(Un-)bounded transition fronts for the parabolic Anderson model and the randomized F-KPP equation

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January 19, 2021

Abstract

We investigate the uniform boundedness of the fronts of the solutions to the randomized Fisher-KPP equation and to its linearization, the parabolic Anderson model. It has been known that for the standard (i.e. deterministic) Fisher-KPP equation, as well as for the special case of a randomized Fisher-KPP equation with so-called ignition type nonlinearity, one has a uniformly bounded (in time) transition front. Here, we show that this property of having a uniformly bounded transition front fails to hold for the general randomized Fisher-KPP equation. Nevertheless, we establish that this property does hold true for the parabolic Anderson model.

1 Introduction

We consider the random partial differential equation

$$\begin{aligned} &w_t(t,x) = \frac{1}{2}w_{xx}(t,x) + \xi(x,\omega)\,F(w(t,x)), \qquad (t,x) \in (0,\infty)\times\mathbb{R},\ \omega\in\Omega, \\ &w(0,\cdot) = \mathbb{1}_{(-\infty,0]}. \end{aligned} \tag{F-KPP}$$

In our specific setting, $(\xi(x))_{x\in\mathbb{R}} = (\xi(x,\omega))_{x\in\mathbb{R}}$, $\omega\in\Omega$, is a stochastic process on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ fulfilling suitable mixing and sample path regularity conditions (see Section 2), and the non-linearity F is generated by the probability generating function belonging to branching Brownian motion, see condition (PROB) below (2.1).

The investigation of (F-KPP) for the homogeneous case $\xi\equiv 1$ has a long history, dating back to the seminal works of Fisher [Fis37] and Kolmogorov, Petrovskii and Piscunov [KPP37]. The equation has found a plethora of applications, such as describing the dynamics of a randomly mating diploid population in a one-dimensional habitat, or also to model flame propagation, see [AW75].

It is well-known, see [KPP37, Theorem 14], that in the homogeneous case $\xi\equiv 1$ the solution w of (F-KPP) converges to a *traveling wave solution.* More precisely, there exists a function $(0,\infty)\ni t\mapsto m(t)$ such that

$$w(t, m(t)+\cdot\,) \underset{t\to\infty}{\longrightarrow} g \quad \text{uniformly,} \tag{1.1}$$

for some function $g:\mathbb{R}\to[0,1]$ with $g(x)\underset{x\to-\infty}{\longrightarrow} 0$ and $g(x)\underset{x\to\infty}{\longrightarrow} 1$, and which is unique up to spatial translations. In this context, the function $m(t)$ is usually referred to as the *position of the wave.* The convergence in (1.1) implies that the front of the solution to (F-KPP) for the case $\xi\equiv 1$ is bounded, i.e. for every $\varepsilon\in(0,1/2)$ there exist $\underline{x},\overline{x}\in\mathbb{R}$, such that for all t large enough,

$$\inf_{x\le\underline{x}} w(t,x+m(t)) \ge 1-\varepsilon \quad \text{and} \quad \sup_{x\ge\overline{x}} w(t,x+m(t)) \le \varepsilon. \tag{1.2}$$

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Therefore, a question arising naturally in our context is whether a behavior similar to (1.2) is observed in the setting of a random nonlinearity in (F-KPP) as well. It turns out that in the investigation of this question, for a variety of reasons the linearization of (F-KPP), which goes under the name parabolic Anderson model and which is of independent interest,

$$\begin{aligned} u_t(t,x) &= \frac{1}{2}u_{xx}(t,x) + \xi(x,\omega)\,u(t,x), \qquad (t,x) \in (0,\infty)\times\mathbb{R},\ \omega\in\Omega, \\ u(0,x) &= u_0(x), \qquad x\in\mathbb{R}, \end{aligned} \tag{PAM}$$

plays an important role as well.

2 Model and results

We will assume $\xi = (\xi(x))_{x\in\mathbb{R}}$ to be a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having Hölder continuous paths. I.e., there exists $\alpha = \alpha(\xi) > 0$ and $C = C(\xi) > 0$, such that

$$|\xi(x) - \xi(y)| \le C\,|x-y|^{\alpha} \quad \forall x,y\in\mathbb{R}. \tag{HÖL}$$

We will consider throughout the standard model of Ω being the space of Hölder continuous functions and $\mathcal{F}$ to be the σ-algebra generated by point evaluations. Furthermore, we assume the following conditions to be fulfilled:

- ξ is *uniformly bounded away from* 0 *and* ∞:

$$0 < \mathsf{ei} := \operatorname*{ess\,inf}_{\omega}\xi(x,\omega) < \operatorname*{ess\,sup}_{\omega}\xi(x,\omega) =: \mathsf{es} < \infty \qquad \text{for all } x\in\mathbb{R}; \tag{BDD}$$

- ξ is *stationary*: For every $h\in\mathbb{R}$,

$$(\xi(x))_{x\in\mathbb{R}} \text{ and } (\xi(x+h))_{x\in\mathbb{R}} \text{ have the same distribution;} \tag{STAT}$$

- ξ fulfills a ψ*-mixing* condition: Let $\mathcal{F}_x := \sigma(\xi(z) : z\le x)$ and $\mathcal{F}^y := \sigma(\xi(z) : z \ge y)$, $x,y\in\mathbb{R}$, and assume that there exists a continuous, non-increasing function $\psi : [0,\infty) \to [0,\infty)$, such that for all $j \le k$ as well as integrable $\mathcal{F}_j$-measurable X and integrable $\mathcal{F}^k$-measurable Y, we have

$$\begin{gathered} \big|\mathbb{E}\big[X - \mathbb{E}[X]\,\big|\,\mathcal{F}^k\big]\big| \le \mathbb{E}[|X|]\cdot\psi(k-j), \\ \big|\mathbb{E}\big[Y - \mathbb{E}[Y]\,\big|\,\mathcal{F}_j\big]\big| \le \mathbb{E}[|Y|]\cdot\psi(k-j), \quad \text{and} \\ \sum_{k=1}^{\infty}\psi(k) < \infty. \end{gathered} \tag{MIX}$$

Note that (MIX) implies the ergodicity of ξ with respect to the shift operator θ_y acting on Ω via $\xi(\cdot)\circ\theta_y = \xi(\cdot + y)$, $y\in\mathbb{R}$.

In order to specify the initial conditions for (PAM) under consideration, for $\delta' \in (0,1)$ and $C' > 1$ consider the condition

$$\delta' \mathbb{1}_{[-\delta',0]} \le u_0 \le C'\mathbb{1}_{(-\infty,0]}, \tag{PAM-INI}$$

and we define the class of initial conditions to (PAM) as

$$\mathcal{I}_{\mathrm{PAM}} := \mathcal{I}_{\mathrm{PAM}}(\delta', C') := \{u_0 : \mathbb{R}\to[0,\infty) \text{ measurable}:\ u_0 \text{ fulfills (PAM-INI) for } \delta' \text{ and } C'\}.$$

In order to describe the admissible non-linearities for (F-KPP), let $(p_k)_{k\in\mathbb{N}}$ be an arbitrary sequence of reals in $[0,1]$ such that

$$\sum_{k=1}^{\infty} p_k = 1, \quad \sum_{k=1}^{\infty} kp_k = 2, \quad \text{and} \quad \sum_{k=1}^{\infty} k^2 p_k =: m_2 < \infty. \tag{2.1}$$

Then define $F: [0,1] \to [0,1]$ via

$$F(u) := (1-u) - \sum_{k=1}^{\infty} p_k (1-u)^k, \quad u \in [0,1]. \tag{PROB}$$

In passing, we note that $F'(0) = 1$. The reason for considering this type of non-linearity is its suitability for being investigated using techniques from branching processes. In particular, the solutions to (F-KPP) can then be expressed as a functionals of a branching Brownian motion, see Proposition 5.1.

On top of the above, we need a further technical condition to be fulfilled. In order to be able to formulate it, note that Lemma A.2 states the existence of a critical velocity $v_c \geq 0$ and Proposition A.3 that of another velocity $v_0 > 0$; here, the former pertains to the characteristics of the Lyapunov exponent while, under suitable assumptions, the latter essentially is the speed of the front of the solutions to (PAM) and (F-KPP). In order for our approach to be effective, we need to perform a change of measure that requires

$$v_0 > v_c \tag{VEL}$$

to be fulfilled. For the time being, we content ourselves with referring to Section 2.1, where we argue that there do exist potentials fulfilling (VEL), alongside all other conditions required for our results to hold. For further details and a more profound discussion of condition (VEL), as well as for examples of potentials which do or do not entail (VEL) to be satisfied, we refer to [DS20].

In order to investigate the position of the front, we introduce for $\varepsilon \in (0,1)$, $M > 0$ and $t \geq 0$ the quantities

$$\begin{aligned} m^{\varepsilon}(t) &:= \sup\{x \in \mathbb{R} : w(t,x) \geq \varepsilon\}, \\ m^{\varepsilon,-}(t) &:= \inf\{x \geq 0 : w(t,x) \leq \varepsilon\}, \\ \overline{m}^{M}(t) &:= \sup\{x \in \mathbb{R} : u(t,x) \geq M\}, \\ \overline{m}^{M,-}(t) &:= \inf\{x \geq 0 : u(t,x) \leq M\}. \end{aligned} \tag{2.2}$$

Note that all these quantities are random variables (and their distributions depend on the initial conditions of the respective equations).

Definition 2.1. The solution to (F-KPP) is said to have a *uniformly bounded transition front* if for each $\varepsilon \in (0, \frac{1}{2})$ there exists a constant $C_{\varepsilon} \in (0,\infty)$ such that $\mathbb{P}$-a.s., for all t large enough we have

$$m^{\varepsilon}(t) - m^{1-\varepsilon,-}(t) \leq C_{\varepsilon}.$$

The solution to (PAM) is said to have a *uniformly bounded transition front* if for all $\varepsilon, M \in (0,\infty)$ with $\varepsilon < M$, there exists a constant $C_{\varepsilon,M} \in (0,\infty)$ such that $\mathbb{P}$-a.s., for all t large enough,

$$\overline{m}^{\varepsilon}(t) - \overline{m}^{M,-}(t) \leq C_{\varepsilon,M}. \tag{2.3}$$

We can now state our two main results. The first one is for the solution to (PAM) and states that its transition front stays bounded uniformly in time.

Theorem 2.2. *If* (HÖL)*,* (BDD)*,* (STAT)*,* (MIX) *and* (VEL) *are fulfilled, the solution to* (PAM) *has a uniformly bounded transition front. Furthermore, for $\delta', C' > 0$ fixed, the corresponding constant $C_{\varepsilon,M}$ in* (2.3) *is independent of $u_0 \in \mathcal{I}_{\text{PAM}}(\delta', C')$.*

Our second, and more important, main result states that an analogous statement is in general *not* true for the solution to (F-KPP).

Theorem 2.3. *There exist potentials ξ fulfilling* (HÖL)*,* (BDD)*,* (STAT) *and* (MIX) *such that the transition front of the solution to* (F-KPP) *is not uniformly bounded in time. More precisely, such ξ can be chosen so that for any $\delta \in (0,1)$ and any $\varepsilon > 0$ we find a sequence $(x_n, t_n)_{n\in\mathbb{N}}$ in $\mathbb{R} \times [0,\infty)$ as well as a function $\varphi \in \Theta(\ln n)$ such that*

(a) $x_n, t_n \to \infty$ as $n \to \infty$, and $(x_n)_{n\in\mathbb{N}} \in \Theta(n)$,

(b) for all $n \in \mathbb{N}$,

$$\delta = w(t_n, x_n) \leq w(t_n, x_n + \varphi(n)) + \varepsilon. \tag{2.4}$$

This means that, at least along a subsequence of times, the interval of transition in which the solution changes from being locally unstable ($w \approx 0$) to locally stable ($w \approx 1$), grows at least logarithmically in time as $t \to \infty$.

While the previous result will be derived using probabilistic techniques, we will enhance it employing analytic techniques to show that the statement of Theorem 2.3 is true even for some "negative ε". In particular, this entails the non-monotonicity of the solution in space.

Theorem 2.4. *There exist potentials ξ fulfilling* (HÖL), (BDD), (STAT) *and* (MIX)*, some $\varepsilon > 0$ small enough, and sequences $(t'_n)_{n\in\mathbb{N}}$, $(l'_n)_{n\in\mathbb{N}}$ and $(r'_n)_{n\in\mathbb{N}}$ in $[0, \infty)$ such that $t'_n, r'_n, l'_n \in \Theta(n)$, $l'_n < r'_n$ for all n, $r_n - l_n \in \Theta(\ln n)$ and for all $n \in \mathbb{N}$,*

$$w(t'_n, l'_n) \leq w(t'_n, r'_n) - \varepsilon.$$

Let us already mention here that at a first glance, it may seem slightly difficult to reconcile the statement of Theorem 2.2 with the the statements of Theorems 2.3 and 2.4. In particular, it might seem surprising given that oftentimes the linearization of a non-linear PDE is considered to be a good approximation for the original PDE, at least in the domain where the solutions remain small. We will address this issue in more detail towards the end of Section 2.1.

Remark 2.5. It will become apparent from the respective proofs that Theorem 2.2–2.4 have immediate discrete space analogues for the respective stochastic partial difference equations. These are obtained as follows:

(a) In equations (PAM) and (F-KPP), $x \in \mathbb{R}$ is replaced by $x \in \mathbb{Z}$, and the Laplace operator Δ is replaced by the discrete Laplace operator $\Delta_{\mathrm{d}} f(x) = \frac{1}{2}(f(x+1) + f(x-1) - 2f(x))$.

(b) The potential $(\xi(x))_{x\in\mathbb{R}}$ is replaced by $(\xi(x))_{x\in\mathbb{Z}}$, the assumption (BDD) is replaced by $\mathsf{ei} \leq \xi(x) \leq \mathsf{es}$ for all $x \in \mathbb{Z}$, and condition (STAT) is replaced by $(\xi(x))_{x\in\mathbb{Z}} \overset{d}{=} (\xi(x+1))_{x\in\mathbb{Z}}$.

(c) In (2.2) and Definition 2.1, $x \in \mathbb{R}$ is again substituted by $x \in \mathbb{Z}$.

Then the statements of Theorems 2.2, 2.3 and 2.4 still hold verbatim.

2.1 Discussion and previous results

As already explained in the Introduction, the homogeneous case of constant ξ has been well-understood by now (and, in fact, to a much finer extent than illustrated in the Introduction, see e.g. [Bov16] and references therein for further details). Also the heterogeneous case of random non-linearities we are dealing with has been investigated before. Specifically, under fairly general assumptions, the existence and characterization of the propagation speed (i.e., the linear order of the position of the front $\lim_{t\to\infty} m^\varepsilon(t)/t$) have been derived by Freidlin and Gärtner, see e.g. [GF79] as well as [Fre85, Chapter VII] using large deviation principles. Incidentally, the Feynman-Kac formula (see also Section 3.1 below), which characterizes the solution to the linearization (PAM), also played an important role in the derivation.

In the setting described in the Introduction, second order corrections to the position $m^\varepsilon(t)$ of the front are obtained in [DS20], where it has been shown that the suitably centered and rescaled front fulfills an invariance principle. Again, the proof takes advantage of analyzing (PAM) first. Let us note here that in [Nol11b], a corresponding invariance principle has been derived for non-linearities that are either ignition type or bistable; note however, that – as will be explained below – on a logarithmic in time scale these fronts behave quite differently from the fronts to (F-KPP) in our context. For a different and due technical reasons restricted set of initial conditions, Nolen [Nol11a] has derived a central limit theorem for the position of the front of the solution to (F-KPP) by analytic means. The initial condition $w_0(x,\xi)$ of [Nol11a] is required to depend on the randomness of the environment.

When it comes to the boundedness of transition fronts, Nolen and Ryzhik [NR09] consider the setting of a stationary, ergodic and bounded ξ. The nonlinearity F is assumed to be of ignition-type. I.e., there exists $\theta \in (0,1)$ such that

$$F(w) = 0 \text{ for all } w \notin (\theta_0, 1),\ F(w) > 0 \text{ for all } w \in (\theta_0, 1), \text{ and } F'(1) < 0. \tag{2.5}$$

They find that the solution to (F-KPP) has a uniformly bounded transition front, see [NR09, Proposition 2.3]. Our main result Theorem 2.3 entails that condition (2.5) is crucial here, since otherwise one cannot expect uniformly bounded transition fronts.

Also note that in [DS20, Theorem 1.4] it has been shown that in the setting of the current article, the front of the solution to (F-KPP) lags behind the front of the solution to (PAM) at most logarithmically in t. More precisely, for $\varepsilon \in (0,1)$,

$$\overline{m}^{\varepsilon}(t) - m^{\varepsilon}(t) \in O(\ln t), \quad t \to \infty.$$

Therefore, it immediately arises the question whether this upper bound is sharp. Theorem 2.3 provides the following partial affirmative answer: There exists an increasing sequence (t_n) of times with $t_n \in (0,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and a sequence (x_n) of reals such that $\overline{m}^{\frac{1}{2}}(t_n) - x_n \geq c_0 \log t_n$ such that for all $n \in \mathbb{N}$:

$$w(t_n, x_n) < \frac{1}{2} \quad \text{and (by definition)} \quad u(t_n, \overline{m}^{\frac{1}{2}}(t_n)) = \frac{1}{2}.$$

As in the homogeneous context, there are profound and interesting links to branching processes (in random environment). In [ČD20], in the setting of discrete space, invariance principles have been derived for the position of the front of the PAM as well as the position of the maximum of BRWRE. Furthermore, it has been shown that the distance between these two quantities is in $O(\ln t)$ as $t \to \infty$. In this context, a subtle but important difference to the homogeneous setting is that the solution to (F-KPP) and the maximum of branching Brownian motion in random environment (BBMRE; see Section 3.2 below for the precise definition) do exhibit a slightly more involved interrelation. In particular, neither can we directly transfer the sub-sequential tightness result of Kriechbaum [Kri20] for the law of the maximum of branching random walk in random environment (BRWRE) in the context of [ČD20] to the setting of (F-KPP), nor can we directly obtain a respective non-tightness result for BBMRE from our unbounded transition fronts for the solution to (F-KPP). Furthermore, it is trivial that the distribution function $w^{\xi\equiv\text{const}}(t,\cdot)$ of the maximum of a BBMRE at time t, which is the solution to (F-KPP) with $\xi \equiv \text{const}$, is non-increasing in space. This again is in stark contrast to Theorem 2.4, which states that this is not the case for the solution to (F-KPP) anymore if ξ exhibits "enough" irregularity.

As already alluded to above, Theorem 2.2 as well as Theorems 2.3 and 2.4 might seem slightly surprising in the light of each other, since they imply that the front of (F-KPP) behaves qualitatively quite differently from that of (PAM). In this context, note that Theorem 2.2 requires condition (VEL) to be fulfilled, while the potential ξ satisfying the properties stated in Theorems 2.3 and 2.4 is constructed in (5.3) from the sole assumption $\mathsf{es}/\mathsf{ei} > 2$ of (5.2). In Section B below, cf. Proposition B.2, we show that these conditions can be fulfilled simultaneously and hence this regime of qualitatively different behaviors for the solutions of (F-KPP) and (PAM) is non-trivial.

While from a PDE point of view we lack the experience as well as a good enough control of the fronts that would enable us to explain this phenomenon, it becomes more tractable from a probabilistic point of view. Indeed, we will see below, cf. Proposition 3.1, that the solution to (PAM) can be represented in terms of expectations of a Brownian motion in random potential, i.e. as

$$u(t,x) = E_x\Big[\exp\Big\{\int_0^t \xi(B_s)\,\mathrm{d}s\Big\}\mathbb{1}_{(-\infty,0]}(B_t)\Big].$$

Here, x which are of linear order in time t, such as $\overline{m}^{\varepsilon}(t)$, turn out to be probabilistically "costly" in the sense that for large $C > 0$, Brownian motion in the expectation corresponding to $u(t, x-C)$, i.e. starting in $x - C$ and being to the left of the origin at time t, has to make less of an effort in terms of large deviations than Brownian motion starting in x and being to the left of the origin at time t. Nevertheless, the former can still collect at least as high potential values as the latter, since, typically between $x - C$ and 0 there are enough locations where ξ is large. As a consequence, $u(t, x) \ll u(t, x - C)$ for C large, which at least on a heuristic level explains how the uniform boundedness of the transition fronts to (PAM) stated in Theorem 2.2 comes about.

On the other hand, regarding the solution to (F-KPP) one has a representation in terms of a maximum of branching Brownian motion in random environment (to be introduced in Section 3.2), see Proposition 5.1. The coupling we will construct below in Section 5.2 demonstrates that when it comes to the displacement of this maximum from the starting site of the process, a crucial role is played by the values of the potential in an environment of the starting point. Exploiting this fact in a subtle manner, we arrive at the diverging sequence of times given in Theorem 2.4 at which the front of (F-KPP) is getting wider and wider. What is more, this result can be strengthened to even deduce the non-monotonicity stated in Theorem 2.4.

Open Questions:

(i) We expect that the front of the solution to (F-KPP) shifts from exhibiting unbounded transition fronts (essentially when $\mathsf{es} - \mathsf{ei}$ large, and maybe further conditions, cf. Theorem 2.3) to exhibiting bounded transition fronts (essentially if $\mathsf{es} - \mathsf{ei}$ small, and maybe further conditions, cf. (1.1)). While it is not clear if "small" means "vanishes" in this context, let us point out here that—while periodic media are oftentimes taken to be a simple instance for heterogeneous or random media, cf. also [Fre85, HNRR16, LTZ20]—it is clear from our proofs that the phenomenon of long stretches of areas of high and low potential, which is crucial in our proof, is not observed for periodic media.

(ii) Is there a logarithmic upper bound corresponding to the result of Theorem 2.3 as well, in the sense that $m^{\varepsilon}(t) - m^{1-\varepsilon,-}(t) \leq C \log t$ for all t large enough?

Organization of the article: In Section 3, we recall the well-known Feynman-Kac formula for the solutions to (F-KPP) and (PAM), and introduce branching Brownian motion in random environment, which plays the role of a key tool in this article. Section 4 contains the proof of Theorem 2.2, together with some preparatory results concerning the perturbation of the solution to (PAM) in space and concentration results for the logarithmic moment generating functions. Finally, Section 5 deals with the proofs of the main results about the F-KPP equation, Theorems 2.3 and 2.4.

This article is closely related to [DS20]. While it takes advantage of some results derived in [DS20], it also provides suitable results such as Lemma 4.1 in a natural context, and which are also taken advantage of in [DS20].

3 Preliminaries

In this section we recall two important well-known results which are used to prove our main theorems, and introduce the related notation.

3.1 Feynman-Kac representation

An important tool for the investigation of the solutions to (F-KPP) and (PAM) are their Feynman-Kac representations. Here and in what follows, for $x \in \mathbb{R}$ arbitrary, we denote by E_x the expectation operator with respect to the probability measure P_x under which the process $(B_t)_{t \geq 0}$ is a standard Brownian motion starting in x.

Proposition 3.1 (Feynman-Kac formula, [Bra83, (1.32)]). *Under the assumptions of Section 2, the (unique) non-negative solution u to (PAM) is given by*

$$u(t, x) = E_x\Big[\exp\Big\{\int_0^t \xi(B_s)\,\mathrm{d}s\Big\}\,u_0(B_t)\Big], \tag{3.1}$$

while the (unique) non-negative solution w to (F-KPP) fulfills

$$w(t,x) = E_x\Big[\exp\Big\{\int_0^t \xi(B_s)F(w(t-s,B_s))/w(t-s,B_s)\,\mathrm{d}s\Big\}\,w_0(B_t)\Big]. \tag{3.2}$$

Remark 3.2. In fact, we will take (3.1) and (3.2) as the definition of the solution to (PAM) and (F-KPP), respectively. Indeed, while the function (3.1) is given explicitly, there exists a unique function satisfying (3.2) (see e.g. [Fre85, Theorem 7.4.1]). If the solution to (PAM) and (F-KPP) exist, it can be shown (see e.g. [KS91, Corollary 4.4.5] for (PAM) and [Fre85, (1.4), p. 354, and (a), p. 355] for (F-KPP)) that they satisfy (3.1) and (3.2), respectively.

3.2 Branching Brownian motion in random environment

A key tool for proving Theorems 2.3 and 2.4 is the correspondence between the solution to (F-KPP) and branching Brownian motion in random environment, cf. Proposition 5.1 below. Branching Brownian motion in random environment ξ (BBMRE) started at $x \in \mathbb{R}$ is defined as follows: Conditionally on the realization of ξ, we place one particle at x at time 0. As time evolves, all particles move independently according to standard Brownian motion. In addition, and independently of everything else, while at y, a particle splits at rate $\xi(y)$. Once a particle splits, this particle is removed and, randomly and independently from everything else with probability p_k, replaced by k new particles that are put at the position y of the removed particle. These k new particles evolve independently according to the same diffusion-branching mechanism as the remaining particles. This defines *branching Brownian motion in the branching environment* ξ with offspring distribution (p_k). For every $x \in \mathbb{R}$ and ξ, $\mathtt{E}_x^\xi$ denotes the corresponding expectation of the probability measure $\mathtt{P}_x^\xi$ of a BBMRE, starting in x.

If the respective BBMRE is evident from the context, we use $N(t)$ to denote the set of particles alive at time t in this BBMRE. For any particle $Y \in N(t)$, we denote by $(Y_s)_{s\in[0,t]}$ the trajectory of itself and its ancestors up to time t. We will also call $(Y_s)_{s\in[0,t]}$ the *genealogy* of Y. For $t \geq 0$ and $x \in \mathbb{R}$, we define

$$N^{\geq}(t,x) := \{Y \in N(t) \,:\, Y_t \geq x\} \quad \text{and} \quad N^{\leq}(t,x) := \{Y \in N(t) \,:\, Y_t \leq x\} \tag{3.3}$$

as the number of particles in the process at time t which are located to the right or to the left of x. Furthermore, in a slight abuse of notation, we also use N to denote an entire BBMRE process.

To complete the list of notation, for a stochastic process $X = (X_t)_{t\geq 0}$ and some Borel set $B \subset \mathbb{R}$, we denote $H_B(X) := \inf\{t \geq 0 : X_t \in B\}$ and set $H_x(X) := H_{\{x\}}(X)$, $x \in \mathbb{R}$. For a particle $Y \in N(t)$ of a BBMRE, we set $H_B(Y) = \inf\{s \in [0,t] : Y_s \in B\}$, where $(Y_s)_{s\geq 0}$ is the genealogy of Y and as usual $\inf \emptyset = \infty$.

4 Boundedness of the front for PAM

In this section we show our first main result, the boundedness of the front for the equation (PAM), that is Theorem 2.2.

4.1 A perturbation estimate

The main tool in the proof is a space perturbation result for the solution to (PAM) in a regime of sub-linear perturbation, see Lemma 4.1 bellow.

To state this lemma we need to introduce some notation. Let $\zeta(x) := \xi(x) - \mathtt{es} \leq 0$ with $\mathtt{es}$ defined in (BDD). For $\eta < 0$, define the logarithmic moment generating function as well as the related quantities

$$
\begin{aligned}
L_x^\zeta(\eta) &:= \ln E_x\Big[\exp\Big\{\int_0^{H_{\lceil x\rceil-1}} (\zeta(B_s)+\eta)\,\mathrm{d}s\Big\}\Big], \quad x\in\mathbb{R},\\
\overline{L}_x^\zeta(\eta) &:= \frac{1}{x}\ln E_x\Big[\exp\Big\{\int_0^{H_0} (\zeta(B_s)+\eta)\,\mathrm{d}s\Big\}\Big], \quad x>0,\\
L(\eta) &:= \mathbb{E}[L_1^\zeta(\eta)],\\
S_x^{\zeta,v}(\eta) &:= x\Big(\frac{\eta}{v}-\overline{L}_x^\zeta(\eta)\Big), \quad x>0,\ v>0.
\end{aligned}
\tag{4.1}
$$

Some elementary properties of these functions are recalled in the Appendix. Here we note that, under the assumptions (BDD), (STAT), and (MIX) on the potential ξ, we have $\mathbb{E}[\overline{L}_x^\zeta(\eta)] = L(\eta)$ for all $\eta<0$ and all $x>0$. Further observe that using the strong Markov property one easily shows that for any $x\ge 1$,

$$
x\overline{L}_x^\zeta(\eta) = L_x^\zeta(\eta) + \sum_{i=1}^{\lceil x\rceil-1} L_i^\zeta(\eta) =: \sum_{i=1}^{x} L_i^\zeta(\eta), \tag{4.2}
$$

where the last equality should be seen as the definition of the sum on the right-hand side. For convenience, for $1\le x\le y$, we also define

$$
\sum_{i=x+1}^{y} L_i^\zeta(\eta) := \sum_{i=1}^{y} L_i^\zeta(\eta) - \sum_{i=1}^{x} L_i^\zeta(\eta), \qquad \text{and} \qquad \sum_{i=y+1}^{x} L_i^\zeta(\eta) := -\sum_{i=x+1}^{y} L_i^\zeta(\eta). \tag{4.3}
$$

Furthermore, it essentially follows from Lemma A.1(b) that

$$
\begin{aligned}
&\big(\eta\mapsto L_x^\zeta(\eta) : x\in\mathbb{R}, \zeta\in\Omega \text{ with } \mathsf{ei}-\mathsf{es}\le\zeta\le 0\big) \text{ is a family of}\\
&\text{equicontinuous functions on every compact interval } I\subset(-\infty,0).
\end{aligned}
\tag{4.4}
$$

We further define tilted probability measures under which the process $(B_t)_{t\ge0}$ moves *on average* with speed v up to time t, cf. (4.6) below. We start with introducing the family of tilted probability measures

$$
P_x^{\zeta,\eta}(\cdot) := \exp\big\{-x\overline{L}_x^\zeta(\eta)\big\}\cdot E_x\Big[\exp\Big\{\int_0^{H_0}(\zeta(B_s)+\eta)\,\mathrm{d}s\Big\};\ \cdot\ \Big], \quad x>0, \tag{4.5}
$$

on the space of continuous functions mapping the (initial) argument 0 to x and vanishing only at their (variable) terminal argument. We denote the corresponding expectation operator by $E_x^{\zeta,\eta}$. Then we fix a compact interval $V\subset(v_c,\infty)$ (see Lemma A.2 (d) for the notation) containing v_0 in its interior. It is known, see Lemma A.4, that there exists a compact interval $\triangle\subset(-\infty,0)$, such that $\mathbb{P}$-a.s., for all t large enough and all $v\in V$, there exists a unique $\eta_{vt}^\zeta(v)\in\triangle$ fulfilling

$$
E_{vt}^{\zeta,\eta_{vt}^\zeta(v)}[H_0] = vt(\overline{L}_{vt}^\zeta)'(\eta_{vt}^\zeta) = t. \tag{4.6}
$$

As consequence, there exists a $\mathbb{P}$-a.s. finite random position $\mathcal{N}_1 = \mathcal{N}_1(\xi,V,\triangle)$ such that the event

$$
\mathcal{H}_x := \mathcal{H}_x(V,\triangle) := \big\{\eta_x^\zeta(v)\in\triangle \text{ for all } v\in V\big\} \quad \text{occurs for all } x\ge\mathcal{N}_1. \tag{4.7}
$$

Further, by Lemma A.2 (d) there exists $\overline{\eta}(v)<0$, $v\in V$, such that

$$
L'(\overline{\eta}(v)) = \frac{1}{v}.
$$

Finally, we have that

$$
\overline{\eta}(V)\subset\triangle \quad \text{and} \quad \overline{\eta} \text{ is uniformly Lipschitz continuous on } V, \tag{4.8}
$$

cf. [DS20, (2.22) and below (3.30)].

We can now state our main perturbation lemma.

Lemma 4.1. *Let $\varepsilon(t)$ be a positive function such that $\varepsilon(t) \to 0$ and $\frac{t\varepsilon(t)}{\ln t} \to \infty$ as $t \to \infty$. Then for all $\delta > 0$ there exists $C = C(\delta) > 0$ such that $\mathbb{P}$-a.s., for all $u_0 \in \mathcal{I}_{\mathrm{PAM}}$ we have*

(a)

$$\limsup_{t\to\infty} \sup \Big\{ \Big| \frac{1}{h} \ln \Big(\frac{u(t, vt+h)}{u(t, vt)} \Big) - L(\overline{\eta}(v)) \Big| : (v, h) \in \mathcal{E}_t \Big\} \le \delta, \tag{4.9}$$

where $\mathcal{E}_t := \Big\{(v,h): \ v, v+\frac{h}{t} \in V, \ C(\delta)\ln t \le |h| \le t\varepsilon(t)\Big\}$.

(b) Let $\varepsilon(t)$ be a positive function such that $\varepsilon(t) \to 0$. Then there exists a constant $C_1 < \infty$ and a $\mathbb{P}$-a.s. finite random variable $\mathcal{T}_1$ such that for all $t \ge \mathcal{T}_1$, uniformly in $0 \le h \le t\varepsilon(t)$, $v \in V$, $v + \frac{h}{t} \in V$ and $u_0 \in \mathcal{I}_{\mathrm{PAM}}$ we have

$$C_1^{-1} e^{-C_1 h} u(t, vt) \le u(t, vt+h) \le C_1 e^{-h/C_1} u(t, vt). \tag{4.10}$$

Remark 4.2. Lemma 4.1 is a continuous-space version of [ČD20, Lemma 5.1] and its proof follows similar lines. We will only need part (b) of this lemma in this paper, for the proof of Theorem 2.2. Part (a) is required in [DS20], where an invariance principle for $\overline{m}(t)$ is proved. However, the proof of Lemma 4.1(b) heavily builds on that of (a), which is why it is natural to provide it here.

Proof of Lemma 4.1. (a) It is shown in [DS20, (3.24)] that there exists a constant $\widetilde{C} = \widetilde{C}(\delta', C')$, with δ', C' from (PAM-INI), such that for all $u_0 \in \mathcal{I}_{\mathrm{PAM}}$, all $v \in V$, and all t large enough

$$\widetilde{C}^{-1} u^{\mathbb{1}_{(-\infty,0]}}(t, vt) \le u^{u_0}(t, vt) \le C' u^{\mathbb{1}_{(-\infty,0]}}(t, vt). \tag{4.11}$$

where u^{u_0} denotes the solution to (PAM) with initial condition u_0. Therefore, in order to establish (4.9), it is enough to consider $u_0 = \mathbb{1}_{(-\infty,0]}$.

For this u_0, the solution to (PAM) can be represented by the Feynman-Kac formula (see Proposition 3.1)

$$u(t, vt) = E_{vt}\Big[e^{\int_0^t \xi(B_s)\,\mathrm{d}s}; B_t \le 0\Big].$$

If follows from [DS20, Corollary 3.8 and (3.9)] that if $\mathcal{H}_{vt}$ occurs, then, up to a universal multiplicative constant, this can be approximated by

$$E_{vt}\Big[e^{\int_0^{H_0} \xi(B_s)\,\mathrm{d}s}; H_0 \le t\Big].$$

We now consider t large enough such that $\mathcal{H}_{vt}$ occurs for all $v \in V$. Taking $(v,h) \in \mathcal{E}_t$ and defining $v' := v + \frac{h}{t} \in V$, we see that $\mathcal{H}_{v't}$ occurs as well. Therefore the fraction in (4.9), up to a positive multiplicative constant, is equal to

$$\frac{E_{v't}\big[\exp\big\{\int_0^{H_0} \zeta(B_s)\,\mathrm{d}s\big\}; H_0 \le t\big]}{E_{vt}\big[\exp\big\{\int_0^{H_0} \zeta(B_s)\,\mathrm{d}s\big\}; H_0 \le t\big]} = \frac{E_{v't}\big[\exp\big\{\int_0^{H_0} (\zeta(B_s) + \eta^{\zeta}_{v't}(v'))\,\mathrm{d}s\big\} e^{-\eta^{\zeta}_{v't}(v') H_0}; H_0 \le t\big]}{E_{vt}\big[\exp\big\{\int_0^{H_0} (\zeta(B_s) + \eta^{\zeta}_{vt}(v))\,\mathrm{d}s\big\} e^{-\eta^{\zeta}_{vt}(v) H_0}; H_0 \le t\big]}.$$

Using that $E_{vt}^{\zeta,\eta_{vt}(v)}[H_0] = E_{v't}^{\zeta,\eta_{v't}(v')}[H_0] = t$, recalling (4.1) and (4.5), the latter fraction can be written as

$$\frac{E_{v't}^{\zeta,\eta^{\zeta}_{v't}(v')}\big[e^{-\eta^{\zeta}_{v't}(v')(H_0 - E_{v't}^{\zeta,\eta^{\zeta}_{v't}(v')}[H_0])}; H_0 - E_{v't}^{\zeta,\eta^{\zeta}_{v't}(v')}[H_0] \le 0\big]}{E_{vt}^{\zeta,\eta^{\zeta}_{vt}(v)}\big[e^{-\eta^{\zeta}_{vt}(v)(H_0 - E_{vt}^{\zeta,\eta^{\zeta}_{vt}(v)}[H_0])}; H_0 - E_{vt}^{\zeta,\eta^{\zeta}_{vt}(v)}[H_0] \le 0\big]} \cdot \exp\Big\{S^{\zeta,v}_{vt}(\eta^{\zeta}_{vt}(v)) - S^{\zeta,v'}_{v't}(\eta^{\zeta}_{v't}(v'))\Big\},$$

Since $\mathcal{H}_{vt}$ and $\mathcal{H}_{v't}$ occur, the first fraction is bounded from below and above by positive constants (see [DS20, Lemma 3.6]). The logarithm of the second factor divided by h can be written as

$$\frac{1}{h}\big(S^{\zeta,v}_{vt}(\eta^{\zeta}_{vt}(v)) - S^{\zeta,v'}_{v't}(\eta^{\zeta}_{vt}(v))\big) + \frac{1}{h}\big(S^{\zeta,v'}_{v't}(\eta^{\zeta}_{vt}(v)) - S^{\zeta,v'}_{v't}(\eta^{\zeta}_{v't}(v'))\big). \tag{4.12}$$

We claim that the second summand in (4.12) tends to 0 uniformly in $(v,h)\in\mathcal{E}_t$ as $t\to\infty$, $\mathbb{P}$-a.s. Indeed, by a Taylor expansion we get

$$\begin{aligned} &S^{\zeta,v'}_{v't}(\eta^\zeta_{vt}(v)) - S^{\zeta,v'}_{v't}(\eta^\zeta_{v't}(v')) \\ &\qquad = (S^{\zeta,v'}_{v't})'(\eta^\zeta_{v't}(v'))\cdot(\eta^\zeta_{vt}(v)-\eta^\zeta_{v't}(v')) + \frac{1}{2}(S^{\zeta,v'}_{v't})''(\widetilde{\eta})\big(\eta^\zeta_{vt}(v)-\eta^\zeta_{v't}(v')\big)^2 \end{aligned} \tag{4.13}$$

for some $\widetilde{\eta}\in\triangle$ between $\eta^\zeta_{v't}(v')$ and $\eta^\zeta_{vt}(v)$. By (4.6) and Lemma A.1 we have $(S^{\zeta,v'}_{v't})'(\eta^\zeta_{v't}(v'))=0$. Lemma A.2 (b) entails that $(\overline{L}^\zeta_{v't})''(\cdot)$ is uniformly bounded on $\triangle$ and thus

$$(S^{\zeta,v'}_{v't})''(\widetilde{\eta}) = -v't(\overline{L}^\zeta_{v't})''(\widetilde{\eta}) \in [-v'tc_1^{-1}, -v'tc_1]. \tag{4.14}$$

Furthermore, by Lemma A.5 we have

$$|\eta^\zeta_{vt}(v)-\eta^\zeta_{v't}(v)| \le c_2\frac{|h|}{vt} \le c_3\frac{|h|}{t}, \tag{4.15}$$

and by [DS20, (3.31)]

$$|\eta^\zeta_{v't}(v)-\eta^\zeta_{v't}(v')| \le c_4|v-v'| = c_4\frac{|h|}{t}. \tag{4.16}$$

Thus, for all t large enough, uniformly in $(v,h)\in\mathcal{E}_t$, we get

$$\Big|\frac{1}{h}\big(S^{\zeta,v'}_{v't}(\eta^\zeta_{vt}(v)) - S^{\zeta,v'}_{v't}(\eta^\zeta_{v't}(v'))\big)\Big| \le c_5\frac{|h|}{t} \le \varepsilon(t), \tag{4.17}$$

which tends to zero by assumption.

It remains to show convergence of the first summand in (4.12). We first note that, using the notation introduced in (4.2), (4.3),

$$\frac{1}{h}\big(S^{\zeta,v}_{vt}(\eta^\zeta_{vt}(v)) - S^{\zeta,v'}_{v't}(\eta^\zeta_{vt}(v))\big) = \frac{1}{h}\sum_{i=vt+1}^{v't} L^\zeta_i(\eta^\zeta_{vt}(v)). \tag{4.18}$$

To finish the proof, we will use the following lemma. Recall $\mathcal{N}_1$ from definition (4.7) and let $\varepsilon^*(t) := \sup_{s\in[\lfloor t\rfloor,\lceil t\rceil]}\varepsilon(s)$.

Lemma 4.3 (cf. [ČD20, Claim 5.2])**.** *For every $\delta>0$ and every $q\in\mathbb{N}$, there exists $C_0=C_0(q,\delta)>0$ such that for all $t\ge 1$*

$$\mathbb{P}\Big(\sup_{\substack{C_0\ln\lfloor t\rfloor\le|h|\le\lceil t\rceil\cdot\varepsilon^*(t),\\ v\in V}}\Big|L(\overline{\eta}(v)) - \frac{1}{h}\sum_{i=vt+1}^{vt+h} L^\zeta_i(\eta^\zeta_{vt}(v))\Big| > \delta, \big(vt\ge\mathcal{N}_1\ \forall v\in V\big)\Big) \le ct^{-q}. \tag{4.19}$$

To not disturb the flow of reading, we postpone the proof of Lemma 4.3 to Section 4.2 below. We let A_t be the first event and B_t be the second event on the left-hand side of (4.19). By Lemma 4.3 with $q=2$ and $C_0=C_0(2,\delta/3)$, $\sum_n\mathbb{P}(A_n,B_n)<\infty$ and thus, by the first Borel-Cantelli lemma, $\mathbb{P}$-a.s. the event $A_n\cap B_n$ occurs only finitely often. Because $\mathcal{N}_1$ is $\mathbb{P}$-a.s. finite, we get

$$\sup_{\substack{C_0\ln t\le|h|\le t\varepsilon(t),\\ v\in V}}\Big|L(\overline{\eta}(v)) - \frac{1}{h}\sum_{i=v\lfloor t\rfloor+1}^{v\lfloor t\rfloor+h} L^\zeta_i(\eta^\zeta_{v\lfloor t\rfloor}(v))\Big| \le \frac{\delta}{3} \tag{4.20}$$

$\mathbb{P}$-a.s. for all t large enough.

To bound the right-hand side of (4.18), we need to replace $v\lfloor t\rfloor$ in (4.20) by vt. First note that for all $x,y,z\in\mathbb{R}$ such that $x\le y\le z$, due to the strong Markov property at H_y, similarly as (4.2), we have $\sum_{i=x+1}^z L^\zeta_i(\eta) = \sum_{i=x+1}^y L^\zeta_i(\eta) + \sum_{i=y+1}^z L^\zeta_i(\eta)$ and thus

$$\sum_{i=v\lfloor t\rfloor+1}^{v\lfloor t\rfloor+h} L^\zeta_i(\eta) - \sum_{i=vt+1}^{vt+h} L^\zeta_i(\eta) = \ln E_{vt}\big[e^{\int_0^{H_{v\lfloor t\rfloor}}(\zeta(B_s)+\eta)}\big] - \ln E_{vt+h}\big[e^{\int_0^{H_{v\lfloor t\rfloor+h}}(\zeta(B_s)+\eta)}\big].$$

By [BS15, (2.0.1), p. 204] we have

$$\ln E_x\big[e^{-cH_y}\big] = \sqrt{2c}|y-x|, \qquad \text{for all } c \geq 0 \text{ and } x, y \in \mathbb{R}. \tag{4.21}$$

Therefore, for all t large enough, for every $\eta \in \triangle \subset (-\infty, 0)$ and $0 \geq \zeta(x) \geq -(\mathsf{es} - \mathsf{ei})$,

$$\sup_{\substack{C_0 \ln t \leq |h| \leq t\varepsilon(t),\\ v \in V}} \Big| \frac{1}{h}\Big(\sum_{i=v\lfloor t \rfloor+1}^{v\lfloor t \rfloor+h} L_i^\zeta(\eta) - \sum_{i=vt+1}^{vt+h} L_i^\zeta(\eta)\Big)\Big| \leq \frac{2v\sqrt{2(|\eta| + (\mathsf{es}-\mathsf{ei}))}}{C_0 \ln t} \leq \frac{\delta}{3}. \tag{4.22}$$

In particular, since $\eta^\zeta_{v\lfloor t \rfloor}(v) \in \triangle \subset (-\infty, 0)$ (cf. (4.8)), (4.22) holds with η replaced by $\eta^\zeta_{v\lfloor t \rfloor}(v)$. Moreover, By Lemma A.5, there exists $C > 0$ such that $\mathbb{P}$-a.s. for all x large enough we have $\sup_{v \in V} |\eta^\zeta_{x+h}(v) - \eta^\zeta_x(v)| \leq C\frac{h}{x}$ for all $h \in [0, x]$. Using the equicontinuity (4.4) of $L^\zeta_x(\cdot)$, we get that $\mathbb{P}$-a.s. for all t large enough,

$$\sup_{\substack{C_0 \ln t \leq |h| \leq t\varepsilon(t),\\ v \in V}} \Big| \frac{1}{h} \sum_{i=vt+1}^{vt+h} \big(L_i^\zeta(\eta^\zeta_{v\lfloor t \rfloor}(v)) - L_i^\zeta(\eta^\zeta_{vt}(v))\big)\Big| \leq \frac{\delta}{3}. \tag{4.23}$$

Applying the triangle inequality to the inequalities (4.20)–(4.23), the absolute value of the difference of the right-hand side of (4.18) and $L(\overline{\eta}(v))$ is bounded from above by δ, uniformly in $(v, h) \in \mathcal{E}_t$ for all t large enough, completing the proof of claim (a).

(b) Analogously to the first steps in the proof of (a), it is enough to consider the case $u_0 = \mathbb{1}_{(-\infty,0]}$, and then to show that the expression in (4.12) is bounded from above and below by negative constants, uniformly for all $0 < h \leq t\varepsilon(t)$. Performing the same calculations as in the proof of (a), i.e. using equations (4.13) to (4.16), one can observe that the second summand in (4.12) is contained in the interval $[-c_5\frac{h}{t}, 0]$ for c_5 from (4.17) uniformly for all $v \in V$ and $v' := v + \frac{h}{t} \in V$ and all t large enough.

For the first summand in (4.12), we mention that due to the strong Markov property at time H_{vt}, we have

$$\begin{aligned} S^{\zeta,v}_{vt}(\eta^\zeta_{vt}(v)) - S^{\zeta,v'}_{v't}(\eta^\zeta_{vt}(v)) &= \ln E_{vt+h}\big[e^{\int_0^{H_0}(\zeta(B_s)+\eta^\zeta_{vt}(v))\,\mathrm{d}s}\big] - \ln E_{vt}\big[e^{\int_0^{H_0}(\zeta(B_s)+\eta^\zeta_{vt}(v))\,\mathrm{d}s}\big] \\ &= \ln E_{vt+h}\big[e^{\int_0^{H_{vt}}(\zeta(B_s)+\eta^\zeta_{vt}(v))\,\mathrm{d}s}\big]. \end{aligned}$$

Using (4.21), (BDD) and $\eta^\zeta_{vt}(v) \in \triangle \subset (-\infty, 0)$, for all t large enough, we get

$$-\sqrt{2(|\eta^\zeta_{vt}(v)| + \mathsf{es} - \mathsf{ei})h} \leq \ln E_{vt+h}\big[e^{\int_0^{H_{vt}}(\zeta(B_s)+\eta^\zeta_{vt}(v))\,\mathrm{d}s}\big] \leq -\sqrt{2|\eta^\zeta_{vt}(v)|h} \tag{4.24}$$

and we can conclude. □

4.2 Proof of Lemma 4.3

To finish the proof of Lemma 4.1, we still have to provide the proof of Lemma 4.3.

Proof of Lemma 4.3. We decompose the difference in (4.19) as

$$L(\overline{\eta}(v)) - \sum_{i=vt+1}^{vt+h} L_i^\zeta(\eta^\zeta_{vt}(v)) = L(\overline{\eta}(v)) - \sum_{i=vt+1}^{vt+h} L_i^\zeta(\overline{\eta}(v)) + \sum_{i=vt+1}^{vt+h} \big(L_i^\zeta(\eta^\zeta_{vt}(v)) - L_i^\zeta(\overline{\eta}(v))\big). \tag{4.25}$$

To bound the last summand on the right-hand side, we again recall that the family $\big(L_i^\zeta(\cdot) : i \in \mathbb{R}, 0 \geq \zeta(\cdot) \geq \mathsf{ei} - \mathsf{es}\big)$ is bounded and uniformly equicontinuous on $\triangle$. Therefore, by Lemma A.4, we have

$$\mathbb{P}\Big(\sup_{\substack{\ln\lfloor t \rfloor \leq |h| \leq \lceil t \rceil \varepsilon^*(t),\\ v \in V}} \Big| \frac{1}{h} \sum_{i=vt+1}^{vt+h} \big(L_i^\zeta(\eta^\zeta_{vt}(v)) - L_i^\zeta(\overline{\eta}(v))\big)\Big| > \frac{\delta}{2},\ vt \geq \mathcal{N}_1\ \forall v \in V\Big) \leq ct^{-q}$$

for t large enough. It thus suffices to bound the first summand in (4.25), i.e. to show that there exists $C_0 = C_0(\varepsilon, q) > 0$ such that for all t large enough we have

$$\mathbb{P}\Big(\sup_{\substack{C_0 \ln\lfloor t\rfloor \le |h| \le \lceil t\rceil \varepsilon^*(t),\\ v\in V}} \Big|L(\overline{\eta}(v)) - \frac{1}{h}\sum_{i=vt+1}^{vt+h} L_i^{\zeta}(\overline{\eta}(v))\Big| > \frac{\delta}{2}\Big) \le ct^{-q}. \tag{4.26}$$

Hence, for every h we write $hL(\eta) - \sum_{i=vt+1}^{vt+h} L_i^{\zeta}(\eta) = \sum_{i=1}^{\lfloor h\rfloor+2} \widetilde{L}_i^{\zeta,h,v}(\eta)$, where $(\widetilde{L}_i^{\zeta,h,v}(\eta))_{i=1}^{\lfloor |h|\rfloor+2}$ is a sequence of centered random variables, which are $\mathbb{P}$-a.s. uniformly bounded in $v \in V$, $h \in \mathbb{R}$, $t \in \mathbb{R}$ and $\eta \in \triangle$, as well as fulfill the mixing condition from [DS20, Lemma A.2]. Thus, we can apply Lemma A.6 to show that there exist constants $C > 0$ and $C_0(\varepsilon, q) > 0$, such that for all $v \in V$ and all h fulfilling $|h| \ge C_0 \ln t$ we have

$$\mathbb{P}\Big(\Big|L(\eta) - \frac{1}{h}\sum_{i=vt+1}^{vt+h} L_i^{\zeta}(\eta(v))\Big| > \frac{\delta}{2}\Big) \le \sqrt{e}\exp\Big\{ - \frac{1}{2C\lfloor h\rfloor}\Big(\frac{|h|\varepsilon}{2}\Big)^2\Big\} \le ct^{-q-3}$$

for all t large enough.

To get the "uniform bound" from (4.26), we first show it on the grid $V_n := (\frac{1}{n}\mathbb{Z}) \cap V$ and $C_n^{(t)} := (\frac{1}{n}\mathbb{Z}) \cap [\ln\lfloor t\rfloor, \lceil t\rceil\varepsilon^*(t)]$, $n \in \mathbb{N}$. Indeed, because $|V_n| \le (\operatorname{diam}(V) + 1)n$ and $|C_n^{(t)}| \le nt$, we get

$$\mathbb{P}\Big(\sup_{|h|\in C_{\lfloor t\rfloor}^{(t)},\ v\in V_{\lfloor t\rfloor}} \Big|L(\overline{\eta}(v)) - \frac{1}{h}\sum_{i=vt+1}^{vt+h} L_i^{\zeta}(\overline{\eta}(v))\Big| > \frac{\delta}{2}\Big) \le \operatorname{diam}(V)C \cdot t^{-q} \tag{4.27}$$

for all t large enough. To control all $v \in V$ and $|h| \in [\ln\lfloor t\rfloor, \lceil t\rceil\varepsilon^*(t)]$, we note that for all $s \ge 0$ we have

$$\begin{aligned} &\ln E_{vt+\frac{k}{n}+s}\big[e^{\int_0^{H_{vt}}(\zeta(B_s)+\eta)\,\mathrm{d}s}\big] - \ln E_{vt+\frac{k}{n}}\big[e^{\int_0^{H_{vt}}(\zeta(B_s)+\eta)\,\mathrm{d}s}\big] \\ &\quad = \ln E_{vt+\frac{k}{n}+s}\big[e^{\int_0^{H_{vt+\frac{k}{n}}}(\zeta(B_s)+\eta)\,\mathrm{d}s}\big] \in \Big[- s\sqrt{2(\mathsf{es} - \mathsf{ei} + |\eta|)}, 0\Big] \end{aligned}$$

where the last display is again a consequence of (4.21). Thus all h not on the grid the terms in (4.27) differ at most by a term of order $1/t$. A similar statement holds for all $v \in V$ not on the grid, because $\eta(\cdot)$ is uniformly Lipschitz continuous on V (see (4.8)). Thus the uniform bound in (4.27) can be extended to be valid for all h such that $C_0 \ln\lfloor t\rfloor \le |h| \le \lceil t\rceil\varepsilon^*(t)$. This completes the proof. □

4.3 Proof of Theorem 2.2

We can now finally return to our first main result: the boundedness of the front of (PAM). Its proof builds on the perturbation estimate from Lemma 4.1 (b), and is rather straightforward.

Proof of Theorem 2.2. Due to (VEL), we can choose a compact interval $V \subset (v_c, \infty)$ such that v_0 is in the interior of V. Observe first that the existence of the Lyapunov exponent for the solution of (PAM) (see Proposition A.3) directly implies that the left front $\overline{m}^{M,-}(t)$ as well as the right front $\overline{m}^{\varepsilon}(t)$ of the solution to (PAM) (as defined in (2.2)) satisfy, for arbitrary initial condition $u_0 \in \mathcal{I}_{\mathrm{PAM}}$ and every $\varepsilon, M > 0$, $\mathbb{P}$-a.s.

$$\lim_{t\to\infty} \frac{\overline{m}^{\varepsilon}(t)}{t} = \lim_{t\to\infty} \frac{\overline{m}^{M,-}(t)}{t} = v_0. \tag{4.28}$$

In particular, $\overline{m}^{\varepsilon}(t)/t \in V$ and $\overline{m}^{M,-}(t)/t \in V$ for t large enough, since we assume that v_0 is in the interior of V, and $a_t := \overline{m}^{\varepsilon}(t) - \overline{m}^{M,-}(t) \in o(t)$.

By Lemma 4.1 (b), uniformly in $u_0 \in \mathcal{I}_{\mathrm{PAM}}$ and $v \in V$, for all t large enough such that $vt + a_t \in V$, we get

$$\begin{aligned}\frac{u(t, vt + a_t)}{u(t, vt)} &= \prod_{k=1}^{\lfloor\sqrt{t}\rfloor} \frac{u(t, vt + ka_t/\lfloor\sqrt{t}\rfloor)}{u(t, vt + (k-1)a_t/\lfloor\sqrt{t}\rfloor)} \\ &\leq \left(C_1 e^{-a_t/(C_1\lfloor\sqrt{t}\rfloor)}\right)^{\lfloor\sqrt{t}\rfloor} = e^{-a_t/C_1(1-\frac{\lfloor\sqrt{t}\rfloor}{a_t}\cdot C_1 \ln C_1)}.\end{aligned} \tag{4.29}$$

Now we have all we need to prove Theorem 2.2. Set $C_{\varepsilon,M} := 2C_1 \ln\left(\frac{2MC_1}{\varepsilon}\right)$ and $\varepsilon(t) := 2t^{-1/2}C_1 \ln C_1$. Assume by contradiction that the claim of the theorem does not hold. Then there exist $0 < \varepsilon \leq M$ and a (random) sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \underset{n\to\infty}{\longrightarrow} \infty$ and $a_{t_n} = \overline{m}^{\varepsilon}(t_n) - \overline{m}^{M,-}(t_n) \geq C_{\varepsilon,M}$ for all $n \in \mathbb{N}$. Recalling that $\overline{m}^{\varepsilon}(t)/t \in V$, we get for all n large enough that

$$\varepsilon = u(t_n, \overline{m}^{\varepsilon}(t_n)) = u(t, \overline{m}^{M,-}(t_n) + a_{t_n}) \leq u(t_n, \overline{m}^{M,-}(t_n)) \cdot C_1 e^{-a_{t_n}/2C_1} \leq \varepsilon/2,$$

where in the first inequality we used Lemma 4.1 (b) if $a_{t_n} \leq t_n\varepsilon(t_n)$ and (4.29) if $a_{t_n} > t_n\varepsilon(t_n)$. This is a contradiction. As a consequence, we must have $0 \leq \overline{m}^{\varepsilon}(t) - \overline{m}^{M}(t) \leq C_{\varepsilon,M}$ for all t large enough. Furthermore, this inequality holds uniformly for all $u_0 \in \mathcal{I}_{\mathrm{PAM}}(\delta', C')$, because C_1 is independent of $u_0 \in \mathcal{I}_{\mathrm{PAM}}(\delta', C')$, proving the claim of the theorem. □

5 Unbounded transition front for randomized F-KPP equation

In this section we show our main results about the transition front for the solution to (F-KPP), Theorems 2.3 and 2.4. The proofs are based on the following branching process representation of the solution.

Proposition 5.1 ([DS20, Proposition 2.1])**.** *Let $\xi : \mathbb{R} \to [0, \infty)$ be a non-negative bounded function satisfying (HÖL), F as in (PROB), and let $f : \mathbb{R} \to [0, 1]$ be a function which can be pointwise approximated by an increasing sequence of continuous functions. Then the function*

$$w(t, x) := 1 - \mathtt{E}_x^{\xi}\Big[\prod_{Y\in N(t)} f(Y_t)\Big]$$

solves the equation

$$w_t = \frac{1}{2}w_{xx} + \xi(x)F(w)$$

with initial condition $w(0, \cdot) = 1 - f$. In particular,

$$w(t, x) = \mathtt{P}_x^{\xi}(N^{\leq}(t, 0) \neq \emptyset) \tag{5.1}$$

solves this equation with $f = \mathbb{1}_{(0,\infty)}$, i.e. $w(0, \cdot) = \mathbb{1}_{(-\infty,0]}$.

Remark 5.2. Note that Proposition 5.1 slightly differs from the usual McKean representation in homogeneous branching environment. More precisely, for $\xi \equiv c$ being a constant function and $w(0, \cdot) = \mathbb{1}_{(-\infty,0]}$, the canonical representation is given by $w(t, x) = \mathtt{P}_0^c(N^{\geq}(t, x) \neq \emptyset)$. This representation follows from Proposition 5.1 using the symmetry $\mathtt{P}_x^c(N^{\leq}(t, 0) \neq \emptyset) = \mathtt{P}_0^c(N^{\geq}(t, x) \neq \emptyset)$ which is a consequence of the reflection symmetry of the Brownian motion and the homogeneity of the environment. However, this identity fails to hold if ξ is non-homogeneous.

5.1 The potential

We start the proof of Theorem 2.3 by constructing a suitable potential ξ, for which we then show the unboundedness of the transition front of the solution to (F-KPP). We fix two positive finite constants es and ei such that

$$\frac{\mathsf{es}}{\mathsf{ei}} > 2. \tag{5.2}$$

We further let $\delta_1, \delta_2 \in (0,1)$ be small positive constants, which will be fixed at the end of the proof of Lemma 5.5, see the paragraph below (5.28).

It is an interesting open question whether the condition (5.2) is necessary for the unboundedness of the front. We could not improve it using the methods of this paper, see in particular after (5.25) where the condition (5.2) is crucially needed.

Let furthermore $\chi : [0,\infty) \to [0,1]$ be a continuous non-increasing function with $\chi(x) = 1$ for $x \le 1$ and $\chi(x) = 0$ for $x \ge 2$, and let $\omega = (\omega^i)_{i\in\mathbb{Z}}$ be a Poisson point process on $\mathbb{R}$ with intensity 1 constructed on $(\Omega, \mathcal{F}, \mathbb{P})$. We then define our potential via

$$\xi(x) := \mathsf{ei} + (\mathsf{es} - \mathsf{ei}) \cdot \sup\{\chi(|x - \omega^i|) : i \in \mathbb{Z}\}. \tag{5.3}$$

Observe that the map $x \mapsto \xi(x)$ is a continuous function, $\xi(x) \in [\mathsf{ei}, \mathsf{es}]$ for all $x \in \mathbb{R}$, $\xi(x) = \mathsf{ei}$ if $|x - \omega^i| > 2$ for all i, and $\xi(x) = \mathsf{es}$ if there exists ω^i such that $|x - \omega^i| \le 1$. Also, using the properties of the Poisson point process, ξ fulfills (BDD), (STAT) and (MIX). See Figure 1 for an illustration of this potential.

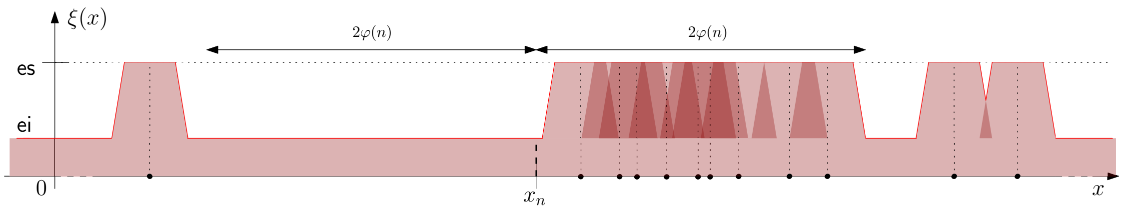

Figure 1: Realization of a potential ξ (top red line) fulfilling (5.4) with $\varphi(n) = c_0 \ln n$. Here we chose $\chi(x) = ((3 - 2x) \wedge 1) \vee 0$.

The crucial property of this potential is that it has long stretches where it equals ei that are adjacent to comparably long stretches where it equals es, as is proved in the next lemma.

Lemma 5.3. *There is a constant $c_0 > 0$ such that $\mathbb{P}$-a.s. there exists a (random) increasing sequence $(x_n)_{n\in\mathbb{N}}$ of reals tending to infinity, such that*

$$\begin{aligned} \xi(x) &= \mathsf{ei} \quad \forall x \in [x_n - 2c_0 \ln n, x_n], \\ \xi(x) &= \mathsf{es} \quad \forall x \in [x_n + 2, x_n + 2c_0 \ln n - 2], \end{aligned} \tag{5.4}$$

and $\xi(\cdot)$ is non-decreasing on $[x_n - 2c_0 \ln n, x_n + 2c_0 \ln n - 2]$. Moreover, $\mathbb{P}$-a.s.,

$$1 \le \liminf_{n\to\infty} n^{-1} x_n \le \limsup_{n\to\infty} n^{-1} x_n \le 2. \tag{5.5}$$

Proof. The proof is an easy application of the Borel-Cantelli lemma. For $k \in \mathbb{N}$, let $A_{k,n}$ be the event

$$A_{k,n} = \left\{ \begin{array}{l} \omega : \omega \cap [n + (4k-2)c_0 \ln n - 2, n + 4kc_0 \ln n + 2) = \emptyset \quad \text{and} \\ \qquad \omega \cap [n + 4kc_0 \ln n + \ell, n + 4kc_0 \ln n + \ell + 1) \neq \emptyset \text{ for all } \ell = 2, \dots, \lfloor 2c_0 \ln n \rfloor - 3 \end{array} \right\}.$$

Observe that if $A_{k,n}$ occurs, then ξ satisfies (5.4) with $x_n = n + 4kc_0 \ln n$, and that $A_{k,n}$ only depends on ω in the interval $[n + (4k-2)\ln n - 2, n + 4k \ln n + \lfloor 2 \ln n \rfloor - 2)$. Therefore, the events $(A_{k,n})_{k\in\mathbb{N}}$ are independent. Moreover,

$$\mathbb{P}(A_{k,n}) = e^{-2c_0 \ln n - 4} \prod_{\ell=2}^{\lfloor 2c_0 \ln n \rfloor - 3} (1 - e^{-1}) \ge \alpha^{-c_0 \ln n} = n^{-c_0 \ln \alpha}$$

for some $\alpha > 1$ independent of c_0. Therefore, using $1 - x \le e^{-x}$,

$$\mathbb{P}\Big(\bigcap_{k=0}^{n/(4c_0 \ln n) - 1} A_{k,n}^c \Big) \le (1 - n^{-c_0 \ln \alpha})^{n/(4c_0 \ln n)} \le \exp\{-n^{1 - c_0 \ln \alpha} (4c_0 \ln n)^{-1}\}.$$

For $c_0 < 1/\ln \alpha$, the right-hand side is summable and thus by the Borel-Cantelli lemma, almost surely for n large enough, there exists $k \in [0, n/(4c_0 \ln n) - 1]$ such that $A_{k,n}$ occurs. This implies that $\mathbb{P}$-a.s. for n large enough there is $x \in [n, 2n]$ satisfying (5.4), completing the proof. □

In the following, if not mentioned otherwise, we will always refer to the sequence (x_n) as the one the existence of which is provided by (5.3).

5.2 The coupling

In the next step towards a proof of Theorem 2.3, we construct a coupling of two BBMREs started in the vicinity of the points x_n where the potential satisfies the conditions (5.4) of Lemma 5.3.

Throughout this section, we assume that the constant c_0 and the random sequence x_n are as in Lemma 5.3, and write

$$\varphi(n) = c_0 \ln n. \tag{5.6}$$

In order to emphasize the dependence of the BBMRE on the starting point, we write $N_x = (N_x(t))_{t\geq 0}$ for the BBMRE started from x, that is for the process whose distribution is $\mathtt{P}^\xi_x$.

The content of the next proposition is the coupling alluded to above. Its statement is slightly more general than needed to show Theorem 2.3, since we construct couplings for many different starting points. This additional control will be useful in the proof of Theorem 2.4. Recall that the (possibly small but) positive parameter δ_1 is fixed below (5.28).

Proposition 5.4. *For every $\varepsilon > 0$ there exists $C_2 = C_2(\varepsilon) \in (0,\infty)$ such that for all n large enough, $l \in [x_n - 5\delta_1\varphi(n), x_n - 4\delta_1\varphi(n)]$, and $r \in [x_n + \delta_1\varphi(n), x_n + 2\delta_1\varphi(n)]$, there exists a coupling $\mathtt{Q}^\xi_{l,r}$ of the BBMREs N_l and N_r such that*

$$\mathtt{Q}^\xi_{l,r}(N_l(t) \subset N_r(t)\, \forall t \geq C_2 \ln n) \geq 1 - \varepsilon. \tag{5.7}$$

For an illustration of the coupling and an explanation of the strategy to show that the event in (5.7) occurs with high probability, we refer to Figure 2.

Before proving Proposition 5.4, let us first show that it implies Theorem 2.3.

Proof of Theorem 2.3. Using the notation from Proposition 5.4 we set

$$t_n := \inf\{t \geq 0 : w(t, x_n - 4\delta_1\varphi(n)) = \delta\}.$$

Note that $t_n \geq C_2 \ln n$ for all n large enough (using $x_n \geq n$ and the fact that the front moves linearly, see Proposition A.3). By (5.5) and (5.6) we get $\varphi \in \Omega(\ln n)$, $x_n, t_n \to \infty$, $(x_n)_{n\in\mathbb{N}} \in \mathcal{O}(n)$ and it remains to show (2.4). Let us abbreviate $l := x_n - 4\delta_1\varphi(n)$ and $r := x_n + 2\delta_1\varphi(n)$. By definition of the coupling $\mathtt{Q}^\xi_{l,r}$ and the representation $w(t,x) = \mathtt{P}^\xi_x(N^{\leq}(t,0) \neq \emptyset)$ of the solution to (F-KPP) (see Proposition 5.1), we have for all n large enough that

$$\begin{aligned}
\delta = w(t_n, x_n - 4\delta_1\varphi(n)) &= \mathtt{P}^\xi_l(N^{\leq}(t_n,0) \neq \emptyset) = \mathtt{Q}^\xi_{l,r}(N^{\leq}_l(t_n,0) \neq \emptyset) \\
&\leq \mathtt{Q}^\xi_{l,r}(N^{\leq}_l(t_n,0) \neq \emptyset, N_l(t) \subset N_r(t)\, \forall t \geq C_2 \ln n) + \varepsilon \\
&\leq \mathtt{Q}^\xi_{l,r}(N^{\leq}_r(t_n,0) \neq \emptyset) + \varepsilon = \mathtt{P}^\xi_r(N^{\leq}(t_n,0) \neq \emptyset) + \varepsilon \\
&= w(t_n, x_n + 2\delta_1\varphi(n)) + \varepsilon,
\end{aligned}$$

where we used (5.7) in the first inequality. Adapting the notation to that of the statement, we can conclude. □

Proof of Proposition 5.4. To construct the coupling, we endow every particle in N_l and N_r at every time with a type. The type of the particle does not influence its dynamics within N_l or N_r, but rather helps to encode the dependence between N_l and N_r under $\mathtt{Q}^\xi_{l,r}$. At any given time, every particle in N_l can have either of the types *l-mirrored*, *l-coupled*, or *bad*. Similarly, every particle in N_r can have either of the types *r-mirrored*, *r-coupled*, or *free*. We denote $\mathtt{LM}(t), \mathtt{LC}(t), \mathtt{B}(t)$ and $\mathtt{RM}(t)$, $\mathtt{RC}(t)$ and $\mathtt{F}(t)$ the sets of particles with those respective types at time t. A particle is given a type when it is created, and its type can change only if it branches, meets another particle or hits some special point in space, as we will describe later. The assignment of the type is a right-continuous function in times, in the sense that if, e.g., a particle Y changes its type from l-mirrored to bad at time t, then $Y \in \mathtt{B}(t)$ and $Y \in \mathtt{LM}(t-)$.

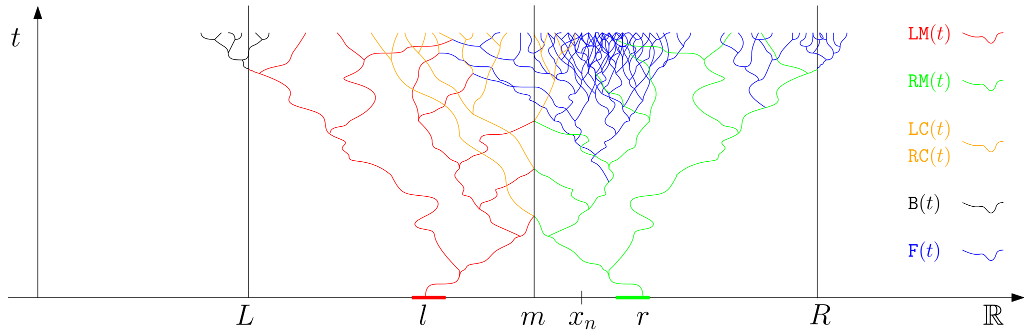

Figure 2: An illustration of the coupling mechanism. l-mirrored particles are illustrated in red, r-mirrored particles in green, while l- and r-coupled particles are illustrated in orange. Free particles are blue and bad particles are black. The fat red (resp. green) line on the $\mathbb{R}$-axis denotes the set $[x_n - 5\delta_1\varphi(n), x_n - 4\delta_1\varphi(n)]$ (resp. $[x_n + \delta_1\varphi(n), x_n + 2\delta_1\varphi(n)]$). Note that x_n is nearer to the green domain, forcing a particle $Y \in N_l$ to go a long way to reach high branching-potential. The event in (5.7) occurs, if at time $t = c \ln n$, all l-mirrored particles (red) are already turned into l-coupled ones (orange) and no l-mirrored particles have crossed L yet. But then there will be no bad particles (black) either, which already implies the event in (5.7).

In addition, under the coupling, at every time $t \geq 0$, there are bijections $\mu_t : \mathtt{LM}(t) \to \mathtt{RM}(t)$ and $\gamma_t : \mathtt{LC}(t) \to \mathtt{RC}(t)$. The bijections μ_t "mirror" the positions of the particles:

$$\text{If } Y \in \mathtt{LM}(t) \text{ and } Y' = \mu_t(Y) \in \mathtt{RM}(t), \text{ then } m - Y_t = Y'_t - m, \tag{5.8}$$

where m is the midpoint of the segment (l, r),

$$m := \frac{1}{2}(l + r) \in [x_n - 2\delta_1\varphi(n), x_n - \delta_1\varphi(n)].$$

On the other hand, coupled particles are at the same position:

$$\text{If } Y \in \mathtt{LC}(t) \text{ and } Y' = \gamma_t(Y) \in \mathtt{RC}(t), \text{ then } Y_t = Y'_t. \tag{5.9}$$

As time evolves, the bijections μ_t and γ_t naturally follow the particles. That is, for the mirrored particles, if $Y \in \mathtt{LM}(t) \cap \mathtt{LM}(t')$, $Y' \in \mathtt{RM}(t) \cap \mathtt{RM}(t')$ and $Y' = \mu_t(Y)$, then also $Y' = \mu_{t'}(Y)$, and similarly for the coupled particles.

We set

$$L := x_n - \varphi(n) \qquad \text{and} \qquad R := 2m - L. \tag{5.10}$$

It will turn out that under the coupling constructed below, the l-mirrored particles will always be in the interval (L, m), that is $\{Y_t : Y \in \mathtt{LM}(t)\} \subset (L, m)$, see (A) and (C) below. As a consequence of (5.8) and (5.10), we then have $\{Y_t : Y \in \mathtt{RM}(t)\} \subset (m, R)$. In particular, in combination with (5.4), we infer that the potential is always larger at the position of an r-mirrored particle than at the position of the corresponding l-mirrored particle:

$$\text{If } Y \in \mathtt{LM}(t) \text{ and } Y' = \mu_t(Y), \text{ then } \xi(Y_t) \leq \xi(Y'_t). \tag{5.11}$$

We can now describe the dynamics of N_l, N_r and of the types under the coupling $\mathtt{Q}^{\xi}_{l,r}$. At time 0, there is one (l-mirrored) particle at position l in N_l and one (r-mirrored) particle at position r in N_r; this determines the bijection μ_0 uniquely. Every particle in N_l (resp. N_r) performs Brownian motion, independently of the other particles in N_l (resp. N_r). The corresponding mirrored and coupled particles are required to satisfy (5.8) and (5.9) respectively, which is possible, since the law of Brownian motion is invariant by reflection; besides these two conditions the motion of particles in N_l is independent of the motion of particles in N_r.

The branching events occur according to the following rules.

(a) At time t, every $Y \in N_l$ branches with rate $\xi(Y_t)$. It is replaced by k new particles, with probability p_k, independently of remaining randomness. The type of the new particles is the same as of Y.

If a particle Y is l-mirrored (resp. l-coupled), $Y \in \mathtt{LM}(t-)$ (resp. $Y \in \mathtt{LC}(t-)$) before time t, then the corresponding r-mirrored particle $Y' = \mu_{t-}(Y)$ (resp. r-coupled particle, $Y' = \gamma_{t-}(Y)$) branches as well. It is replaced by the same number k of particles. The newly created particles are set to be r-mirrored (resp. r-coupled) and the bijection μ_t (resp. γ_t) is a natural extension of μ_{t-} (resp. γ_{t-}) to the newly created particles.

(b) At time t, every r-mirrored particle $Y' \in \mathtt{RM}(t-)$ (mirrored with $Y = \mu_{t-}^{-1}(Y')$) branches with rate $\xi(Y'_t) - \xi(2m - Y'_t) = \xi(Y'_t) - \xi(Y_t)$, in addition to the branching occurring in (a). This rate is non-negative due to (5.8) and (5.11). It is replaced by k new particles, with probability p_k, independently of everything else. One of the newly created particles, say Z', is set to be r-mirrored, and we set $\mu_t(Y) := Z'$. The type of the remaining newly created particles is free.

(c) At time t, every free particle $Y' \in \mathtt{F}(t)$ branches with rate $\xi(Y'_t)$. It is replaced by k new particles, with probability p_k, independently of everything else. The type of the new particles is free.

It can be easily checked that, as a result of the rules (a)–(c), every $Y' \in N_r$ branches with rate $\xi(Y'_t)$ at time t, as it should.

Finally, the particles can change their type if one of the following events occur:

(A) If an l-mirrored particle hits m, that is $Y \in \mathtt{LM}(t-)$ and $Y_t = m$, then, by consequence of (5.8), the corresponding particle $Y' = \mu_{t-}(Y)$ satisfies $Y'_t = m$ as well. We thus change the types of Y and Y' to l-coupled and r-coupled, respectively, and define $\gamma_t(Y) := Y'$.

(B) If an l-mirrored particle $Y \in \mathtt{LM}(t-)$ meets a free particle at time t, that is there is $Z' \in \mathtt{F}(t-)$ with $Z'_t = Y_t$, then we change the types of Y and Z' to l-coupled and r-coupled, respectively, and define with $\gamma_t(Y) := Z'$. The type of the r-mirrored particle $Y' = \mu_{t-}(Y)$ that was mirrored with Y is changed to free.

(C) If an l-mirrored particle hits L, that is $Y \in \mathtt{LM}(t-)$ and $Y_t = L$, then the type of Y is changed to bad, and the type of the corresponding r-mirrored particle $Y' = \mu_{t-}(Y)$ is changed to free.

To show that the coupling succeeds, i.e. that (5.7) holds, it is sufficient to show that with probability at least $1 - \varepsilon$, there are no l-mirrored and bad particles after time $C_2 \log n$. In this vein, we define two good events:

$$\mathcal{G}_1(t) := \{N_l^{\le}(s, L) = \emptyset \, \forall s \le t\}, \tag{5.12}$$

i.e., on $\mathcal{G}_1(t)$ no particle from N_l enters $(-\infty, L)$ before time t, and

$$\mathcal{G}_2(t) := \{N_r^{\le}(t, L) \neq \emptyset\}; \tag{5.13}$$

i.e., there is a (necessarily free, if $\mathcal{G}_1(t)$ occurs as well) particle to the left of L at time t. We now need the following lemma which ensures that we can find t such that those events are typical.

Lemma 5.5. *For any $\varepsilon > 0$ there exists $t' < 1$ such that for all n large enough, with $t = t'\varphi(n)/\sqrt{2\mathsf{ei}}$,*

$$\mathtt{Q}_{l,r}^{\xi}(\mathcal{G}_1(t) \cap \mathcal{G}_2(t)) \ge 1 - \varepsilon. \tag{5.14}$$

We postpone the proof of this lemma and complete the proof of Proposition 5.4 first. Let t be as in Lemma 5.5. We claim that

$$\{N_l(t) \subset N_r(t)\} \supset \mathcal{G}_1(t) \cap \mathcal{G}_2(t). \tag{5.15}$$

If we show this, then the claim of Proposition 5.4 follows with $C_2 = t/\ln n = t'c_0/\sqrt{2\mathsf{ei}}$.

To prove (5.15), recall first that bad particles can only be created if an l-mirrored particle hits L. As a consequence,

$$\text{on } \mathcal{G}_1(t) \text{ there cannot be any bad particles at time } t. \tag{5.16}$$

Next, we show that

$$\text{on } \mathcal{G}_1(t)\cap\mathcal{G}_2(t) \text{ there are no l-mirrored particles at time } t \tag{5.17}$$

either. To this end define $\mathcal{R}(t) = \inf\{Y'_t : Y \in \mathtt{F}(t)\}$ to be the position of the leftmost free particle, and $\mathcal{L}(t) = \sup\{Y_t : Y \in \mathtt{LM}(t)\}$ to be the position of the rightmost l-mirrored particle, with the convention $\inf\emptyset = +\infty$, $\sup\emptyset = -\infty$; in the remaining cases, a.s., the infimum and supremum are attained, since $\mathtt{F}(t)$ and $\mathtt{LM}(t)$ are a.s. finite sets). Let

$$\tau := \inf\{t \geq 0 : \mathcal{L}(t) > \mathcal{R}(t)\}.$$

We claim that $\tau = \infty$, $\mathtt{Q}^{\xi}_{l,r}$-a.s. Indeed, we first note that $\mathcal{L}$ and $\mathcal{R}$ are right-continuous. In addition, the only jumps that $\mathcal{L}$ has are downward jumps. They occur a.s. iff the rightmost l-mirrored particle changes its type due to (A)–(C). (If one of (A)–(C) occurs, then a.s. there is only one l-mirrored particle at position $\mathcal{L}(t)$. At branching events, $\mathcal{L}$ is unchanged, as l-mirrored particles are created only at positions where l-mirrored particles are already present, see (a)). Similarly, with the exception of the first jump from $+\infty$, the only jumps that the function $\mathcal{R}$ has are upwards jumps, occurring a.s. iff the leftmost free particle becomes r-coupled due to (B). Therefore, it follows that a.s. $\tau \geq \inf\{t \geq 0 : \mathcal{L}(t) = \mathcal{R}(t)\}$. However, the event $\{\exists t \in [0,\infty) : \mathcal{L}(t) = \mathcal{R}(t)\}$ cannot occur by the construction of the coupling, since if an l-mirrored and a free particle meet, then at this instant they become l-/r-coupled immediately. Hence, $\tau = \infty$ almost surely, as claimed.

Assume now that $\mathcal{G}_1(t)\cap\mathcal{G}_2(t)$ occurs. At time t, there is thus a particle from N_r and no particle from N_l to the left of L. From the construction, this particle is neither r-coupled (since on $\mathcal{G}_1(t)$ there is no corresponding l-coupled particle there), nor r-mirrored (as all r-mirrored particles are always in (m,R)). Therefore, it must be free and thus $\mathcal{R}(t) < L$. Since $\tau = \infty$ a.s., $\mathcal{L}(t) < L$ as well. However, by construction, l-mirrored particles are always located in (L,m), and thus $\mathcal{L}(t) < L$ implies $\mathcal{L}(t) = -\infty$, that is $\mathtt{LM}(t) = \emptyset$, establishing (5.17).

All in all, from the above it follows that on $\mathcal{G}_1(t)\cap\mathcal{G}_2(t)$, (5.16) as well as (5.17) hold true, i.e., there do not exist any l-mirrored or bad particles at time t. Hence, all particles in $N_l(t)$ are necessarily l-coupled, which proves (5.15). This completes the proof of Proposition 5.4. □

It remains to show Lemma 5.5.

Proof of Lemma 5.5. We first estimate the probability of $\mathcal{G}_1(t)$ as a function of $t \in [0, \varphi(n)/\sqrt{2\mathsf{ei}}]$. To this end we write $\mathcal{N}(t)$ for the number of particles from N_l that hit L before t; here, we only count the first hit of L by any particle. That is, we disregard possible successive hits of L by the same particle, and also the fact that this particle could branch between the hitting of L and the time t, and thus produce more particles at time t that hit L. The expectation of $\mathcal{N}(t)$ can be written as

$$\mathtt{E}^{\xi}_l[\mathcal{N}(t)] = E_l\Big[e^{\int_0^{H_L}\xi(X_s)\mathrm{d}s}; H_L < t\Big] \leq E_l\Big[e^{\int_0^{H_L}\tilde{\xi}(X_s)\mathrm{d}s}; H_L < t\Big], \tag{5.18}$$

where the potential $\tilde{\xi}$ is given by $\tilde{\xi}(x) = \mathsf{es}$ if $x \geq x_n$, and $\tilde{\xi}(x) = \mathsf{ei}$ if $x < x_n$. To estimate the right-hand side, note that there are two possible scenarios for a particle to hit L. Either, it stays all the time in the interval (L, x_n) where the potential equals ei and hits L (i.e., it displaces by altogether at least $l - L \geq (1-5\delta_1)\varphi(n)$). Or, it spends some s units of time in the interval $[x_n,\infty)$, where the potential is es, but then it should displace by at least $(x_n - l) + (x_n - L) \geq (1+4\delta_1)\varphi(n)$ in $t-s$ units of time. Ignoring prefactors which are sub-exponential in $\varphi(n)$ and using standard Gaussian tail bounds, we thus arrive at the following upper bound:

$$\begin{aligned}\mathtt{E}^{\xi}_l[\mathcal{N}(t)] &\lesssim \exp\Big\{t\mathsf{ei} - \frac{(1-5\delta_1)^2\varphi(n)^2}{2t}\Big\} + \sup_{s\leq t}\exp\Big\{(t-s)\mathsf{ei} + s\mathsf{es} - \frac{(1+4\delta_1)^2\varphi(n)^2}{2(t-s)}\Big\} \\ &= \exp\Big\{\sigma(n)\Big(t' - \frac{(1-5\delta_1)^2}{t'}\Big)\Big\} + \sup_{s'<t'}\exp\Big\{\sigma(n)\Big(t' + s'\,\frac{\mathsf{es}-\mathsf{ei}}{\mathsf{ei}} - \frac{(1+4\delta_1)^2}{t'-s'}\Big)\Big\},\end{aligned} \tag{5.19}$$

where we introduced

$$\sigma(n) = \varphi(n)\sqrt{\frac{\mathsf{ei}}{2}} \qquad \text{and} \qquad t' = \frac{t\,\mathsf{ei}}{\sigma(n)} \tag{5.20}$$

in order to put the various terms on the same scale. Using Markov's inequality, to show that $\mathsf{P}_l^\xi(\mathcal{G}_1(t)^c) \to 0$, it is sufficient to show that both summands on the right-hand side of (5.19) tend to 0. For this to be the case for the first one, it is sufficient to require

$$t' < (1 - 5\delta_1). \tag{5.21}$$

Before dealing with the second term (which we will do below (5.24)), we turn our attention to the event $\mathcal{G}_2(t)$.

To control the probability of the event $\mathcal{G}_2$, we need two claims.

Claim 5.6. *For every $\varepsilon > 0$ there exists $t_0 < \infty$ such that for all n large enough,*

$$\mathsf{P}_r^\xi\Big(\big|\{Y \in N_r(t) : Y_t \in [x_n + \delta_1\varphi(n), x_n + 2\delta_1\varphi(n)]\}\big| \ge e^{(1-\delta_2)\mathsf{es}\,t}\Big) \ge 1 - \varepsilon/2, \qquad \textit{for all } t \ge t_0. \tag{5.22}$$

In order not to hinder the flow of reading, we postpone the proof of Claim 5.6 to the end of the proof of Lemma 5.5.

Claim 5.7. *Let $t = t'\varphi(n)/\sqrt{2\mathsf{ei}}$ with $t' < 1$ and $\eta > 0$. Then for every $y \in [x_n + \delta_1\varphi(n), x_n + 2\delta_1\varphi(n)]$ and all n large enough*

$$\mathsf{P}_y^\xi(N^{\le}(t, L) \neq \emptyset) \ge \exp\Big\{\sigma(n)\Big(t' - \frac{(1+2\delta_1)^2}{t'} - \eta\Big)\Big\}. \tag{5.23}$$

Proof. Obviously $\mathsf{P}_y^\xi(N^{\le}(t, L) \neq \emptyset) \ge \mathsf{P}_y^{\mathsf{ei}}(N^{\le}(t, L) \neq \emptyset) \ge \mathsf{P}_{x_n+2\delta_1\varphi(n)}^{\mathsf{ei}}(N^{\le}(t, L) \neq \emptyset)$. Moreover, by the large deviation lower bound from [CR88, Thm. 1], for every $v > \sqrt{2\mathsf{ei}}$ and $\eta > 0$, if t is sufficiently large, then

$$\mathsf{P}_0^{\mathsf{ei}}(N^{\le}(t, -vt) \neq \emptyset) \ge \exp\{t(\mathsf{ei} - v^2/2 - \eta)\}.$$

Using this estimate with $v = (x_n + 2\delta_1\varphi(n) - L)/t = (1 + 2\delta_1)\varphi(n)/t = (1 + 2\delta_1)\sqrt{2\mathsf{ei}}/t' > \sqrt{2\mathsf{ei}}$, and by rewriting it using the notation introduced in (5.20), the claim follows. □

Using these two claims, we have that for any $0 < s' < t' < 1$ as well as for $t = t'\varphi(n)/\sqrt{2\mathsf{ei}}$ and $s = s'\varphi(n)/\sqrt{2\mathsf{ei}}$, that

$$\begin{aligned}
\mathsf{P}_r^\xi(\mathcal{G}_2(t)^c) &\le \mathsf{P}_r^\xi\Big(\big|\{Y \in N_r(s) : Y_s \in [x_n + \delta_1\varphi(n), x_n + 2\delta_1\varphi(n)]\} \le e^{(1-\delta_2)\mathsf{es}\,s}\Big) \\
&\quad + \mathsf{P}_r^\xi\Big(\mathcal{G}_2(t)^c \,\big|\, \big|\{Y \in N_r(s) : Y_s \in [x_n + \delta_1\varphi(n), x_n + 2\delta_1\varphi(n)]\}\big| \ge e^{(1-\delta_2)\mathsf{es}\,s}\Big) \\
&\le \frac{\varepsilon}{2} + \Big(1 - \exp\Big\{\sigma(n)\Big(t' - s' - \frac{(1+2\delta_1)^2}{t'-s'} - \eta\Big)\Big\}\Big)^{\exp\{(1-\delta_2)\mathsf{es}\,s\}}.
\end{aligned}$$

The second summand on the right-hand side converges to 0 as $n \to \infty$ if

$$\begin{aligned}
&\exp\Big\{\sigma(n)\Big(t' - s' - \frac{(1+2\delta_1)^2}{t'-s'} - \eta\Big)\Big\} \cdot \exp\{(1-\delta_2)\mathsf{es}\,s\} \\
&= \exp\Big\{\sigma(n)\Big(t' + s'\frac{\mathsf{es}(1-\delta_2) - \mathsf{ei}}{\mathsf{ei}} - \frac{(1+2\delta_1)^2}{t'-s'} - \eta\Big)\Big\} \to \infty.
\end{aligned} \tag{5.24}$$

The factors in the exponents of (5.19) and (5.24) are both of the form $t' + As' - B/(t' - s')$ such that (for $\delta_2 > 0$ small) $A > 0$ and $B > 1$. For A, B and t', fixed, this function is maximized for $s \in [0, t']$ by

$$s' = \begin{cases} t' - \sqrt{B/A}, \\ 0, \end{cases} \qquad \text{with a maximum value of } \begin{cases} (1+A)t' - 2\sqrt{AB}, & \text{if } t' > \sqrt{B/A}, \\ t' - B/t', & \text{otherwise.} \end{cases} \tag{5.25}$$

Ignoring for a moment the constants δ_2 and η, we write $A = (\mathsf{es} - \mathsf{ei})/\mathsf{ei}$, $B_1 = (1 + 4\delta_1)^2$, and $B_2 = (1 + 2\delta_1)^2$. Observe that $A > 1$ by (5.2). In order to satisfy (5.24) and let (5.19) tend to 0, we must fix t' and δ_1 so that (5.21) holds, and at the same time

$$\sup_{0<s'<t'} t' + s'A - B_1/(t' - s') < 0, \tag{5.26}$$

$$\sup_{0<s'<t'} t' + s'A - B_2/(t' - s') > 0. \tag{5.27}$$

Since $B_2 > 1$ and $t' < 1$, the analysis in (5.25) implies that the supremum in (5.27) can be positive only if

$$t' > \max\left(\sqrt{\frac{B_2}{A}}, \frac{2\sqrt{AB_2}}{1+A}\right) = \frac{2\sqrt{AB_2}}{1+A}, \tag{5.28}$$

where to obtain the equality we used the fact that $A > 1$. We thus fix $\delta_1 > 0$ small enough so that $1 - 5\delta_1 > 2\sqrt{AB_2}/(1 + A)$ and (5.21) as well as (5.28) can be both satisfied; this is possible only if $A > 1$ which is true by assumption. We then fix t' satisfying (5.21) and (5.28), so that the supremum in (5.27) is positive (this is by construction), but small enough, so that the supremum in (5.26) is negative; this is possible since $B_1 > B_2$. Finally, we fix $\delta_2 > 0$, $\eta > 0$ so that the validity of the established inequalities is not modified. With this choice of constants, (5.24) holds and the right-hand side of (5.19) tends to 0, as required. Hence, for $t = t'\varphi(n)/\sqrt{2\mathsf{ei}}$ we have $\mathsf{Q}^{\xi}_{r,l}(\mathcal{G}_1(t)^c \cup \mathcal{G}_2(t)^c) \le \mathsf{P}^{\xi}_{l}(\mathcal{G}_1(t)^c) + \mathsf{P}^{\xi}_{r}(\mathcal{G}_2(t)^c) \to 0$ as $n \to \infty$. This completes the proof. □

It remains to prove Claim 5.6.

Proof of Claim 5.6. The proof follows by a comparison with branching processes split into two phases. For the first phase we recall that by Lemma [DS20, Lemma 4.7] there exist $\kappa > 1$ and $t_1 < \infty$ such that, $\mathbb{P}$-a.s.,

$$\sup_{x\in\mathbb{R}} \mathsf{P}^{\xi}_x\big(|\{Y \in N(t) : Y_t \in [x-1, x+1]\}| \le \kappa^t\big) \le \kappa^{-t} \qquad \text{for all } t \ge t_1. \tag{5.29}$$

For the second phase we need few preparatory steps. We fix $T > 0$ such that

$$e^{(1-\frac{\delta_2}{2})\mathsf{es}\,T} \le \frac{1}{4}e^{\mathsf{es}\,T} \quad \text{and} \quad P_0(B_T > 1) \ge \frac{7}{16}. \tag{5.30}$$

We further fix $K_1 > 1$ large enough so that

$$\inf_{x\in[-K_1-1,K_1+1]} P_x(B_T \in [-K_1, K_1]) \ge \frac{3}{8}, \tag{5.31}$$

which is possible due to the second part of (5.30). Finally, we fix $K_2 > K_1$ large enough so that

$$\sup_{x\in[-K_1-1,K_1+1]} P_x(B_s \notin [-K_2, K_2] \text{ some } s \in [0, T]) \le \frac{1}{16}, \tag{5.32}$$

so (5.31) in combination with (5.32) entail that

$$\inf_{x\in[-K_1-1,K_1+1]} P_x(B_T \in [-K_1, K_1], B_s \in [-K_2, K_2]\, \forall s \le T) \ge \frac{5}{16}. \tag{5.33}$$

Next, assume that n is large enough, so that $\delta_1\varphi(n) > K_2/2$, and in particular ξ equals es on $[x_n + \delta_1\varphi(n) - K_2, x_n + \delta_1\varphi(n) + K_2]$. For $x \in [x_n + \delta_1\varphi(n) - 1, x_n + \delta_1\varphi(n) + 1]$, define

$$x' = \begin{cases} x_n + \delta_1\varphi(n) + K_1, & \text{if } x < x_n + \delta_1\varphi(n) + K_1, \\ x_n + 2\delta_1\varphi(n) - K_1, & \text{if } x > x_n + 2\delta_1\varphi(n) - K_1, \\ x, & \text{otherwise,} \end{cases} \tag{5.34}$$

and set $I_i = [x' - K_i, x' - K_i]$, $i = 1, 2$, so that $I_1 \subset I_2$.

We now consider the BBMRE started at x and for $k \geq 1$ we define

$$Z_k = |\{Y \in N(kT) : Y_{lT} \in I_1 \, \forall 1 \leq l \leq K, Y_s \in I_2 \, \forall s < kT\}|. \tag{5.35}$$

Z_k can be interpreted as the number of particles in the k-th generation of a multi-type branching process; here, the type corresponds to the position of the particle in I_1 at which it is born (with exception of the initial particle which is at most at distance 1 from I_1), and where the number of offspring of a particle of type y is distributed as $|\{Y \in N(T) : Y_T \in I_1, Y_s \in I_2 \, \forall s \leq T\}|$ under $\mathtt{P}^{\mathtt{es}}_y$. In particular, using the Feynman-Kac formula as well as (5.33) and then (5.30), the expected offspring number of a particle of type y satisfies

$$\begin{aligned} &\mathtt{E}^{\mathtt{es}}_y[|\{Y \in N(T) : Y_T \in I_1, Y_s \in I_2 \, \forall s \leq T\}|] \\ &\quad = e^{\mathtt{es}\, T} P_y(B_T \in I_1, B_s \in I_2 \, \forall s < T) \geq \frac{5}{16} e^{\mathtt{es}\, T} \geq e^{(1-\frac{\delta_2}{2})\mathtt{es}\, T}, \end{aligned} \tag{5.36}$$

uniformly over all admissible types y. In addition, the second moment of the same quantity is finite, again uniformly over all admissible types, by comparison with branching process with branching rate $\mathtt{es}$. It thus follows by the standard results on multi-type branching processes that for some $\rho \geq e^{(1-\frac{\delta_2}{2})\mathtt{es}\, T}$ finite, Z_k/ρ^k converges in distribution to a non-negative random variable W with $P(W > 0) > 0$ (see e.g. [Har63, Theorem 14.1], where ρ is the principal eigenvalue of the expectation operator of the multi-type branching process; observe also that Condition 10.1 of this theorem is easily checked for V being the Lebesgue measure). In particular, one can find $\varepsilon_2 > 0$ and k_0 large such that

$$\mathtt{P}^{\mathtt{es}}_x\big(Z_k \geq \varepsilon_2 e^{(1-\frac{\delta_2}{2})\mathtt{es}\, kT}\big) \geq \mathtt{P}^{\mathtt{es}}_x(Z_k \geq \varepsilon_2 \rho^k) \geq \varepsilon_2 \qquad \text{for all } k \geq k_0, \tag{5.37}$$

uniformly in $x \in [x_n + \delta_1\varphi(n) - 1, x_n + \delta_1\varphi(n) + 1]$. This terminates the investigation of the second phase of comparison with BRW, and we may now proceed to the proof of Claim 5.6.

To this end, fix K such that $(1-\varepsilon_2)^K < \varepsilon/4$ and set (for κ and t_1 from (5.29))

$$t' = \inf\{s \in [t_1, t], \kappa^s > K \vee (4/\varepsilon), t - s = kT \text{ for some } k \in \mathbb{N}\}. \tag{5.38}$$

Observe that there is $c < \infty$ such that $t' < c$ for all $t \geq c$. Setting $\mathcal{N} = \{Y \in N(t') : Y_{t'} \in [r-1, r+1]\}$, we have, using (5.29) and (5.38) for the last inequality, that

$$\begin{aligned} &\mathtt{P}^{\xi}_r\Big(|\{Y \in N_r(t) : Y_t \in [x_n + \delta_1\varphi(n), x_n + 2\delta_1\varphi(n)]\}| \leq e^{(1-\delta_2)\mathtt{es}\, t}\Big) \\ &\quad \leq \mathtt{P}^{\xi}_r(|\mathcal{N}| < \kappa^{t'}) + \mathtt{P}^{\xi}_r(\{|\mathcal{N}| \geq \kappa^{t'}\} \cap \mathcal{A}) \leq \frac{\varepsilon}{4} + \mathtt{P}^{\xi}_r(\{|\mathcal{N}| \geq \kappa^{t'}\} \cap \mathcal{A}), \end{aligned} \tag{5.39}$$

where $\mathcal{A}$ denotes the event that each particle in $\mathcal{N}$ produces less than $e^{(1-\delta_2)\mathtt{es}\, t}$ particles in $[x_n + \delta_1\varphi(n), x_n + 2\delta_1\varphi(n)]$ at time t. For a particle at position $x \in [r-1, r+1]$, we then fix the intervals I_1, I_2 as above, and observe that the number of its children in $[x_n + \delta_1\varphi(n), x_n + 2\delta_1\varphi(n)]$ at time $t - t' =: k_t T$ dominates Z_{k_t} under $\mathtt{P}^{\mathtt{es}}_x$. Since the offspring of different particles are independent, for t large enough such that $e^{(1-\delta_2)\mathtt{es}\, t} \leq \varepsilon_2 e^{(1-\frac{\delta_2}{2})\mathtt{es}\, k_t T}$, we obtain

$$\begin{aligned} \mathtt{P}^{\xi}_r(\{|\mathcal{N}| \geq \kappa^{t'}\} \cap \mathcal{A}) &\leq \mathtt{E}^{\xi}_r\Big[\prod_{Y \in \mathcal{N}} \mathtt{P}_{Y_{t'}}\big(Z_{k_t} \leq e^{(1-\delta_2)\mathtt{es}\, t}\big);\, |\mathcal{N}| \geq \kappa^{t'}\Big] \\ &\leq \mathtt{E}^{\xi}_r\Big[\prod_{Y \in \mathcal{N}} \mathtt{P}_{Y_{t'}}\big(Z_{k_t} \leq \varepsilon_2 e^{(1-\frac{\delta_2}{2})\mathtt{es}\, k_t T}\big);\, |\mathcal{N}| \geq \kappa^{t'}\Big] \\ &\leq \mathtt{E}^{\xi}_r\Big[(1-\varepsilon_2)^{|\mathcal{N}|};\, |\mathcal{N}| \geq \kappa^{t'}\Big] \leq (1-\varepsilon_2)^{\kappa^{t'}} \leq (1-\varepsilon_2)^K \leq \frac{\varepsilon}{4}, \end{aligned} \tag{5.40}$$

where for the third inequality we used (5.37) and for the last two inequalities we applied (5.38). Combining (5.39) with the last display completes the proof of the claim. □

5.3 Non-monotonicity of the solution to randomized F-KPP equation

In this section we prove Theorem 2.4. Its proof is based on the simple idea that if there are two adjacent long stretches, the left one with potential $\mathtt{ei}$ and the right one with $\mathtt{es}$, where the values of w are comparable at some time t_n, as proved in Theorem 2.3, then at some later time $t_n + s$ the function w must be non-monotone, since it grows faster on the right stretch.

Proof of Theorem 2.4. For every $\varepsilon > 0$ we choose $K = K(\varepsilon)$ such that

$$f(K) := e^{\mathtt{es}} P_0\Big(\sup_{0\le u\le 1} |B_u| > K\Big) \le \varepsilon. \tag{5.41}$$

Recall that by Proposition 5.4, the definition of the coupling $\mathtt{Q}^{\xi}_{l,r}$ and the representation $w(t,x) = \mathtt{P}^{\xi}_x(N^{\le}(t,0) \neq \emptyset)$ of the solution to (F-KPP) (see Proposition 5.1), for $\delta \in (0,1)$ there exist l_n, r_n, t_n such that $t_n \to \infty$, $w(t_n,l_n) = \delta$, $r_n - l_n \underset{n\to\infty}{\longrightarrow} \infty$ and such that for all n large enough

$$\sup_{l\in[l_n-K,l_n+K]} w(t_n,l) \le \inf_{r\in[r_n-K,r_n+K]} w(t_n,r) + \varepsilon \tag{5.42}$$

holds. We will prove the result by contradiction and therefore assume for the time being that the claim of the theorem does not hold. Then, for all $\varepsilon > 0$, all n large enough and all $s \in [0,1]$, we have

$$\inf_{l\in[l_n-K,l_n+K]} w(t_n+s,l) \ge \sup_{r\in[r_n-K,r_n+K]} w(t_n+s,r) - \varepsilon. \tag{5.43}$$

Let us choose $\varepsilon \in (0,\delta)$, $s' \in (0,1]$ small enough and $b \in (0,1)$ such that for all $s \in [0,s']$,

$$e^{\mathtt{es}\, s}(\delta + 3\varepsilon) \le b. \tag{5.44}$$

Recall that the solution can be represented by the Feynman-Kac formula (3.2) with some $F : [0,1] \to [0,1]$ fulfilling (PROB) for some sequence (p_k) fulfilling (2.1). Let us abbreviate $c(w) := \frac{F(w)}{w}$, $w \in (0,1]$. It is easy to see that c is strictly decreasing, can be extended continuously to $w = 0$, i.e. $c(0) = \lim_{w\downarrow 0} c(w) = \sup_{w\in(0,1]} c(w) = 1$, $c(1) = 0$ and the function $c : [0,1] \to [0,1]$ is Lipschitz continuous with Lipschitz constant $H \in (0,\infty)$. Among others, due to (5.43) and $w \in [0,1]$, for all $s \in [0,1]$ we have

$$\sup_{l\in[l_n-K,l_n+K]} c\big(w(t_n+s,l)\big) \le \inf_{r\in[r_n-K,r_n+K]} c\big(w(t_n+s,r)\big) + H\varepsilon. \tag{5.45}$$

Furthermore, by the Feynman-Kac formula (3.2) and the Markov property, for all $s \ge 0$ we have

$$w(t_n+s,l_n) = E_{l_n}\Big[\exp\Big\{\int_0^s \xi(B_u) c(w(t_n+s-u,B_u))\,\mathrm{d}u\Big\} w(t_n,B_s)\Big].$$

Then due to $\xi \le \mathtt{es}$, $w \in [0,1]$, $c \le 1$, (5.42), (5.43), (5.41), and (5.44), for all n large enough we have for all $s \in [0,s']$ that

$$w(t_n+s,l_n) \le e^{\mathtt{es}\, s}\Big(P_{l_n}\big(\sup_{0\le u\le 1} |B_u - l_n| > K\big) + \sup_{l\in[l_n-K,l_n+K]} w(t_n,l)\Big) \le b. \tag{5.46}$$

Furthermore, using $\xi \le \mathtt{es}$, $w \in [0,1]$ and $c(w) \in [0,1]$ for $w \in [0,1]$ we get that for all $s \in [0,1]$ we have

$$\begin{aligned} w(t_n+s,l_n) \le\; & E_{l_n}\Big[\exp\Big\{\int_0^s \xi(B_u) c(w(t_n+s-u,B_u))\,\mathrm{d}u\Big\} w(t_n,B_s);\ \sup_{0\le u\le 1} |B_u - l_n| \le K\Big] \\ & + e^{\mathtt{es}} P_0\Big(\sup_{0\le u\le 1} |B_u| > K\Big). \end{aligned}$$

To bound the first summand, we recall (by definition of l_n, r_n) that $\xi(l) = \mathsf{ei}$ for all $l \in [l_n - K, l_n + K]$ and $\xi(r) = \mathsf{es}$ for all $r \in [r_n - K, r_n + K]$. Using (5.42) and (5.45), we see that the first summand can be bounded from above by

$$\begin{aligned}
&E_{l_n}\Big[\exp\Big\{\frac{\mathsf{ei}}{\mathsf{es}}\int_0^s \xi(B_u - l_n + r_n)\big(c(w(t_n+s-u, B_u - l_n + r_n)) + H\varepsilon\big)\,\mathrm{d}u\Big\} \\
&\qquad\qquad \times (w(t_n, B_s - l_n + r_n) + \varepsilon);\ \sup_{0\le u\le 1}|B_u - l_n| \le K\Big] \\
&= e^{\mathsf{ei}H\,\varepsilon s}E_{r_n}\Big[\exp\Big\{\frac{\mathsf{ei}}{\mathsf{es}}\int_0^s \xi(B_u)c(w(t_n+s-u, B_u))\,\mathrm{d}u\Big\}(w(t_n, B_s)+\varepsilon);\ \sup_{0\le u\le 1}|B_u - r_n| \le K\Big].
\end{aligned}$$

Recall the inequality $e^{ax} \le e^x - (1-a)x$ for all $a \in [0,1]$ and $x \ge 0$. Then, since $\frac{\mathsf{ei}}{\mathsf{es}} \in (0,1)$, we get

$$\begin{aligned}
w(t_n+s, l_n) \le f(K) + e^{\mathsf{ei}H\,\varepsilon s}\Big(&\varepsilon e^{\mathsf{eis}} + E_{r_n}\Big[\exp\Big\{\int_0^s \xi(B_u)c(w(t_n+s-u,B_u))\,\mathrm{d}u\Big\}w(t_n, B_s)\Big] \\
&-(1-\mathsf{ei}/\mathsf{es})E_{r_n}\Big[\int_0^s \xi(B_u)c(w(t_n+s-u,B_u))\,\mathrm{d}u\ w(t_n,B_s);\ \sup_{0\le u\le 1}|B_u - r_n|\le K\Big]\Big).
\end{aligned} \tag{5.47}$$

Recalling (5.42), we also have $\inf_{r\in[r_n-K, r_n+K]} w(t_n, r) \ge \delta - \varepsilon$. Furthermore, using the properties of c, for ε small enough such that $\varepsilon + b < 1$, we have that $\underline{c} = \underline{c}(\varepsilon, b) := \inf_{v\in[0,b+\varepsilon]} c(v) > 0$. Using (5.41), $\xi \ge \mathsf{ei}$, (5.43), (5.46), the inequality $e^x \le 1 + 2x$ for $x \ge 0$ small enough, and $w \in [0,1]$, we get, choosing $s = s'$ from (5.44) and continuing the bound from (5.47),

$$\begin{aligned}
w(t_n+s', l_n) &\le \varepsilon(1 + e^{(1+H\varepsilon)\mathsf{eis}'}) + (1 + 2H\varepsilon\mathsf{eis}')w(t_n+s', r_n) - (1-\mathsf{ei}/\mathsf{es})\,\mathsf{ei}\,\underline{c}\,(\delta-\varepsilon)(1-\varepsilon)s' \\
&\le w(t_n+s', r_n) + \varepsilon(1 + 2\mathsf{ei}(1+2H\varepsilon)) - (1-\mathsf{ei}/\mathsf{es})\,\mathsf{ei}\,\underline{c}\,(\delta-\varepsilon)(1-\varepsilon)s'
\end{aligned}$$

and the right-hand side can made smaller than $w(t_n+s', r_n) - 2\varepsilon$ if we choose s' (say) of order $\sqrt{\varepsilon}$ and ε small enough. But this is a contradiction to (5.43), which hence proves Theorem 2.4. □

A Appendix: Further auxiliary results

We collect here a couple of results needed primarily for the proof of Lemma 4.1, and start with several lemmas concerning the logarithmic moment generating functions defined in (4.1) as well as related objects. They are proved in [DS20] and are modifications of the corresponding discrete-space statements proved in [ČD20].

Lemma A.1 ([DS20, Lemma A.1]). *We recall that $P_x^{\zeta,\eta}$ has been defined in (4.5).*

(a) The functions L, L_x^ζ, and $\overline{L}_x^\zeta$, for $x \in \mathbb{R}$, defined in (4.1), are infinitely differentiable on $(-\infty, 0)$. Furthermore, for all $\eta < 0$ we have

$$(L_x^\zeta)'(\eta) = \frac{E_x\Big[e^{\int_0^{H_{\lceil x\rceil-1}}(\zeta(B_r)+\eta)\,\mathrm{d}r}H_{\lceil x\rceil -1}\Big]}{E_x\Big[e^{\int_0^{H_{\lceil x\rceil-1}}(\zeta(B_r)+\eta)\,\mathrm{d}r}\Big]} = E_x^{\zeta,\eta}[\tau_{\lceil x\rceil-1}], \quad x\in\mathbb{R}, \tag{A.1}$$

$$(\overline{L}_x^\zeta)'(\eta) = \frac{1}{x}E_x^{\zeta,\eta}[H_0], \quad x>0, \tag{A.2}$$

$$L'(\eta) = \mathbb{E}\Big[\frac{E_1\big[e^{\int_0^{H_0}(\zeta(B_r)+\eta)\,\mathrm{d}r}H_0\big]}{E_1\big[e^{\int_0^{H_0}(\zeta(B_r)+\eta)\,\mathrm{d}r}\big]}\Big] = \mathbb{E}\big[E_1^{\zeta,\eta}[H_0]\big], \tag{A.3}$$

and

$$(L_x^\zeta)''(\eta) = E_x^{\zeta,\eta}[\tau_{\lceil x\rceil-1}^2] - \big(E_x^{\zeta,\eta}[\tau_{\lceil x\rceil-1}]\big)^2 = \mathrm{Var}_x^{\zeta,\eta}(\tau_{\lceil x\rceil-1}) > 0, \quad x\in\mathbb{R}, \tag{A.4}$$

$$(\overline{L}_x^\zeta)''(\eta) = \frac{1}{x}\mathrm{Var}_x^{\zeta,\eta}(H_0), \quad x>0, \tag{A.5}$$

$$L''(\eta) = \mathbb{E}\Big[E_1^{\zeta,\eta}[H_0^2] - \big(E_1^{\zeta,\eta}[H_0]\big)^2\Big] = \mathbb{E}\big[\mathrm{Var}_1^{\zeta,\eta}(H_0)\big] > 0. \tag{A.6}$$

(b) *For each compact interval $\triangle \subset (-\infty, 0)$, there exists a constant $C_3 = C_3(\triangle) > 1$, such that the following inequalities hold $\mathbb{P}$-a.s.:*

$$-C_3 \le \inf_{\eta\in\triangle, x\ge 1} \{L^{\zeta}_{\lfloor x\rfloor}(\eta), \overline{L}^{\zeta}_x(\eta), L(\eta)\} \le \sup_{\eta\in\triangle, x\ge 1} \{L^{\zeta}_{\lfloor x\rfloor}(\eta), \overline{L}^{\zeta}_x(\eta), L(\eta)\} \le -C_3^{-1},$$
$$C_3^{-1} \le \inf_{\eta\in\triangle, x\ge 1} \{(L^{\zeta}_{\lfloor x\rfloor})'(\eta), (\overline{L}^{\zeta}_x)'(\eta), L'(\eta)\} \le \sup_{\eta\in\triangle, x\ge 1} \{(L^{\zeta}_{\lfloor x\rfloor})'(\eta), (\overline{L}^{\zeta}_x)'(\eta), L'(\eta)\} \le C_3,$$
$$C_3^{-1} \le \inf_{\eta\in\triangle, x\ge 1} \{(L^{\zeta}_{\lfloor x\rfloor})''(\eta), (\overline{L}^{\zeta}_x)''(\eta), L''(\eta)\} \le \sup_{\eta\in\triangle, x\ge 1} \{(L^{\zeta}_{\lfloor x\rfloor})''(\eta), (\overline{L}^{\zeta}_x)''(\eta), L''(\eta)\} \le C_3.$$

Lemma A.2 ([DS20, Lemma 2.4]). *(a) The function $(-\infty, 0) \ni \eta \mapsto L(\eta)$ is infinitely differentiable and its derivative $L'(\eta)$ is positive and monotonically strictly increasing.*

(b) *We have $\mathbb{P}$-a.s. that*

$$\lim_{x\to\infty} \overline{L}^{\zeta}_x(\eta) = L(\eta) \quad \textit{for all } \eta < 0. \tag{A.7}$$

(c) *$L'(\eta) \downarrow 0$ as $\eta \downarrow -\infty$*

(d) *For every $v > v_c := \frac{1}{L'(0-)}$ (where $\frac{1}{+\infty} := 0$), which we call* critical velocity, *there exists a*

$$\textit{unique solution } \overline{\eta}(v) < 0 \textit{ to the equation } L'(\overline{\eta}(v)) = \frac{1}{v}. \tag{A.8}$$

$\overline{\eta}(v)$ can be characterized as the unique maximizer to $(-\infty, 0] \ni \eta \mapsto \frac{\eta}{v} - L(\eta)$, i.e.

$$\sup_{\eta\le 0} \Big(\frac{\eta}{v} - L(\eta)\Big) = \frac{\overline{\eta}(v)}{v} - L(\overline{\eta}(v)). \tag{A.9}$$

The function $(v_c, \infty) \ni v \mapsto \overline{\eta}(v)$ is continuously differentiable and strictly decreasing.

We now recall the well-known existence of the Lyapunov exponent for the solutions to (PAM).

Proposition A.3 ([DS20, Proposition A.3, Corollary 3.10]). *Assume* (BDD)–(PAM-INI). *For all $v \ge 0$ and all $u_0 \in \mathcal{I}_{PAM}$ the limit*

$$\Lambda(v) = \lim_{t\to\infty} \frac{1}{t} \ln u^{u_0}(t, vt) \tag{A.10}$$

exists $\mathbb{P}$-a.s., is non-random and independent of u_0. We have $\Lambda(0) = \mathsf{es}$, Λ is nondecreasing, linear on $[0, v_c]$, strictly concave on (v_c, ∞) and $\lim_{v\to\infty} \frac{\Lambda(v)}{v} = -\infty$. In particular, there exists a unique $v_0 > 0$ such that $\Lambda(v_0) = 0$. Furthermore, the convergence in (A.10) *holds uniformly on any compact interval $K \subset [0, \infty)$.*

Lemma A.4 ([DS20, Lemma 2.5 (b)]). *(a) For every $v > v_c$ there exists a finite random variable $\mathcal{N} = \mathcal{N}(v)$ such that for all $x \ge \mathcal{N}$ the solution $\eta^{\zeta}_x(v) < 0$ to $E^{\zeta,\eta^{\zeta}_x(v)}_x[H_0] = \frac{x}{v}$ exists.*

(b) *For each $q \in \mathbb{N}$ and each compact interval $V \subset (v_c, \infty)$, there exists $C_4 := C_4(V, q) \in (0, \infty)$ such that*

$$\mathbb{P}\Big(\sup_{v\in V} \sup_{x\in[n,n+1)} |\eta^{\zeta}_x(v) - \overline{\eta}(v)| \ge C_4\sqrt{\frac{\ln n}{n}}\Big) \le C_4 n^{-q} \qquad \textit{for all } n \in \mathbb{N}. \tag{A.11}$$

Lemma A.5 ([DS20, Lemma 2.7]). *There exists a constant $C_5 > 0$ such that $\mathbb{P}$-a.s., for all $x \in (0, \infty)$ large enough, uniformly in $v \in V$ and $0 \le h \le x$,*

$$|\eta^{\zeta}_x(v) - \eta^{\zeta}_{x+h}(v)| \le C_5 \frac{h}{x}. \tag{A.12}$$

In the final lemma we recall a Hoeffding-type inequality for mixing random variables, which is a consequence of [Rio17, Theorem 2.4].

Lemma A.6 ([DS20, Corollary A.5]). *Let $(Y_i)_{i\in\mathbb{Z}}$ be a sequence of real-valued bounded random variables, $\widetilde{\mathcal{F}}^k := \sigma(Y_j : j \geq k)$, and let $(m_1, \dots, m_n)$ be an n-tuple of positive real numbers such that for all $i \in \{1, \dots, n\}$,*

$$\sup_{j\in\{1,\dots,i\}} \Big(\|Y_i^2\|_\infty + 2\Big\| Y_i \sum_{k=j}^{i-1} \mathbb{E}[Y_k|\widetilde{\mathcal{F}}^i] \Big\|_\infty \Big) \leq m_i,$$

with the convention $\sum_{k=i}^{i-1} \mathbb{E}[Y_k|\widetilde{\mathcal{F}}^i] = 0$. Then for every $x > 0$,

$$\mathbb{P}\Big(\Big| \sum_{i=1}^n Y_i \Big| \geq x \Big) \leq \sqrt{e} \exp\Big\{ -x^2/(2m_1 + \cdots + 2m_n) \Big\}.$$

B Appendix: Non-triviality of the regime of validity

The next lemma is used to show that there are potentials ξ that simultaneously satisfy the assumptions of Theorem 2.2 as well as of Theorems 2.3 and 2.4.

Lemma B.1. *Let ξ be the potential constructed in (5.3) for real numbers* es *and* ei *satisfying* $0 < \mathsf{ei} < \mathsf{es}$ *(with (5.2) not necessarily fulfilled). Then, making the dependence of L explicit in writing $L = L_\xi$, we have that the family of real numbers $\frac{1}{L'_{C\xi}(0-)}$, $C \in [1, \infty)$, is upper bounded away from infinity.*

Proof. Equation (A.3) and monotone convergence entail that for all $C \in [1, \infty)$ we have

$$L'_{C\xi}(0-) = \mathbb{E}\Bigg[\frac{E_1\big[e^{C\int_0^{H_0}(\xi(B_r)-\mathsf{es})\,\mathrm{d}r} H_0\big]}{E_1\big[e^{C\int_0^{H_0}(\xi(B_r)-\mathsf{es})\,\mathrm{d}r}\big]} \Bigg].$$

Since the expectation in the denominator on the right-hand side of the previous display is $\mathbb{P}$-a.s. upper bounded by 1, we can continue the above to infer that for some positive constant $c > 0$ and all $C \in [1, \infty)$ we have

$$\begin{aligned} L'_{C\xi}(0-) &\geq \mathbb{E}\Big[E_1\big[e^{C\int_0^{H_0}(\xi(B_r)-\mathsf{es})\,\mathrm{d}r} H_0\big] \cdot \mathbb{1}_{\{\xi(x)=\mathsf{es}\,\forall x\in[0,2]\}} \Big] \\ &\geq \mathbb{E}\Big[E_1[H_0 \cdot \mathbb{1}_{\{B_r\in[0,2]\,\forall r\in[0,H_0]\}}] \cdot \mathbb{1}_{\{\xi(x)=\mathsf{es}\,\forall x\in[0,2]\}} \Big] \geq c > 0, \end{aligned}$$

which finishes the proof of the lemma. □

Proposition B.2. *There exist potentials ξ that satisfy the assumptions of Theorem 2.2 as well as of Theorems 2.3 and 2.4.*

Proof. It is sufficient to find a potential ξ as in (5.3), under the sole assumption $\mathsf{es}/\mathsf{ei} > 2$ of (5.2), such that at the same time (VEL) holds true for the respective potential.

For this purpose, we choose an arbitrary potential ξ as in (5.3) satisfying (5.2) We then infer that for such a potential and $C \in (0, \infty)$ large enough, one has—making explicit the dependence of the respective quantities on the potential—that $v_0(C\xi) > v_c(C\xi)$. Indeed, note that Lemma A.2 entails $v_c(\xi) = \frac{1}{L'(0-)}$, and Lemma B.1 implies that $\frac{1}{L'(0-)}$ is upper bounded away from infinity for the potentials $C\xi$ as $C \to \infty$. Regarding v_0, a comparison with the constant potentials $C\mathsf{ei}$ yields that $v_0(C\xi) \to \infty$ as $C \to \infty$, so (VEL) holds true for all C large enough, which is sufficient for the assumptions of Theorem 2.2 to be fulfilled.

At the same time, the potential $C\xi$ still satisfies (5.2) and hence fulfills the assumptions of Theorems 2.3 and 2.4. □

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