Sheet 1

Exercises for September 27

Level A exercises are not difficult and should be understood by everybody. Level B exercises are more difficult but doable. Level C exercises are optional, solving them is a bonus for the final exam. *Please hand in only Question 1 this week.*

Question 1. Normal random variables (Level A, 5 pts)

(a) Let $X \sim \mathcal{N}(0,1)$, $m, \sigma \in \mathbb{R}$. Show that $Y = m + \sigma X \sim \mathcal{N}(m, \sigma^2)$. *Hint:* Use characteristic functions.

(b) Let $X \sim \mathcal{N}(0,1)$. Show that its moments satisfy, for $n \in \mathbb{N}$,

$$E[X^{2n}] = (2n)!/(2^n n!), \qquad E[X^{2n+1}] = 0.$$

Hint: Recall the connections between moments and derivatives of characteristic functions; or use the induction and integration by parts.

(c) Let $X_i \sim \mathcal{N}(m_i, \sigma_i^2)$, $i = 1, \dots, n$, be independent random variables. Show that $\sum_{i=1}^n X_i \sim \mathcal{N}\Big(\sum_{i=1}^n m_i, \sum_{i=1}^n \sigma_i^2 \Big)$.

(d) Let $X = (X_1, X_2)$ be a Gaussian vector. Show that X_1 and X_2 are independent iff $\mathrm{Cov}(X_1, X_2) = 0$. *Hint:* You may use the form of density of X proved in the lecture

(e) In (d), it is necessary that X_1, X_2 form a Gaussian vector. To see this consider $Y \sim \mathcal{N}(0,1)$ and Z Bernoulli with $P(Z = \pm 1) = 1/2$ independent. Take $X_1 = Y$, $X_2 = YZ$ and prove that $X_1 \sim \mathcal{N}(0,1)$, $X_2 \sim \mathcal{N}(0,1)$, $\mathrm{Cov}(X_1, X_2) = 0$ but X_1, X_2 are not independent. In particular show that (X_1, X_2) is not a Gaussian vector.

Question 2. Convergence of normal random variables (Level B, 4 pts)
Let $X_n \sim \mathcal{N}(m_n, \sigma_n^2)$ be a sequence of random variables which converges in distribution to a random variable X. Show that

(a) X is also a normal random variable, $X \sim N(m, \sigma^2)$, $m = \lim_{n\to\infty} m_n$, $\sigma^2 = \lim_{n\to\infty} \sigma_n^2$ *Hint:* Recall that convergence in distribution implies the convergence of characteristic function. Use this convergence to deduce that the above limits exist.

(b) if X_n converge in probability, then they converge also in L^p, $p \in [1, \infty)$.

Hint. This is Lemma (2.4), in the handwritten notes, but try to do the proof yourself, or at least really understand its details.

Question 3. Kolmogorov extension theorem (Level B, 0 pts)
Kolmogorov extension theorem is a basic tool to construct stochastic processes. Read this theorem in your favourite book on probability or measure theory (e.g.: Billingsley: Probability and measure, Ch. 36, or for a weaker version Durrett: Probability Theory and Examples, Ch. A.3). Reading the proof is optional (since it requires some non-trivial measure theory), but try to understand the statement.