]$ and $\eta < 0$, the function $Z^{\zeta,\eta}$ is non-decreasing, strictly positive, twice continuously differentiable and satisfies*

$$\frac{1}{2}\Delta Z^{\zeta,\eta}(x) + (\zeta(x)+\eta)Z^{\zeta,\eta}(x) = 0, \qquad x \in \mathbb{R}. \tag{4.10}$$

Furthermore, we have

$$b^{\zeta,\eta}(x) := \frac{\mathrm{d}}{\mathrm{d}x}\ln Z^{\zeta,\eta}(x) \in \big[\underline{v}(\eta), \overline{v}(\eta)\big], \tag{4.11}$$

where $\underline{v}(\eta) := \sqrt{2|\eta|}$ and $\overline{v}(\eta) := \sqrt{2(\mathtt{es} - \mathtt{ei} + |\eta|)}$.

Remark 4.2. Let us note here that the notation $\underline{v}(\eta)$ and $\overline{v}(\eta)$ introduced in the above lemma is suggestive of velocities. This will be made precise in Lemma 4.4 below.

Proof of Lemma 4.1. The monotonicity and the strict positivity of $Z^{\zeta,\eta}$ follow directly from its definition (4.8), using also (4.4).

To show (4.10), we observe that, for any interval $[x_1, x_2]$, the equation $\frac{1}{2}\Delta u(x) + (\zeta(x)+\eta)u(x) = 0$, $x \in [x_1, x_2]$, with boundary conditions $u(x_i) = Z^{\zeta,\eta}(x_i)$, $i = 1, 2$, has a unique classical solution (see, e.g., [GT01, Corollary 6.9]). Denoting by T the exit time of X from $[x_1, x_2]$, this solution can be represented as (see [Bas98, Theorem II(4.1), p.48])

$$u(x) = E_x\big[Z^{\zeta,\eta}(X_T)e^{\int_0^T(\zeta(X_s)+\eta)\,\mathrm{d}s}\big]. \tag{4.12}$$

On the other hand, for $x \in [x_1, x_2]$, taking $y = x_2$ in (4.9), using (4.4), and the strong Markov property at time T,

$$\begin{aligned} Z^{\zeta,\eta}(x) &= Z^{\zeta,\eta}(y)Z^{\zeta,\eta}_{x,y} \\ &= Z^{\zeta,\eta}(y)E_x\big[e^{\int_0^T(\zeta(X_s)+\eta)\,\mathrm{d}s}Z^{\zeta,\eta}_{X_T,y}\big] \\ &= E_x\big[Z^{\zeta,\eta}(X_T)e^{\int_0^T(\zeta(X_s)+\eta)\,\mathrm{d}s}\big]. \end{aligned} \tag{4.13}$$

Therefore, $Z^{\zeta,\eta}$ satisfies (4.10) on $[x_1, x_2]$. Since the interval $[x_1, x_2]$ is arbitrary, (4.10) holds for every $x \in \mathbb{R}$.

To show (4.11), note first that $b^{\zeta,\eta}$ is well-defined since $Z^{\zeta,\eta}$ is strictly positive and differentiable, by (4.10). Therefore, with $y \geq x$, by (4.9) and the strong Markov property again,

$$\begin{aligned} b^{\zeta,\eta}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\ln Z^{\zeta,\eta}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln Z^{\zeta,\eta}_{x,y} \\ &= \lim_{\varepsilon\to 0^+}\varepsilon^{-1}\Big(\ln E_x\Big[e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}\Big] - \ln E_{x-\varepsilon}\Big[e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}\Big]\Big) \\ &= -\lim_{\varepsilon\to 0^+}\varepsilon^{-1}\ln E_{x-\varepsilon}\Big[e^{\int_0^{H_x}(\zeta(X_s)+\eta)\,\mathrm{d}s}\Big]. \end{aligned} \tag{4.14}$$

It is a known fact that for $\alpha > 0$ and $z_1, z_2 \in \mathbb{R}$, it holds that

$$\ln E_{z_1}[e^{-\alpha H_{z_2}}] = -\sqrt{2\alpha}|z_1 - z_2| \tag{4.15}$$

(cf. [BS02, (2.0.1), p. 204]). In combination with the bounds (4.2), the expectation on the right-hand side of (4.14) thus satisfies

$$\begin{aligned} -\varepsilon\sqrt{2|\eta|} = \ln E_{x-\varepsilon}\big[e^{H_x\eta}\big] &\geq \ln E_{x-\varepsilon}\Big[e^{\int_0^{H_x}(\zeta(X_s)+\eta)\,\mathrm{d}s}\Big] \\ &\geq \ln E_{x-\varepsilon}\big[e^{H_x(\mathtt{ei}-\mathtt{es}+\eta)}\big] = -\varepsilon\sqrt{2(\mathtt{es}-\mathtt{ei}+|\eta|)}, \end{aligned} \tag{4.16}$$

which together with (4.14) implies (4.11). □

The function $b^{\zeta,\eta}(x)$ introduced in (4.11) is useful in describing the law of X under the tilted measure, as it allows an interpretation of the tilted process as a Brownian motion with an inhomogeneous drift, by constructing an appropriate SDE as follows.

Proposition 4.3. *Let $x_0 \in \mathbb{R}$, $\eta < 0$ and let $\zeta : \mathbb{R} \to (-\infty, 0]$ be a locally Hölder continuous function that is uniformly bounded from below. Furthermore, denote by B a standard Brownian motion. Then the distribution of the solution to the SDE*

$$\begin{aligned} \mathrm{d}X_t &= \mathrm{d}B_t + b^{\zeta,\eta}(X_t)\,\mathrm{d}t, \qquad t > 0, \\ X_0 &= x_0, \end{aligned} \tag{4.17}$$

agrees with $P^{\zeta,\eta}_{x_0}$.

Proof. The proof is based on an exponential change of measure for diffusion processes. For the sake of simplicity we write b for $b^{\zeta,\eta}$ and Z for $Z^{\zeta,\eta}$ if no confusion is to arise. By (4.11) we obtain that

$$b' = (\ln Z)'' = \Big(\frac{Z'}{Z}\Big)' = \frac{\Delta Z}{Z} - \Big(\frac{Z'}{Z}\Big)^2 = -2(\zeta + \eta) - b^2. \tag{4.18}$$

Therefore, the bounds (4.11) and (4.2) imply that b is a bounded Lipschitz function and thus there is a strong solution to (4.17), whose distribution we denote by $Q_{x_0} = Q^{\zeta,\eta}_{x_0}$. Let further, as previously, P_{x_0} be the distribution of Brownian motion started from x_0, and let $Q^t_{x_0}$ and $P^t_{x_0}$ be the restrictions of those distributions to the time interval $[0,t]$, $t > 0$. As a consequence of the Cameron-Martin-Girsanov theorem (see, e.g., [RW00, Theorem V.27.1] for a suitable formulation), it is well known that

$$\frac{\mathrm{d}Q^t_{x_0}}{\mathrm{d}P^t_{x_0}} = \exp\Big\{ \int_0^t b(X_s)\,\mathrm{d}X_s - \frac{1}{2}\int_0^t b^2(X_s)\,\mathrm{d}s \Big\} =: M_t, \tag{4.19}$$

for a P_{x_0}-martingale M. (The fact that M_t is a martingale follows, e.g., from [RW00, Theorem IV.37.8], since b is a bounded function.)

With the aim of arriving at a comparison with (4.3), we claim that

$$M(t) = \frac{Z(X_t)}{Z(X_0)} e^{\int_0^t (\zeta(X_s)+\eta)\,\mathrm{d}s}. \tag{4.20}$$

To see this, note first that applying Itô's formula to $\ln Z(x) = \int_{x_0}^x b(t)\,\mathrm{d}t$ yields

$$\frac{Z(X_t)}{Z(X_0)} = \exp\Big\{ \ln Z(X_t) - \ln Z(X_0) \Big\} = \exp\Big\{ \int_0^t b(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t b'(X_s)\,\mathrm{d}s \Big\}. \tag{4.21}$$

Comparing this with (4.19) shows that

$$M(t) = \frac{Z(X_t)}{Z(X_0)} \exp\Big\{ -\frac{1}{2}\int_0^t \big(b'(X_s) + b^2(X_s)\big)\,\mathrm{d}s \Big\}, \tag{4.22}$$

which together with (4.18) implies (4.20).

We can now complete the proof of the proposition. For $y \geq x_0$, let $Q_{x_0,y}$ be the measure Q_{x_0} restricted to the σ-algebra $\mathcal{H}_y = \sigma(X_{s\wedge H_y} : s \geq 0)$. To show that $Q_{x_0} = P^{\zeta,\eta}_{x_0}$, it is sufficient to show that $Q_{x_0,y} = P^{\zeta,\eta}_{x_0,y}$ for all $y > x_0$ (see (4.3)). For this purpose, we observe that by Lemma 4.1, Z is a bounded function on $(-\infty, y]$ and thus the stopped martingale $M^{H_y}_t = M_{t\wedge H_y}$ is uniformly bounded from above. Therefore, by the optional stopping theorem, for any $A \in \mathcal{H}_y$, using (4.19) for the second equality,

$$\begin{aligned}
Q_{x_0,y}(A) &= \lim_{t\to\infty} E^{Q_{x_0,y}}[\mathbf{1}_{A\cap\{H_y\leq t\}}] = \lim_{t\to\infty} E^{P_{x_0}}[M_t \mathbf{1}_{A\cap\{H_y\leq t\}}] \\
&= \lim_{t\to\infty} E^{P_{x_0}}\big[E^{P_{x_0}}[M_t \mathbf{1}_{A\cap\{H_y\leq t\}} \mid \mathcal{H}_y]\big] \\
&= \lim_{t\to\infty} E^{P_{x_0}}\big[\mathbf{1}_{A\cap\{H_y\leq t\}} E^{P_{x_0}}[M_t \mid \mathcal{H}_y]\big] \\
&= \lim_{t\to\infty} E^{P_{x_0}}\big[\mathbf{1}_{A\cap\{H_y\leq t\}} M_{H_y}\big] = E^{P_{x_0}}[M_{H_y}\mathbf{1}_A].
\end{aligned}$$

By (4.20), $M_{H_y} = \frac{Z(y)}{Z(x_0)} e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}$, and thus, also by (4.9),

$$Q_{x_0,y}(A) = (Z^{\zeta,\eta}_{x_0,y})^{-1} E_{x_0}[e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}\mathbf{1}_A] = P^{\zeta,\eta}_{x_0,y}(A)$$

as required. This completes the proof. □

We are now ready to reap the fruits of the above considerations. Proposition 4.3 together with the uniform bounds (4.11) on $b^{\zeta,\eta}$ allows for a comparison between the tilted measures (4.3) and Brownian motion with constant drift. The next lemma provides this desired control and makes it precise. For a given drift $\alpha \in \mathbb{R}$, we write P^α_x for the law of Brownian motion with constant drift α started at x and E^α_x for the corresponding expectation.

Lemma 4.4. *Let $\zeta : \mathbb{R} \to [-(\mathtt{es} - \mathtt{ei}), 0]$ be locally Hölder continuous and let $\eta < 0$. Then, for any starting point $x \in \mathbb{R}$ and any bounded non-decreasing function $g : \mathbb{R} \to \mathbb{R}$,*

$$E_x^{\underline{v}(\eta)}[g(X_t)] \le E_x^{\zeta,\eta}[g(X_t)] \le E_x^{\overline{v}(\eta)}[g(X_t)],$$

where $\underline{v}(\eta)$ and $\overline{v}(\eta)$ have been introduced in Lemma 4.1.

Proof. By Proposition 4.3, the process X_t driven by the tilted measure $P_{x_0}^{\zeta,\eta}$ has generator $L^{\zeta,\eta} = \frac{1}{2}\Delta + b(x)\frac{\mathrm{d}}{\mathrm{d}x}$. Let further $L^v = \frac{1}{2}\Delta + v\frac{\mathrm{d}}{\mathrm{d}x}$ be the generator of the Brownian motion with drift v. Then, for any non-decreasing $g \in C_b^2(\mathbb{R})$, if follows from (4.11) that

$$L^{\underline{v}(\eta)}g \le L^{\zeta,\eta}g \le L^{\overline{v}(\eta)}g.$$

Since, by Kolmogorov's forward equation, $\frac{\mathrm{d}}{\mathrm{d}t}E_x^{\zeta,\eta}[g(X_t)] = E_x^{\zeta,\eta}[(L^{\zeta,\eta}g)(X_t)]$ and analogously for the measures $E_x^{\underline{v}}$ and $E_x^{\overline{v}}$, the statement of the lemma follows for any non-decreasing $g \in C_b^2(\mathbb{R})$. The extension to arbitrary non-decreasing functions g follows by approximating g by a sequence of non-decreasing functions in $C_b^2(\mathbb{R})$ and using the dominated convergence theorem. □

5. Proof of Theorem 2.1

Using the duality between BBMRE and (F-KPP) as well as the various results presented in the last two sections, we are now ready to prove the main theorem.

Proof of Theorem 2.1. By contradiction, assume that the family $(M(t)-m^\xi(t))_{t\ge 0}$ is not tight. Recalling the notation from (2.8), it then follows that there exists $\varepsilon \in (0, 1/2)$ such that

$$\limsup_{t\to\infty} \big(m^\xi_{1-\varepsilon}(t) - m^\xi_\varepsilon(t)\big) = \infty.$$

Hence, we can find a sequence $(t_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ of times with $t_n \to \infty$ as $n \to \infty$, as well as sequences $(r_n)_{n\in\mathbb{N}}, (l_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ of positions, with $r_n, l_n, r_n - l_n \to \infty$ as $n \to \infty$, such that $\mathtt{P}_0^\xi(M(t_n) \ge r_n) = \varepsilon$ and $\mathtt{P}_0^\xi(M(t_n) \ge l_n) < 1-\varepsilon$. By McKean's representation, cf. Proposition 3.1, this is equivalent to $w^{r_n}(t_n, 0) = \varepsilon$ and

$$w^{l_n}(t_n, 0) < 1 - \varepsilon. \tag{5.1}$$

Due to the law of large numbers for M_t (i.e. $\lim_{t\to\infty} M_t/t = v_0$), it also holds that

$$\lim_{n\to\infty} \frac{l_n}{t_n} = \lim_{n\to\infty} \frac{r_n}{t_n} = v_0. \tag{5.2}$$

Setting

$$\Delta_n := r_n - l_n, \tag{5.3}$$

by Corollary 3.8 and the validity of $w^{l_n+\Delta_n}(t_n, 0) = \varepsilon$, there exists a $\mathbb{P}$-a.s. finite time $T < \infty$, such that for all n large enough,

$$w^{l_n+\Delta_n}(t_n + t', 0) \ge 1 - \varepsilon/2 \quad \text{for all } t' \in [T, T+1]. \tag{5.4}$$

Our goal is to infer a contradiction from (5.4) and the fact that $w^{l_n}(t_n, 0) < 1-\varepsilon$. In order to do so, we want to compare the values of $w^{l_n}(t_n, 0)$ with those of $w^{l_n+\Delta_n}(t_n + T, 0)$ for large n, and argue that

$$w^{l_n}(t_n, 0) \ge w^{l_n+\Delta_n}(t_n + T, 0); \tag{5.5}$$

in combination with (5.4) this would immediately yield a contradiction to (5.1). Instead of proving (5.5) directly however, we use the Sturmian principle to relate

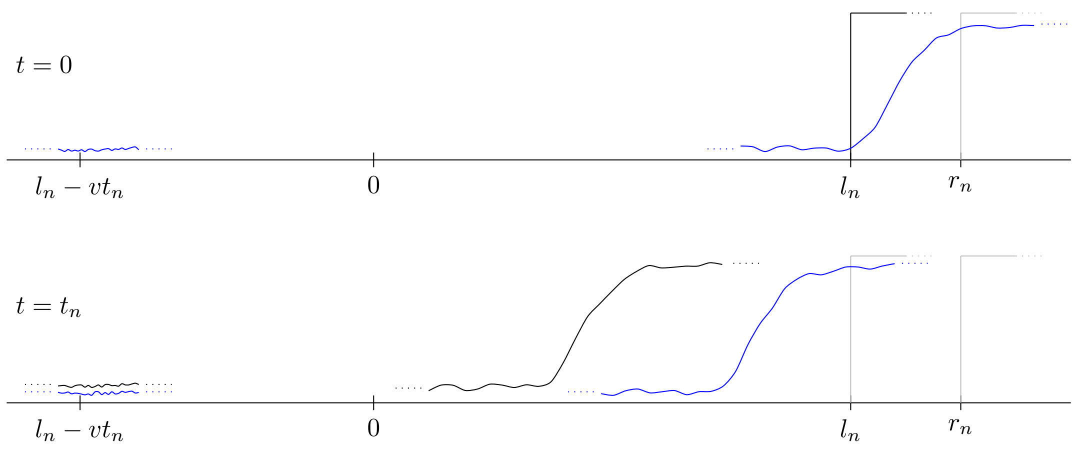

FIGURE 2. The top figure shows the graph of the indicator function $w^{l_n}(0,\cdot) = \mathbf{1}_{[l_n,\infty)}(\cdot)$ in black and the function $w^{r_n}(T,\cdot)$ in blue. The lower figure shows the graph of the same functions $t_n > 0$ time units later. By the Sturmian principle, the region where $w^{l_n}(t_n,\cdot)$ dominates $w^{r_n}(t_n+T,\cdot)$ is an interval that contains $[l_n - vt_n, \infty)$.

the inequality (5.5) at the origin to an inequality at some point on the negative half-line. More precisely, recall that for every admissible n, the difference

$$W_n(t,x) := w^{l_n}(t,x) - w^{l_n+\Delta_n}(t+T,x)$$

solves a linear parabolic equation of the form (3.8). Moreover, since $W_n(0,x) > 0$ for $x > l_n$ and $W_n(0,x) < 0$ for $x < l_n$, we know by Lemma 3.6 that the set

$$\{x \in \mathbb{R} : w^{l_n}(t_n,x) > w^{l_n+\Delta_n}(t_n+T,x)\} = \{x \in \mathbb{R} : W_n(t_n,x) > 0\}$$

is an open interval unbounded to the right; for an illustration of this argument see Figure 2.

Thus, in order to prove (5.5) it suffices to find some $x_n^* < 0$ such that for large n it holds that $W_n(t_n,x_n^*) > 0$, as this implies $0 \in \{x \in \mathbb{R} : W_n(t_n,x) > 0\}$, which in turn implies (5.5). The following lemma ascertains that for large n, this is indeed true and that the choice

$$x_n^* := l_n - vt_n, \tag{5.6}$$

where $v > 0$ is some large value, is adequate. To state it, we introduce two auxiliary velocities,

$$v_1 := \sqrt{2(\mathtt{es}+1)} \quad \text{and} \tag{5.7}$$

$$v_2 := \inf\{v > v_1 + 1 : |\overline{\eta}(v)| \geq 2v_1^2 + 2\}, \tag{5.8}$$

where $\overline{\eta}(v)$ was defined above (4.6); note that display (4.6) also ensures that v_2 is finite. Furthermore, by comparing the BBMRE with the BBM with constant branching rate $\mathtt{es}$, for which the speed of the maximum is $\sqrt{2\mathtt{es}}$, we obtain

$$v_0 < v_1 < v_2$$

Lemma 5.1. *For each $u > 0$ and each $v > v_2$, there exists $\Delta_0 = \Delta_0(u,v) > 0$ as well as a $\mathbb{P}$-a.s. finite random variable $\mathcal{T} = \mathcal{T}(u,v)$, such that $\mathbb{P}$-a.s., for all $\Delta > \Delta_0$, $y \in [0,vt]$ and $t \geq \mathcal{T}$,*

$$w^y(t, y - vt) \geq w^{y+\Delta}(t+u, y-vt). \tag{5.9}$$

We postpone the proof of this crucial lemma to Section 6 and complete the proof of Theorem 2.1 first. Recall the $\mathbb{P}$-a.s. finite random variable T introduced above (5.4), and for $u \in \mathbb{N}$ define the subset $\Omega_u = \{T \in [u-1, u)\}$ of the probability space on which ξ is defined. We now consider $\xi \in \Omega_u$. By (5.4), for such ξ and all n large enough,

$$w^{l_n+\Delta_n}(t_n + u, 0) \geq 1 - \varepsilon/2. \tag{5.10}$$

Let $v > v_2$ be as in Lemma 5.1. Since $v_2 > v_0$ and $l_n/t_n \to v_0$, by (5.2), it follows that $l_n \in [0, vt_n]$ for all n large enough. In addition, for all n large enough, we have $t_n \geq \mathcal{T}(u, v)$ as well as, recalling the notation from (5.3), that $\Delta_n \geq \Delta_0(u, v)$. Therefore, by Lemma 5.1, for such n we in particular deduce that $w^{l_n}(t_n, l_n - vt_n) \geq w^{l_n+\Delta_n}(t_n + u, l_n - vt_n)$, which by the previous discussion and with the choice x_n^* as in (5.6) implies

$$1 - \varepsilon > w^{l_n}(t_n, 0) \geq w^{l_n+\Delta_n}(t_n + u, 0). \tag{5.11}$$

Combining (5.10) and (5.11), we arrive at the desired contradiction. This proves that the family $(M(t) - m^\xi(t))_{t\geq 0}$ is tight for $\mathbb{P}$-a.a. $\xi \in \Omega_u$. As $\Omega = \cup_{u\geq 1}\Omega_u$, this completes the proof. $\square$

6. Proof of Lemma 5.1

It remains to prove Lemma 5.1 which provides the right ordering of the two solutions to (F-KPP). We do so by providing a careful examination of the Feynman-Kac representations of the respective solutions. We aim to apply the tilting of probability measures from Section 4 with an appropriate tilting parameter, which will make the large deviation event in the Feynman-Kac formula typical. Before we can do so, however, we need to bring the Feynman-Kac formulas into a more suitable form. In particular, we are going to compare the two solutions in (5.9) by lower bounding the left-hand side of (5.9) and upper bounding the right-hand side in a suitable way. In order to get these bounds, we apply the perturbation result Lemma 3.4 and introduce a large deviation event which can be dealt with by means of the tilted probability measures.

Proof of Lemma 5.1. We start with upper bounding the right-hand side of (5.9). By the Feynman-Kac representation (3.7) and the fact that $\sup_{w\in[0,1]} \widetilde{F}(w) = 1$, cf. Proposition 3.3, it follows that

$$\begin{aligned} w^{y+\Delta}(t+u, y-vt) &= E_{y-vt}\Big[e^{\int_0^{t+u} \xi(X_s)\widetilde{F}(w^{y+\Delta}(t+u-s,X_s))\,\mathrm{d}s}; X_{t+u} \geq y + \Delta\Big] \\ &\leq E_{y-vt}\Big[e^{\int_0^{t+u} \xi(X_s)\,\mathrm{d}s}; X_{t+u} \geq y+\Delta\Big]. \end{aligned} \tag{6.1}$$

To the right-hand side of (6.1) we now successively apply both parts of the perturbation Lemma 3.4 (with V sufficiently large, as explained before Lemma 3.4 and $A = 2v$). In order to apply them, we let $t \geq u \vee \mathcal{T}_1 \vee \mathcal{T}_2 =: \mathcal{T}$, where $\mathcal{T}_1, \mathcal{T}_2$ are the $\mathbb{P}$-a.s. finite random variables occurring in the statement of the perturbation lemma. For such t, we then obtain

$$\begin{aligned} w^{y+\Delta}(t+u, y-vt) &\leq c_1 e^{c_1 u} E_{y-vt}\Big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \geq y+\Delta\Big] \\ &\leq c_1 c_2 e^{c_1 u - \Delta/c_2} E_{y-vt}\Big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \geq y\Big]. \end{aligned} \tag{6.2}$$

Let us now turn our focus to bounding the left-hand side of (5.9) from below. We start by considering again the non-linearity F of (F-KPP). It is a direct consequence of (3.1) and the normalisation (3.4) of Remark 3.2 that $F(0) = 0$

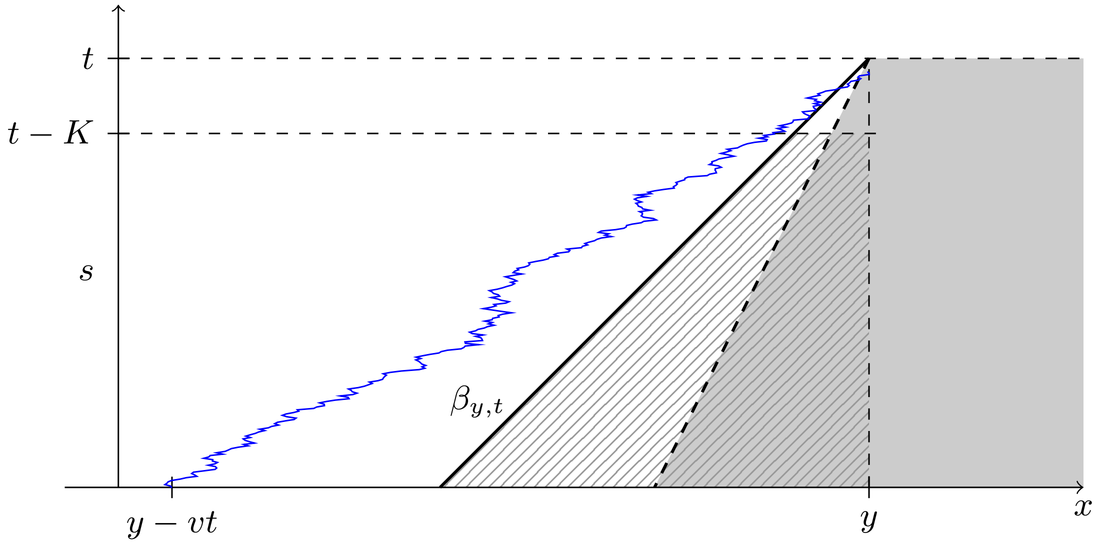

FIGURE 3. Sketch of a trajectory of the Brownian motion $(X_s)_{s\geq 0}$, started at $y-vt$, up until the hitting time H_y of y, which realises the good event $\mathcal{G}$. This trajectory does not hit the moving barrier $\beta_{y,t}(s)$ (thick solid line) in the time interval $[0,t-K]$ and thus avoids the dashed region. The function $w^y(t-s,\cdot)$ is close to 1 in the grey region, close to 0 in its complement, and changes its value from 0 to 1 in the vicinity of the thick dashed line whose slope is v_0.

and $F'(0)=1$. In addition, recall that by (3.5) we have $F''\geq -\mu_2+2$ on $[0,1]$. Therefore, by a first order Taylor approximation with Lagrange remainder,

$$F(w)\geq w+\frac{1}{2}\inf_{w^*\in[0,1]}F''(w^*)w^2=w-\frac{1}{2}(\mu_2-2)w^2.$$

In particular this implies that $\widetilde{F}(w)=F(w)/w\geq 1-\frac{1}{2}(\mu_2-2)w$.

Plugging this into the left-hand side of (5.9), and using the Feynman-Kac representation (3.7) as well as the uniform ellipticity (2.2) from Assumption 1, we arrive at

$$w^y(t,y-vt)\geq E_{y-vt}\Big[e^{\int_0^t\xi(X_s)\,\mathrm{d}s}\ e^{-\frac{\mathrm{es}}{2}(\mu_2-2)\int_0^t w^y(t-s,X_s)\,\mathrm{d}s};X_t\geq y\Big]. \tag{6.3}$$

In order to obtain a suitable control of the second exponential factor in (6.3), we construct an event restricted to which the second exponential is bounded from below in a suitable way. For this purpose, we recall the definition of v_1 from (5.7), and introduce for given t, y the *moving boundary*

$$\beta_{y,t}(s):=y-v_1(t-s),\qquad s\in[0,t]. \tag{6.4}$$

By $\mathcal{T}_{y,t}:=\inf\{s\geq 0: X_s=\beta_{y,t}(s)\}$ we denote the first hitting time of $\beta_{y,t}$ by a Brownian motion started at $y-vt$. We claim that for $K>1\vee v_1^{-2}$ to be fixed later, on the good event $\mathcal{G}:=\{\mathcal{T}_{y,t}\in[t-K,t]\}$, it holds that

$$\int_0^{t-K}w^y(t-s,X_s)\,\mathrm{d}s\leq 1, \tag{6.5}$$

see Figure 3 for an illustration. Indeed, note that using again the Feynman-Kac representation (3.7) as well as the uniform ellipticity (2.2) of Assumption 1, in combination with the fact that $\sup_{w\in[0,1]} \widetilde{F}(w) = 1$ once more, it holds that

$$\begin{aligned} w^y(t-s, X_s) &\le E_{X_s}\Big[e^{\int_0^{t-s} \xi(\widetilde{X}_r)\,\mathrm{d}r}; \widetilde{X}_{t-s} \ge y\Big] \\ &\le e^{\mathtt{es}(t-s)} P_{X_s}\big(\widetilde{X}_{t-s} \ge y\big), \end{aligned}$$

where we write $\widetilde{X}$ for an independent Brownian motion started at X_s in order to avoid confusion of the two processes. On $\mathcal{G}$ one has that $X_s \le y - v_1(t-s)$ for $s \in [0, t-K]$. Hence, by a straightforward coupling argument, on $\mathcal{G}$ we have

$$P_{X_s}\big(\widetilde{X}_{t-s} \ge y\big) \le P_0\big(\widetilde{X}_{t-s} \ge v_1(t-s)\big) = P(Z \ge v_1\sqrt{t-s}),$$

where Z is a standard Gaussian random variable. Using this in combination with a standard Gaussian bound (see e.g. [AT07, (1.2.2)]) and taking advantage of the fact that by assumption $v_1\sqrt{(t-s)} \ge v_1\sqrt{K} \ge 1$, it follows that on $\mathcal{G}$ we can upper bound

$$\begin{aligned} \int_0^{t-K} w^y(t-s, X_s)\,\mathrm{d}s &\le \int_0^{t-K} e^{\mathtt{es}(t-s)} P\big(Z \ge v_1\sqrt{t-s}\big)\,\mathrm{d}s \\ &\le \frac{1}{\sqrt{2\pi}} \int_0^{t-K} e^{-(v_1^2/2-\mathtt{es})(t-s)}\,\mathrm{d}s = \frac{1}{\sqrt{2\pi}} \int_K^t e^{-(v_1^2/2-\mathtt{es})z}\,\mathrm{d}z \\ &\le \frac{1}{\sqrt{2\pi}(v_1^2/2-\mathtt{es})} e^{-K(v_1^2/2-\mathtt{es})} \le 1, \end{aligned} \tag{6.6}$$

where in the last inequality we used $v_1^2/2 - \mathtt{es} = 1$, which holds by (5.7). This proves (6.5).

Coming back to the task of finding a lower bound for the right-hand side of (6.3), we infer by the above discussion that on $\mathcal{G}$ we can use (6.5) to bound the second exponential factor on the right-hand side of (6.3) by

$$\begin{aligned} e^{-\frac{\mathtt{es}}{2}(\mu_2-2)\int_0^t w^y(t-s,X_s)\,\mathrm{d}s} &\ge e^{-\frac{\mathtt{es}}{2}(\mu_2-2)\left(1+\int_{t-K}^t w^y(t-s,X_s)\,\mathrm{d}s\right)} \\ &\ge e^{-\frac{\mathtt{es}}{2}(\mu_2-2)(1+K)}, \end{aligned} \tag{6.7}$$

where in the last inequality we used that $0 \le w^y(s,y) \le 1$ uniformly for all $(s,y) \in [0,\infty)\times\mathbb{R}$. Consequently, by restricting the expectation on the right-hand side of (6.3) to $\mathcal{G}$, it follows by (6.7) that whenever $v > v_1$, then

$$w^y(t, y-vt) \ge e^{-\frac{\mathtt{es}}{2}(\mu_2-2)(1+K)} E_{y-vt}\Big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \ge y,\ \mathcal{G}\Big]. \tag{6.8}$$

In order to finish the proof of (5.9), we need to compare the expectations on the right-hand side of (6.2) and on the right-hand side of (6.8). This is the purpose of the following lemma.

Lemma 6.1. *Let v_2 be as in (5.8). Then for every $v > v_2$ there exists constants $K = K(v)$, $\widetilde{C} = \widetilde{C}(v) \in (0,\infty)$ such that for $\mathbb{P}$-a.a. ξ, for all t large enough and all $y \in [0, vt]$, one has*

$$E_{y-vt}\big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \ge y\big] \le \widetilde{C} E_{y-vt}\big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \ge y, \mathcal{G}\big]. \tag{6.9}$$

We postpone the proof of Lemma 6.1 and complete the proof of Lemma 5.1 first. By combining the lower bound (6.8), the upper bound (6.2) and Lemma 6.1, we obtain.

$$\begin{aligned} &w^y(t, y-vt) - w^{y+\Delta}(t+u, y-vt) \\ &\ge \big(e^{-\frac{\mathtt{es}}{2}(\mu_2-2)(K+1)} - \widetilde{C} c_1 c_2 e^{c_1 u - \Delta/c_2}\big) E_{y-vt}\Big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \ge y, \mathcal{G}\Big]. \end{aligned}$$

For every Δ satisfying

$$\Delta \geq \Delta_0 := c_2\Big(c_1 u + \frac{\mathtt{es}}{2}(\mu_2 - 2)(K+1) + \ln(\widetilde{C} c_1 c_2)\Big),$$

the right-hand side is positive, which proves (5.9) and thus the lemma. $\square$

Proof of Lemma 6.1. To prove the lemma, we use the machinery of tilted measures as introduced in Section 4. We recall the notation $\zeta = \xi - \mathtt{es}$ from (4.1) and observe that, by multiplying both sides of (6.9) by $e^{-\mathtt{es}\, t}$, it is sufficient to show (6.9) with ζ in place of ξ.

We start by proving an upper bound for the left-hand side of (6.9) in terms of tilted measures. By Lemma A.4 there exist constants $C, L < \infty$ such that for any $\eta < 0$, for t large enough uniformly in $y \in [0, vt]$ it holds that

$$\begin{aligned} E_{y-vt}\big[e^{\int_0^t \zeta(X_s)\,\mathrm{ds}}; X_t \geq y\big] &\leq C E_{y-vt}\big[e^{\int_0^{H_y} \zeta(X_s)\,\mathrm{ds}}\,; H_y \in [t-L, t]\big] \\ &\leq C e^{-\eta t} Z^{\zeta,\eta}_{y-vt,y} P^{\zeta,\eta}_{y-vt}\big(H_y \in [t-L, t]\big). \end{aligned} \tag{6.10}$$

In the next step, we bound the expression appearing on the right-hand side of (6.9) from below. To this end, let $p^{\zeta,\eta}_y(t) := P^{\zeta,\eta}_y(X_t \geq y)$. Using the strong Markov property we obtain

$$\begin{aligned} &E_{y-vt}\big[e^{\int_0^t \zeta(X_s)\,\mathrm{ds}}; X_t \geq y, \mathcal{T}_{y,t} \geq t-K\big] \\ &\geq e^{-(\mathtt{es}-\mathtt{ei})K} E_{y-vt}\big[e^{\int_0^{H_y} \zeta(X_s)\,\mathrm{ds}}; H_y \in [t-K, t], X_t \geq y, \mathcal{T}_{y,t} \geq t-K\big] \\ &\geq e^{-(\mathtt{es}-\mathtt{ei}-\eta)K} e^{-\eta t} E_{y-vt}\big[e^{\int_0^{H_y} (\zeta(X_s)+\eta)\,\mathrm{ds}}; H_y \in [t-K, t], X_t \geq y, \mathcal{T}_{y,t} \geq t-K\big] \\ &= e^{-(\mathtt{es}-\mathtt{ei}-\eta)K} e^{-\eta t} Z^{\zeta,\eta}_{y-vt,y} E^{\zeta,\eta}_{y-vt}\big[p^{\zeta,\eta}_y(t-H_y), H_y \in [t-K, t], \mathcal{T}_{y,t} \geq t-K\big] \\ &\geq \frac{1}{2} e^{-(\mathtt{es}-\mathtt{ei}-\eta)K} e^{-\eta t} Z^{\zeta,\eta}_{y-vt,y} P^{\zeta,\eta}_{y-vt}\big(H_y \in [t-K, t], \mathcal{T}_{y,t} \geq t-K\big), \end{aligned} \tag{6.11}$$

where in the last inequality we used Lemma 4.4 to infer that for any $\eta < 0$ and $s \geq 0$ one has $p^{\zeta,\eta}_y(s) \geq P_0^{\sqrt{2|\eta|}}(X_s \geq 0) \geq 1/2$.

In view of (6.10) and (6.11), in order to complete the proof of Lemma 6.1, it is sufficient to show that

$$P^{\zeta,\eta}_{y-vt}\big(H_y \in [t-L, t]\big) \leq C P^{\zeta,\eta}_{y-vt}\big(H_y \in [t-K, t], \mathcal{T}_{y,t} \geq t-K\big), \tag{6.12}$$

for some suitably chosen parameter η and constants C, K, L, $\mathbb{P}$-a.s. for all t large, uniformly in $y \in [0, vt]$.

To this end we will need two further auxiliary lemmas. The first one will be used to upper bound the probability appearing on the right-hand side of (6.12), and also specifies the range of suitable η's.

Lemma 6.2. *Let $\eta < 0$ be such that $\sqrt{2|\eta|} > v_1(1 + \frac{2L}{K})$, and let $0 < L < K$ be such that $L/K \leq 1/3$. Then, $\mathbb{P}$-a.s. for every $y \in \mathbb{R}$ and $v > v_1$,*

$$P^{\zeta,\eta}_{y-vt}\big(H_y \leq t, \mathcal{T}_{y,t} \leq t-K\big) \leq 2P^{\zeta,\eta}_{y-vt}(H_y < t-L). \tag{6.13}$$

The second auxiliary lemma is a quantitative extension of a part of Proposition 3.5 of [DS22]. It states that under the tilted measure, if the tilting is not too strong, the probabilities to cross a large interval in t or $t-L$ time units are comparable.

Lemma 6.3. *For every $v > v_c$ there is $c = c(v) < \infty$ such that for all L large enough and $\eta \in (\overline{\eta}(v) + \frac{c}{L}, 0)$, $\mathbb{P}$-a.s. for all t large enough and $|y| \le 2vt$,*

$$P_{y-vt}^{\zeta,\eta}(H_y \le t - L) \le \frac{1}{4} P_{y-vt}^{\zeta,\eta}(H_y \le t),$$

and as a consequence,

$$P_{y-vt}^{\zeta,\eta}(H_y \le t - L) \le \frac{1}{3} P_{y-vt}^{\zeta,\eta}\big(H_y \in (t - L, t]\big).$$

In order not to hinder the flow or reading, we postpone the proofs of these two lemmas to the end of the current section. We now come back to the proof of Lemma 6.1 and complete it by showing (6.12). To this end we choose the parameters η, K, and L in such a way that the previous two lemmas can be used simultaneously. More precisely, for a given $v \ge v_2$ we fix arbitrary η so that

$$|\overline{\eta}(v)| - 1 > |\eta| > 2v_1^2, \tag{6.14}$$

which is possible by the definition of v_2 in (5.8). Then we fix L as large as required in Lemma 6.3. As consequence, due to (6.14), the required assumptions on η are satisfied in our setting. Finally, we fix $K \ge 3L$ and observe that, in combination with (6.14), we infer $\sqrt{2|\eta|} > 2v_1 \ge v_1(1 + \frac{2L}{K})$, so that the assumptions of Lemma 6.2 are satisfied as well.

With this choice of constants, noting that $\{H_y \in [t-K, t], \mathcal{T}_{y,t} \ge t - K\} = \{H_y \le t, \mathcal{T}_{y,t} \ge t - K\}$ (cf. Figure 3 also), the right-hand side of (6.12) satisfies

$$\begin{aligned} &P_{y-vt}^{\zeta,\eta}\big(H_y \in [t-K, t], \mathcal{T}_{y,t} \ge t - K\big) \\ &\qquad = P_{y-vt}^{\zeta,\eta}\big(H_y \le t\big) - P_{y-vt}^{\zeta,\eta}\big(H_y \le t, \mathcal{T}_{y,t} < t - K\big) \\ &\qquad \ge P_{y-vt}^{\zeta,\eta}\big(H_y \le t\big) - 2P_{y-vt}^{\zeta,\eta}\big(H_y \le t - L\big), \end{aligned} \tag{6.15}$$

where the last inequality follows from Lemma 6.2. This can be written as

$$P_{y-vt}^{\zeta,\eta}\big(H_y \in [t-L, t]\big) - P_{y-vt}^{\zeta,\eta}\big(H_y \le t - L\big) \ge \frac{2}{3} P_{y-vt}^{\zeta,\eta}\big(H_y \in [t-L, t]\big), \tag{6.16}$$

where the last inequality is a direct consequence of Lemma 6.3. Now combining (6.15) and (6.16) we obtain (6.12), which completes the proof. ☐

It remains to provide the proofs of Lemmas 6.2 and 6.3.

Proof of Lemma 6.2. Using the tower property for conditional expectations we obtain

$$\begin{aligned} P_{y-vt}^{\zeta,\eta}(H_y < t - L) &\ge P_{y-vt}^{\zeta,\eta}(H_y < t - L, \mathcal{T}_{y,t} \le t - K) \\ &= E_{y-vt}^{\zeta,\eta}\big[\mathbf{1}_{\{\mathcal{T}_{y,t} \le t-K\}} P_{y-vt}^{\zeta,\eta}(H_y < t - L \mid \mathcal{F}_{\mathcal{T}_{y,t}})\big], \end{aligned} \tag{6.17}$$

where $\mathcal{F}_{\mathcal{T}_{y,t}}$ is the canonical stopped σ-algebra associated to $\mathcal{T}_{y,t}$. It follows from Lemma 4.4 that the drift of X under the tilted measure $P_{y-vt}^{\zeta,\eta}$ is always larger than $\sqrt{2|\eta|}$. On the event $\{0 \le \mathcal{T}_{y,t} \le t - K\}$, by the strong Markov property at time $\mathcal{T}_{y,t}$ and using that $X_{\mathcal{T}_{y,t}} = \beta_{y,t}(\mathcal{T}_{y,t})$, it holds that

$$\begin{aligned} P_{y-vt}^{\zeta,\eta}\big(H_y < t - L \mid \mathcal{F}_{\mathcal{T}_{y,t}}\big) &= P_{X_{\mathcal{T}_{y,t}}}^{\zeta,\eta}\big(H_y < t - L - \mathcal{T}_{y,t}\big) \\ &\ge \inf_{0 \le u \le t-K} P_{\beta_{y,t}(u)}^{\zeta,\eta}(H_y \le t - u - L) \\ &\ge \inf_{0 \le u \le t-K} P_{\beta_{y,t}(u)}^{\sqrt{2|\eta|}}(H_y \le t - u - L). \end{aligned} \tag{6.18}$$

Recalling the assumptions of the lemma, for $u \in [0, t-K]$ we have that

$$\begin{aligned} E^{\sqrt{2|\eta|}}_{\beta_{y,t}(u)}(X_{t-u-L}) &= \beta_{y,t}(u) + \sqrt{2|\eta|}(t-u-L) \\ &\geq y - v_1(t-u) + v_1\big(1+\tfrac{2L}{K}\big)(t-u-L) \\ &\geq y - v_1 L + v_1 \tfrac{2L}{K}(K-L) \geq y + \tfrac{1}{3}v_1 L \geq y, \end{aligned} \tag{6.19}$$

where for the penultimate inequality we used $K - L \geq \frac{2}{3}K$, by assumption. In combination with the fact that X is Brownian motion with drift under $P^{\sqrt{2|\eta|}}_{\beta_{y,t}(u)}$, it follows that the probability on the right-hand side of (6.18) is at least 1/2. Plugging this back into (6.17) we arrive at

$$\begin{aligned} P^{\zeta,\eta}_{y-vt}(H_y < t-L) &\geq \frac{1}{2} P^{\zeta,\eta}_{y-vt}\big(\mathcal{T}_{y,t} \leq t-K\big) \\ &\geq \frac{1}{2} P^{\zeta,\eta}_{y-vt}\big(\mathcal{T}_{y,t} \leq t-K, H_y \leq t\big), \end{aligned}$$

as claimed. □

Next we give the proof of Lemma 6.3.

Proof of Lemma 6.3. The first part of the proof of this lemma follows the same steps as the proof of Proposition 3.5 of [DS22] (see also the proof of Lemma A.3 in the appendix.) By Lemma A.1(a), $\mathbb{P}$-a.s. for all t large enough, and all $|y| \leq 2vt$, there exist constants $\eta^{\zeta}_{y-vt,y}(v)$ so that

$$E^{\zeta,\eta^{\zeta}_{y-vt,y}(v)}_{y-vt}[H_y] = t. \tag{6.20}$$

To simplify the notation we write $\widetilde{\eta} = \eta^{\zeta}_{y-vt,y}(v)$. Using Lemma A.1(b), we can assume that $\widetilde{\eta} < \overline{\eta}(v) + \frac{c}{2L}$, and thus, by the hypothesis of the lemma,

$$\eta - \widetilde{\eta} > \frac{c}{2L}. \tag{6.21}$$

By definition of tilted measures (4.3),

$$\begin{aligned} P^{\zeta,\eta}_{y-vt}(H_y \leq t-L) &= \frac{1}{Z^{\zeta,\eta}_{y-vt,y}} E_{y-vt}\Big[e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}; H_y \leq t-L\Big] \\ &= \frac{Z^{\zeta,\widetilde{\eta}}_{y-vt,y}}{Z^{\zeta,\eta}_{y-vt,y}} \frac{1}{Z^{\zeta,\widetilde{\eta}}_{y-vt,y}} E_{y-vt}\Big[e^{\int_0^{H_y}(\zeta(X_s)+\widetilde{\eta})\,\mathrm{d}s} e^{-H_y(\widetilde{\eta}-\eta)}; H_y \leq t-L\Big]. \\ &= \frac{Z^{\zeta,\widetilde{\eta}}_{y-vt,y}}{Z^{\zeta,\eta}_{y-vt,y}} E^{\zeta,\widetilde{\eta}}_{y-vt}\Big[e^{-H_y(\widetilde{\eta}-\eta)}; H_y \leq t-L\Big]. \end{aligned} \tag{6.22}$$

Define random variables $\tau_i = H_{y-vt+i} - H_{y-vt+i-1}$, $i = 1, \ldots, \lfloor vt \rfloor$, and $\tau_{vt} = H_y - H_{y-vt+\lfloor vt \rfloor}$, so that $\sum_{i=1}^{\lfloor vt \rfloor} \tau_i + \tau_{vt} = H_y$, and their re-centred versions $\widehat{\tau}_i = \tau_i - E^{\zeta,\widetilde{\eta}}_{y-vt}[\tau_i]$ for $i = 1, \ldots, \lfloor vt \rfloor$, and $\widehat{\tau}_{vt} = \tau_{vt} - E^{\zeta,\widetilde{\eta}}_{y-vt}[\tau_{vt}]$. Further, let

$$Y^{\zeta}_{y-vt,y} := \tfrac{(\widetilde{\eta}-\eta)}{\widetilde{\sigma}}\Big(\sum_{i=1}^{\lfloor vt \rfloor} \widehat{\tau}_i + \widehat{\tau}_{vt}\Big), \tag{6.23}$$

where

$$\widetilde{\sigma} = \widetilde{\sigma}^{\zeta}_{y-vt,y}(v) = |\widetilde{\eta} - \eta| \sqrt{\mathrm{Var}_{P^{\zeta,\widetilde{\eta}}_{y-vt}}(H_y)}. \tag{6.24}$$

is chosen so that the variance of $Y^{\zeta}_{y-vt,y}$ is one. Denoting by $\mu^{\zeta}_{y-vt,y}$ the distribution of $Y^{\zeta}_{y-vt,y}$ under $P^{\zeta,\widetilde{\eta}}_{y-vt,y}$, using also the fact that $E^{\widetilde{\eta}}_{y-vt,y}[H_y] = t$, by the definition of $\widetilde{\eta}$, (6.22) can be rewritten as

$$\begin{aligned} &P^{\zeta,\widetilde{\eta}}_{y-vt}(H_y \le t - L) \\ &\qquad = \frac{Z^{\zeta,\widetilde{\eta}}_{y-vt,y}}{Z^{\zeta,\eta}_{y-vt,y}}\, e^{(\eta-\widetilde{\eta})t} E^{\zeta,\widetilde{\eta}}_{y-vt}\Big[e^{-\widetilde{\sigma} Y^{\zeta}_{y-vt,y}}; Y^{\zeta}_{y-vt,y} \in \Big[\frac{L(\eta-\widetilde{\eta})}{\widetilde{\sigma}}, \infty\Big)\Big] \\ &\qquad = \frac{Z^{\zeta,\widetilde{\eta}}_{y-vt,y}}{Z^{\zeta,\eta}_{y-vt,y}}\, e^{(\eta-\widetilde{\eta})t} \int_{L(\eta-\widetilde{\eta})/\widetilde{\sigma}}^{\infty} e^{-\widetilde{\sigma} u} \mu^{\zeta}_{y-vt,y}(\mathrm{d}u). \end{aligned} \tag{6.25}$$

Setting $L = 0$ in the above formula we further obtain

$$P^{\zeta,\eta}_{y-vt}(H_y \le t) = \frac{Z^{\zeta,\widetilde{\eta}}_{y-vt,y}}{Z^{\zeta,\eta}_{y-vt,y}}\, e^{(\eta-\widetilde{\eta})t} \int_{0}^{\infty} e^{-\widetilde{\sigma} u} \mu^{\zeta}_{y-vt,y}(\mathrm{d}u), \tag{6.26}$$

Hence, to finish the proof of the lemma, it suffices to show that the integral on the right-hand side of (6.25) is at most $1/4$ of the integral on the right-hand side of (6.26).

To see this we proceed as in the proof of Lemma 3.6 of [DS22]. By the strong Markov property the random variables $\widehat{\tau}_i$, $i = 1, \dots, \lfloor vt \rfloor$, and $\widehat{\tau}_{vt}$ are independent under $P^{\zeta,\widetilde{\eta}}_{y-vt}$. Further, it is a straightforward consequence of the definitions of the logarithmic moment generating functions in [DS22, (2.7)] and their being well defined for $\eta < 0$ that those random variables have uniform exponential moments. Moreover, recall that $\widetilde{\sigma}$ was chosen such that the variance of $\mu^{\zeta}_{y-vt,y}$ is one. This allows the application of a local central limit theorem for independent normalised sequences [BR10, Theorem 13.3], which infers that

$$\sup_{B} |\mu^{\zeta}_{y-vt,y}(B) - \Phi(B)| \le c_1(\lceil vt \rceil)^{-1/2}, \tag{6.27}$$

where the supremum is taken over all intervals B in $\mathbb{R}$ and Φ denotes the standard Gaussian measure. Note that the constant c_1 in the last display depends only on the uniform bound of the exponential moments of the $\widehat{\tau}_i$'s. Without loss of generality, we can assume that $c_1 > 4$. We also note that by [DS22, (3.8)] (see also (A.8)) the variance $\widetilde{\sigma}^2$ defined in (6.24) satisfies for $\mathbb{P}$-a.a. ζ and t large enough

$$c_2^{-1}\sqrt{\lceil vt \rceil} \le \widetilde{\sigma} \le c_2\sqrt{\lceil vt \rceil}. \tag{6.28}$$

We now have all ingredients to finish the proof. For that purpose, we assume that the constant c from the statement of the lemma satisfies the inequality

$$\ell := \frac{L(\eta-\widetilde{\eta})}{\widetilde{\sigma}} \ge \frac{c}{2c_2\sqrt{vt}} \ge \frac{20c_1}{\sqrt{vt}}. \tag{6.29}$$

To bound the integral in (6.25) from above, we observe that for any interval (a, b) of length ℓ we have $\Phi((a, b)) \le \ell/\sqrt{2\pi}$ and thus $\mu^{\zeta}_{y-vt,y}((a, b)) \le (\ell + c_1/\sqrt{vt}) \le 2\ell$, by (6.29). Therefore, using (6.28) in the last step,

$$\begin{aligned} \int_{L(\eta-\widetilde{\eta})/\widetilde{\sigma}}^{\infty} e^{-\widetilde{\sigma} u} \mu^{\zeta}_{y-vt,y}(\mathrm{d}u) &\le \sum_{i=1}^{\infty} e^{-\widetilde{\sigma} i\ell} \mu^{\zeta}_{y-vt,y}((i\ell, (i+1)\ell)) \\ &\le \frac{2\ell e^{-\widetilde{\sigma}\ell}}{1 - e^{-\widetilde{\sigma}\ell}} \le \frac{2\widetilde{\sigma}\ell e^{-\widetilde{\sigma}\ell}}{1 - e^{-\widetilde{\sigma}\ell}} \cdot \frac{c_2}{\sqrt{vt}}. \end{aligned} \tag{6.30}$$

On the other hand, using the rough bound $\Phi((0,x)) \geq x/5$ which holds for small enough x, and (6.29),

$$\begin{aligned}\int_0^\infty e^{-\widetilde{\sigma}u}\mu^\zeta_{y-vt,y}(\mathrm{d}u) &\geq \int_0^{L(\eta-\widetilde{\eta})/2\widetilde{\sigma}} e^{-\widetilde{\sigma}u}\mu^\zeta_{y-vt,y}(\mathrm{d}u)\\ &\geq e^{-\widetilde{\sigma}\ell/2}\mu^\zeta_{y-vt,y}((0,\ell/2)) \geq e^{-\widetilde{\sigma}\ell/2}\Big(\Phi((0,\ell/2)) - \frac{c_1}{\sqrt{vt}}\Big) \\ &\geq e^{-\widetilde{\sigma}\ell/2}\frac{c_1}{\sqrt{vt}}.\end{aligned} \tag{6.31}$$

By increasing the value of the constant c and thus of $\widetilde{\sigma}\ell \geq c/2$, the right-hand side of (6.30) can be made at most $1/4$ as large as the right-hand side of (6.31). This completes the proof of the lemma. $\square$

Appendix A. Perturbation estimates

The goal of this appendix is to show perturbation Lemma 3.4. Its proof strongly resembles the proofs of Lemma 3.11(b) of [DS22] and Lemma 4.1(b) of [ČDS22]. However, compared to the proofs of these two lemmas, we should take care of two key differences:

(A) Lemma 3.4 requires that its estimates hold uniformly over the "starting point" x and the "target point" y in an interval growing linearly with time t. In the original statements, the target point is always the origin and the starting point satisfies $x = vt$.
(B) Lemma 3.4(b) involves a perturbation by the end point (that is, y changes to $y+h$), while the starting point is perturbed in the original statement.

Proving Lemma 3.4 thus requires checking that these two differences can be dealt with by the original arguments. We will not reproduce the lengthy argument in completeness here, but describe key locations where the arguments of [ČDS22, DS22] have to be adapted.

We start by recalling the definition of the tilted measure $P_x^{\zeta,\eta}$ from below (4.4). Similarly to [DS22, (2.13)], we are interested in the tilting parameter $\eta^\zeta_{x,y}(v)$ such that the mean speed on the way from x to y under the tilted measure is v, that is

$$E_x^{\zeta,\eta^\zeta_{x,y}(v)}[H_y] = \frac{y-x}{v}, \qquad v>0, x<y. \tag{A.1}$$

(If $\eta^\zeta_{x,y}(v)$ does not exist, we set $\eta^\zeta_{x,y}(v) = 0$.) We also recall the definitions of $\overline{\eta}(v) < 0$ from below (4.5) (see also [DS22, (2.10)]), and of the compact interval $V \subset (v_c,\infty)$ containing v_0 in its interior from above Lemma 3.4. By [DS22, Lemma 2.4], there is a compact interval $\triangle \subset (-\infty,0)$ which contains $\overline{\eta}(V)$ in its interior. In particular,

$$\infty < \inf_{v\in V}\overline{\eta}(v) \leq \sup_{v\in V}\overline{\eta}(v) < 0. \tag{A.2}$$

The next lemma shows that $\eta^\zeta_{x,y}(v)$ exists with hight probability and is close to $\overline{\eta}$. It is an extension of Lemma 2.5 of [DS22] and the first step on the way to deal with the difference (A) in the above list.

Lemma A.1. *(a) For every $A>1$ there exists a finite random variable $\mathcal{N} = \mathcal{N}(A)$ such that for all $v \in V$ and $x < y \in \mathbb{R}$ such that $y - x \geq \mathcal{N}$ and $|x|, |y| \leq A(y-x)$, the solution $\eta^\zeta_{x,y}(v)$ to (A.1) exists and satisfies $\eta^\zeta_{x,y}(v) \in \triangle$.*

(b) *For each* $q \in \mathbb{N}$, *and each compact interval* $V \subset (v_c, \infty)$, *there exists* $C = C(V,q) \in (0,\infty)$ *such that, for all* $n \in \mathbb{N}$,

$$\mathbb{P}\Big(\sup_{v \in V} \sup_{x \in [-n,-n+1]} \sup_{y \in [0,1]} |\eta^{\zeta}_{x,y}(v) - \overline{\eta}(v)| \geq C\sqrt{\frac{\ln n}{n}} \Big) \leq Cn^{-q}. \tag{A.3}$$

Proof. As in [DS22], (a) follows directly from (b), using the Borel-Cantelli lemma and (A.2) (with the help of the stationarity and an additional union bound to take care over the uniformity in y). Part (b) looks essentially the same as in [DS22], with the additional supremum over $y \in [0,1]$. Due to (2.3), the proof of this part runs exactly as the proofs of Lemmas 2.5, 2.6 of [DS22], the modifications due to the additional supremum essentially require notational changes only. □

We can further adapt Lemma 2.7 of [DS22], which is used in the proof of Lemma 4.1 in [ČDS22]. Besides [DS22, Lemma 2.5] which we already adapted to our setting in Lemma A.1, its proof only uses steps that are uniform in the potential ξ, and thus requires only notational changes.

Lemma A.2. *There exists a constant* $c > 0$ *and for every* $A > 1$ *there exists a finite random variable* $\mathcal{N}' = \mathcal{N}'(A)$ *such that for all* $x, y \in \mathbb{R}$ *with* $y - x \geq \mathcal{N}'$ *and* $|x|, |y| \leq A(y-x)$, $v \in V$, *and* $h \in [0, y-x]$, *we have*

$$\big|\eta^{\zeta}_{x,y}(v) - \eta^{\zeta}_{x,y+h}(v)\big| \leq \frac{ch}{y-x}. \tag{A.4}$$

Next, we need to extend the second part of Proposition 3.5 in [DS22]. To this end, for $x \leq y \in \mathbb{R}$ and $v > 0$, we introduce the quantities (cf. [DS22, (3.7)])

$$\begin{aligned} Y^{\approx}_v(x,y) &:= E_x\Big[e^{\int_0^{H_y} \zeta(X_s)\,\mathrm{d}s}; H_y \in \Big[\frac{y-x}{v} - K, \frac{y-x}{v}\Big]\Big], \\ Y^{>}_v(x,y) &:= E_x\Big[e^{\int_0^{H_y} \zeta(X_s)\,\mathrm{d}s}; H_y < \frac{y-x}{v} - K\Big], \end{aligned} \tag{A.5}$$

where K is a large constant fixed as in [DS22, (3.17)]. It turns out that $Y^{\approx}_v(x,y)$ and $Y^{<}_v(x,y)$ are comparable uniformly in the admissible choices of x and y.

Lemma A.3. *For* $A > 1$, *let* $\mathcal{N} = \mathcal{N}(A)$ *be as in Lemma A.1. Then there exists a constant* $C \in (1,\infty)$ *such that for all* $v \in V$ *and all* $x < y \in \mathbb{R}$ *such that* $y - x \geq \mathcal{N}$ *as well as* $|x|, |y| \leq A(y-x)$, *we have*

$$\frac{Y^{\approx}_v(x,y)}{Y^{<}_v(x,y)} \in [C^{-1}, C]. \tag{A.6}$$

Proof. The proof of this lemma contains a computation that is also at the heart of the proof of Lemma 6.3, so we include it here. We assume that x, y satisfy the assumptions of the lemma, and, in order to keep the notation simple, we in addition assume that $x, y \in \mathbb{Z}$ (cf. [DS22, Section 1.9]). We write $\eta := \eta^{\zeta}_{x,y}(v)$ and define

$$\sigma = \sigma^{\zeta}_{x,y}(v) := |\eta| \sqrt{\mathrm{Var}_{P^{\zeta,\eta}_x}(H_y)}, \tag{A.7}$$

where the variance is with respect to $P^{\zeta,\eta}_x$. As in [DS22, (3.8)], uniformly in ζ and $v \in V$,

$$c^{-1}\sqrt{y-x} \leq \sigma^{\zeta}_{x,y}(v) \leq c\sqrt{y-x} \tag{A.8}$$

for some $c \in (1,\infty)$. Let further $\tau_z = H_z - H_{z-1}$, $z \in [x+1, y] \cap \mathbb{Z}$, and let $\widehat{\tau}_z := \tau_z - E^{\zeta,\eta}_x[\tau_z]$. Then, by the definition of η, for x, y satisfying the assumptions of Lemma A.1, we have $\sum_{z=x+1}^{y} E_x^{\zeta,\eta}[H_y] = \frac{y-x}{v}$. With this notation, as in [DS22, (3.12)],

$$\begin{aligned} Y_v^{\approx}(x,y) &= E_x\Big[e^{\int_0^{H_y}(\zeta(B_s)+\eta)\,\mathrm{d}s}\, e^{-\eta\sum_{z=x+1}^{y}\widehat{\tau}_z};\ \sum_{i=x+1}^{y}\widehat{\tau}_z \in [-K,0]\Big] e^{-x\eta/v} \\ &= E_x^{\zeta,\eta}\Big[e^{-\sigma\frac{\eta}{\sigma}\sum_{z=x+1}^{y}\widehat{\tau}_z};\ \frac{\eta}{\sigma}\sum_{z=x+1}^{y}\widehat{\tau}_z \in \Big[0,-\frac{K\eta}{\sigma}\Big]\Big] e^{-(y-x)(\frac{\eta}{v}-\overline{L}_{x,y}^{\zeta}(\eta))}, \end{aligned} \tag{A.9}$$

where (cf. [DS22, (2.7)])

$$\begin{aligned} \overline{L}_{x,y}^{\zeta}(\eta) &:= (y-x)^{-1}\sum_{z=x+1}^{y}\ln E_{z-1}\Big[e^{\int_0^{H_z}(\zeta(X_s)+\eta)\,\mathrm{d}s}\Big] \\ &= (y-x)^{-1}E_x\Big[e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}\Big]. \end{aligned} \tag{A.10}$$

Defining $\mu_{x,y}^{\zeta,\eta}$ to be the distribution of $\frac{\eta}{\sigma}\sum_{z=x+1}^{y}\widehat{\tau}_z$ under $P_x^{\zeta,\eta}$, this implies

$$Y_v^{\approx}(x,y) = e^{-(y-x)(\frac{\eta}{v}-\overline{L}_{x,y}^{\zeta}(\eta))}\int_0^{\frac{-K\eta}{\sigma}} e^{-\sigma u}\mu_{x,y}^{\zeta,\eta}(\mathrm{d}u). \tag{A.11}$$

Similarly,

$$Y_v^{<}(x,y) = e^{-(y-x)(\frac{\eta}{v}-\overline{L}_{x,y}^{\zeta}(\eta))}\int_{\frac{-K\eta}{\sigma}}^{\infty} e^{-\sigma u}\mu_{x,y}^{\zeta,\eta}(\mathrm{d}u). \tag{A.12}$$

The upshot of this computation is that under $P_x^{\zeta,\eta}$, the random variables $\widehat{\tau}_z$, $z = x+1,\dots,y$ are centred, independent, have uniform exponential moments, and $\mu_{x,y}^{\zeta,\eta}$ has unit variance. This allows, as in the proof of [DS22, Lemma 3.6], to (uniformly) approximate $\mu_{x,y}^{\zeta,\eta}$ by the standard Gaussian measure, and to show that the integrals appearing on the right-hand side of (A.11) and (A.12) are both comparable to $(\sigma_{x,y}^{\zeta}(v))^{-1}$ and thus, by (A.8), to $(y-x)^{-1/2}$, uniformly in the ζ and $v \in V$ under consideration and for all x, y satisfying the assumptions of Lemma A.1. With this the claim of the lemma follows. □

Lemma A.3 has an important corollary allowing to approximate the Feynman-Kac formula for the PAM by expressions involving $Y_v^{\approx}(x,y)$. This approximation is used in (6.10), and also in the proof of Lemma 3.4 below.

Lemma A.4. *(cf. [DS22, Lemma 3.7]) For each $A > 1$, there exists a constant $C \in (1,\infty)$ such that for all $t \in (0,\infty)$ and all $x < y \in \mathbb{R}$ such that $y - x \geq \mathcal{N}$, $|x|, |y| \leq A(y-x)$ and $\frac{y-x}{t} \in V$,*

$$C^{-1}Y_v^{\approx}(x,y) \leq E_x\big[e^{\int_0^t \zeta(X_s)\,\mathrm{d}s}; X_t \geq y\big] \leq C Y_v^{\approx}(x,y). \tag{A.13}$$

Proof. The proof of the corresponding Lemma 3.7 of [DS22] only uses estimates that are uniform in ζ and the starting/target position, as well as the part of Proposition 3.5 therein which we already extended in Lemma A.3. It can thus directly be adapted to the current setting. □

We can now finally show Lemma 3.4.

Proof of Lemma 3.4. The proof of part (a) involving the perturbation in time follows exactly the same lines as the proof of Lemma 3.11(b) in [DS22]: We denote

$v := (y-x)/t$, $v' := (y-x)/(t+h)$ and observe that by Lemma A.4, for x, y, t and h as in the statement, by choosing $\mathcal{T}_1$ sufficiently large so that $y - x \geq \mathcal{N}$,

$$\frac{E_x\Big[e^{\int_0^{t+h}\xi(X_s)\,\mathrm{d}s}; X_{t+h} \geq y\Big]}{E_x\Big[e^{\int_0^{t}\xi(X_s)\,\mathrm{d}s}; X_{t} \geq y\Big]} \leq C\frac{Y_{v'}^{\approx}(x,y)}{Y_{v}^{\approx}(x,y)}. \tag{A.14}$$

The fraction on the right-hand side can be rewritten with help of (A.11). Using also the fact that the integral in (A.11) is of order $\sqrt{y-x}$, as explained at the end of the proof of Lemma A.3, we obtain (cf. [DS22, (3.36)])

$$\frac{Y_{v'}^{\approx}(x,y)}{Y_{v}^{\approx}(x,y)} \leq \frac{\exp\big\{-(y-x)\big(\frac{\eta_{x,y}^{\zeta}(v')}{v'} - \overline{L}_{x,y}^{\zeta}(\eta_{x,y}^{\zeta}(v'))\big)\big\}}{\exp\big\{-(y-x)\big(\frac{\eta_{x,y}^{\zeta}(v)}{v} - \overline{L}_{x,y}^{\zeta}(\eta_{x,y}^{\zeta}(v))\big)\big\}}. \tag{A.15}$$

Now—cf. [DS22, (3.4)]—denoting

$$S_{x,y}^{\zeta,v}(\eta) := (y-x)\Big(\frac{\eta}{v} - \overline{L}_{x,y}^{\zeta}(\eta)\Big), \tag{A.16}$$

the logarithm of the right-hand side of (A.15) can be written as

$$\big(S_{x,y}^{\zeta,v}(\eta_{x,y}^{\zeta}(v)) - S_{x,y}^{\zeta,v}(\eta_{x,y}^{\zeta}(v'))\big) + \big(S_{x,y}^{\zeta,v}(\eta_{x,y}^{\zeta}(v')) - S_{x,y}^{\zeta,v'}(\eta_{x,y}^{\zeta}(v'))\big) \tag{A.17}$$

Recalling the definitions of v and v', the second summand in (A.17) satisfies

$$\big(S_{x,y}^{\zeta,v}(\eta_{x,y}^{\zeta}(v')) - S_{x,y}^{\zeta,v'}(\eta_{x,y}^{\zeta}(v'))\big) = -h\eta_{x,y}^{\zeta}(v') \leq ch, \tag{A.18}$$

since $\frac{1}{c'} \leq \eta_{x,y}^{\zeta}(v') \leq c' < 0$ for the considered x, y, v', due to Lemma A.1(a). Moreover, the absolute value of the first summand in (A.17) can be upper bounded by $ch^2/t \ll h$ uniformly for x, y, t and h under consideration, exactly as in the paragraph containing [DS22, (3.39)] (this proof uses again only estimates that are uniform in ζ). This completes the proof of part (a).

The proof of part (b) follows the lines of the proof of Lemma 4.1 in [ČDS22], but it should accommodate for the difference (B), as explained above at the beginning of Appendix A. Using the same reasoning as in (A.14)–(A.17), now with the choices $v := (y-x)/t$ and $v' := (y+h-x)/t$, we infer that

$$\frac{E_x\Big[e^{\int_0^{t}\xi(X_s)\,\mathrm{d}s}; X_{t} \geq y+h\Big]}{E_x\Big[e^{\int_0^{t}\xi(X_s)\,\mathrm{d}s}; X_{t} \geq y\Big]} \leq C\frac{Y_{v'}^{\approx}(x,y+h)}{Y_{v}^{\approx}(x,y)} \tag{A.19}$$

as well as

$$\begin{aligned}\ln\frac{Y_{v'}^{\approx}(x,y+h)}{Y_{v}^{\approx}(x,y)} &\leq \big(S_{x,y}^{\zeta,v}(\eta_{x,y}^{\zeta}(v)) - S_{x,y+h}^{\zeta,v'}(\eta_{x,y}^{\zeta}(v))\big)\\ &\quad + \big(S_{x,y+h}^{\zeta,v'}(\eta_{x,y}^{\zeta}(v)) - S_{x,y+h}^{\zeta,v'}(\eta_{x,y+h}^{\zeta}(v'))\big).\end{aligned} \tag{A.20}$$

By (A.16), the first summand on the right-hand side of (A.20) (which differs slightly from the one in [ČDS22], due to the difference (B)) satisfies (with $\eta :=$

$\eta^\zeta_{x,y}(v))$

$$
\begin{aligned}
&\left| S^{\zeta,v}_{x,y}(\eta^\zeta_{x,y}(v)) - S^{\zeta,v'}_{x,y+h}(\eta^\zeta_{x,y}(v)) \right| \\
&\quad = \left| \ln E_x\left[e^{\int_0^{H_{y+h}} (\zeta(X_s)+\eta^\zeta_{x,y}(v))\,\mathrm{d}s} \right] - \ln E_x\left[e^{\int_0^{H_y} (\zeta(X_s)+\eta^\zeta_{x,y}(v))\,\mathrm{d}s} \right] \right| \\
&\quad = \left| \ln E_y\left[e^{\int_0^{H_{y+h}} (\zeta(X_s)+\eta^\zeta_{x,y}(v))\,\mathrm{d}s} \right] \right| \\
&\quad \le h\sqrt{2(\mathtt{es} - \mathtt{ei} + |\eta^\zeta_{x,y}(v)|)} \le ch,
\end{aligned}
\tag{A.21}
$$

where in the second equality we applied the strong Markov property at time H_y, and used (4.16) for the inequality.

The second summand on the right-hand side of (A.20) is bounded by $ch^2/t \ll h$ and is thus negligible. This can be proved exactly as in [ČDS22, (4.13)–(4.16)]. Besides [DS22, Lemma 2.7], which we already extended in Lemma A.2, this proof again only uses uniform estimates and thus does not require any modification. This completes the proof of the lemma. □

References

[ABR09] Louigi Addario-Berry and Bruce Reed, *Minima in branching random walks*, Ann. Probab. **37** (2009), no. 3, 1044–1079. MR2537549

[Aïd13] Elie Aïdékon, *Convergence in law of the minimum of a branching random walk*, Ann. Probab. **41** (2013), no. 3A, 1362–1426.

[Ang88] Sigurd Angenent, *The zero set of a solution of a parabolic equation*, J. Reine Angew. Math. **390** (1988), 79–96. MR953678

[Ang91] Sigurd Angenent, *Nodal properties of solutions of parabolic equations*, Rocky Mountain J. Math. **21** (1991), no. 2, 585–592. MR1121527

[AT07] Robert J. Adler and Jonathan E. Taylor, *Random fields and geometry*, Springer Monographs in Mathematics, Springer, New York, 2007. MR2319516

[Bas98] Richard F. Bass, *Diffusions and elliptic operators*, Probability and its Applications (New York), Springer-Verlag, New York, 1998. MR1483890

[BBH+15] Julien Berestycki, Éric Brunet, John W. Harris, Simon C. Harris, and Matthew I. Roberts, *Growth rates of the population in a branching Brownian motion with an inhomogeneous breeding potential*, Stochastic Process. Appl. **125** (2015), no. 5, 2096–2145. MR3315624

[BDZ11] Erwin Bolthausen, Jean Dominique Deuschel, and Ofer Zeitouni, *Recursions and tightness for the maximum of the discrete, two dimensional Gaussian free field*, Electron. Commun. Probab. **16** (2011), 114–119. MR2772390

[BH14] Anton Bovier and Lisa Hartung, *The extremal process of two-speed branching Brownian motion*, Electron. J. Probab. **19** (2014), no. 18, 28. MR3164771

[BH15] Anton Bovier and Lisa Hartung, *Variable speed branching Brownian motion 1. Extremal processes in the weak correlation regime*, ALEA Lat. Am. J. Probab. Math. Stat. **12** (2015), no. 1, 261–291. MR3351476

[BR10] Rabi N. Bhattacharya and R. Ranga Rao, *Normal approximation and asymptotic expansions*, Classics in Applied Mathematics, vol. 64, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2010. MR3396213

[Bra78] Maury D. Bramson, *Maximal displacement of branching Brownian motion*, Comm. Pure Appl. Math. **31** (1978), no. 5, 531–581. MR494541

[Bra83] Maury Bramson, *Convergence of solutions of the Kolmogorov equation to travelling waves*, Mem. Amer. Math. Soc. **44** (1983), no. 285, iv+190. MR705746

[BS02] Andrei N. Borodin and Paavo Salminen, *Handbook of Brownian motion—facts and formulae*, second ed., Probability and its Applications, Birkhäuser Verlag, Basel, 2002. MR1912205

[BZ07] Maury Bramson and Ofer Zeitouni, *Tightness for the minimal displacement of branching random walk*, J. Stat. Mech. Theory Exp. (2007), no. 7, P07010, 12. MR2335694

[BZ09] Maury Bramson and Ofer Zeitouni, *Tightness for a family of recursion equations*, Ann. Probab. **37** (2009), no. 2, 615–653. MR2510018

[ČD20] Jiří Černý and Alexander Drewitz, *Quenched invariance principles for the maximal particle in branching random walk in random environment and the parabolic Anderson model*, Ann. Probab. **48** (2020), no. 1, 94–146. MR4079432

[ČDS22] Jiří Černý, Alexander Drewitz, and Lars Schmitz, *(Un-)bounded transition fronts for the parabolic Anderson model and the randomized F-KPP equation*, Annals of Applied probability (to appear), 2022.

[DGM14] Arnaud Ducrot, Thomas Giletti, and Hiroshi Matano, *Existence and convergence to a propagating terrace in one-dimensional reaction-diffusion equations*, Trans. Amer. Math. Soc. **366** (2014), no. 10, 5541–5566. MR3240934

[DRZ21] Amir Dembo, Jay Rosen, and Ofer Zeitouni, *Limit law for the cover time of a random walk on a binary tree*, Ann. Inst. Henri Poincaré Probab. Stat. **57** (2021), no. 2, 830–855. MR4260486

[DS22] Alexander Drewitz and Lars Schmitz, *Invariance principles and Log-distance of F-KPP fronts in a random medium*, Arch. Ration. Mech. Anal. (to appear), 2022.

[EW99] Steven N. Evans and Ruth J. Williams, *Transition operators of diffusions reduce zero-crossing*, Trans. Amer. Math. Soc. **351** (1999), no. 4, 1377–1389. MR1615955

[Fre85] Mark Freidlin, *Functional integration and partial differential equations*, Annals of Mathematics Studies, vol. 109, Princeton University Press, Princeton, NJ, 1985. MR833742

[FZ12a] Ming Fang and Ofer Zeitouni, *Branching random walks in time inhomogeneous environments*, Electron. J. Probab. **17** (2012), no. 67, 18. MR2968674

[FZ12b] Ming Fang and Ofer Zeitouni, *Slowdown for time inhomogeneous branching Brownian motion*, J. Stat. Phys. **149** (2012), no. 1, 1–9. MR2981635

[Gal04] Victor A. Galaktionov, *Geometric Sturmian theory of nonlinear parabolic equations and applications*, vol. 3, Chapman & Hall/CRC, Boca Raton, FL, 2004. MR2059317

[GT01] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. MR1814364

[HNRR16] François Hamel, James Nolen, Jean-Michel Roquejoffre, and Lenya Ryzhik, *The logarithmic delay of KPP fronts in a periodic medium*, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 3, 465–505. MR3463416

[HRS22] Haojie Hou, Yan-Xia Ren, and Renming Song, *Invariance principle for the maximal position process of branching brownian motion in random environment*, Preprint, available at *arXiv:2206.07950*, 2022.

[HS09] Yueyun Hu and Zhan Shi, *Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees*, Ann. Probab. **37** (2009), no. 2, 742–789. MR2510023

[INW68] Nobuyuki Ikeda, Masao Nagasawa, and Shinzo Watanabe, *Branching Markov processes. I*, J. Math. Kyoto Univ. **8** (1968), 233–278. MR232439

[Kön16] Wolfgang König, *The parabolic Anderson model*, Pathways in Mathematics, Birkhäuser, 2016. MR3526112

[KPP37] Andrey Kolmogorov, Ivan Petrovsky, and Nikolai Piskunov, *Étude de l'équation de la diffusion avec croissance de la quantite de matière et son application à un problème biologique*, Bull. Univ. État Moscou, Sér. Int., Sect. A: Math. et Mécan. 1, Fasc. 6, 1-25 (1937), 1937.

[Kri21] Xaver Kriechbaum, *Subsequential tightness for branching random walk in random environment*, Electron. Commun. Probab. **26** (2021), Paper No. 16, 12. MR4236686

[Kri22] Xaver Kriechbaum, *Tightness for branching random walk in time-inhomogeneous random environment*, 2022.

[LS88] S. Lalley and T. Sellke, *Traveling waves in inhomogeneous branching Brownian motions. I*, Ann. Probab. **16** (1988), no. 3, 1051–1062. MR942755

[LS89] S. Lalley and T. Sellke, *Travelling waves in inhomogeneous branching Brownian motions. II*, Ann. Probab. **17** (1989), no. 1, 116–127. MR972775

[LSU68] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and quasi-linear equations of parabolic type*, Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1968. MR0241822

[LTZ22] Eyal Lubetzky, Chris Thornett, and Ofer Zeitouni, *Maximum of branching Brownian motion in a periodic environment*, Ann. Inst. Henri Poincaré Probab. Stat. **58** (2022), no. 4, 2065–2093. MR4492971

[Mal15] Bastien Mallein, *Maximal displacement of a branching random walk in time-inhomogeneous environment*, Stochastic Process. Appl. **125** (2015), no. 10, 3958–4019. MR3373310

[McK75] H. P. McKean, *Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov*, Comm. Pure Appl. Math. **28** (1975), no. 3, 323–331. MR400428

[MZ16] Pascal Maillard and Ofer Zeitouni, *Slowdown in branching Brownian motion with inhomogeneous variance*, Ann. Inst. Henri Poincaré Probab. Stat. **52** (2016), no. 3, 1144–1160. MR3531703

[Nad15] Grégoire Nadin, *Critical travelling waves for general heterogeneous one-dimensional reaction-diffusion equations*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **32** (2015), no. 4, 841–873. MR3390087

[NZ21] Eyal Neuman and Xinghua Zheng, *On the maximal displacement of near-critical branching random walks*, Probab. Theory Related Fields **180** (2021), no. 1-2, 199–232. MR4265021

[RW00] L. C. G. Rogers and David Williams, *Diffusions, Markov processes, and martingales. Vol. 2*, Cambridge University Press, Cambridge, 2000. MR1780932

[Sko64] A. V. Skorohod, *Branching diffusion processes*, Teor. Verojatnost. i Primenen. **9** (1964), 492–497. MR0168030

[Stu36] Charles Sturm, *Mémoire sur une classe d'équations à différences partielles*, J. Math. Pures et Appl. **1** (1836), 373–444.

Jiří Černý, Department of mathematics and computer science, University of Basel, Spiegelgasse 1, 4051 Basel, Switzerland
Email address: `jiri.cerny@unibas.ch`

Alexander Drewitz, Universität zu Köln, Department Mathematik/Informatik, Weyertal 86–90, 50931 Köln, Germany
Email address: `adrewitz@uni-koeln.de`

Pascal Oswald, Department of mathematics and computer science, University of Basel, Spiegelgasse 1, 4051 Basel, Switzerland
Email address: `pascalamadeus.oswald@unibas.ch`