Sheet 3

Exercises for October 11

Level A exercises are not difficult and should be understood by everybody. Level B exercises are more difficult but doable. Level C exercises are optional, solving them is a bonus for the final exam.

Question 8. Applications of Blumenthal 0-1 law (Level A, 4 pts)

(a) Let $(B_t)_{t\geq 0}$ be a Brownian motion in $d = 1$, and let t_n be a sequence of strictly positive times with $t_n \to 0$ as $n \to \infty$. Show that $\limsup_{n\to\infty} B_{t_n}/\sqrt{t_n} = -\liminf B(t_n)/\sqrt{t_n} = +\infty$.

(b) Let now $(B_t)_{t\geq 0}$ be a Brownian motion in $d \geq 2$, K be an open cone in $\mathbb{R}^d$ (i.e. $K = \{\lambda x : x \in O, \lambda > 0\}$ for some open set $O \subset \mathbb{R}^d$), and $\tilde{H}_K = \inf\{t > 0 : B_t \in K\}$. Show that $P[\tilde{H}_K = 0] = 1$.

Question 9. Continuity of time inversion (Level B, 2 pts)
Let $(B_t)_{t\geq 0}$ be a Brownian motion. Show that its time inversion defined by $\beta_t = tB_{1/t}$ for $t > 0$ and $\beta_0 = 0$ is a Brownian motion as well. In particular, show that its trajectories are continuous at time $t = 0$.

Question 10. Brownian Bridge (Level B, 4 pts)
Let $(B_t)_{t\geq 0}$ be a Brownian motion and set $W_t = B_t - tB_1$, $t \in [0, 1]$.

(a) Show that $(W_t)_{t\in[0,1]}$ is a centred Gaussian process and give its covariance.

(b) For $0 < t_1 < \cdots < t_n < 1$, show that the vector $(W_{t_1}, \ldots, W_{t_n})$ has a density

$$f(x_1, \ldots, x_n) = \sqrt{2\pi}p_{t_1}(x_1)p_{t_2-t_1}(x_2 - x_1)\ldots p_{t_n-t_{n-1}}(x_n - x_{n-1})p_{1-t_n}(-x_n),$$

where $p_t(x) = \frac{1}{t\sqrt{2\pi}}e^{-x^2/(2t)}$.

(c) Try to explain why the law of $(W_{t_1}, \ldots, W_{t_n})$ can be viewed as the conditional law of $(B_{t_1}, \ldots, B_{t_n})$ knowing that $B_1 = 0$. *Hint:* Consult a definition of conditional density.

Question 11. Lévy-Ciesielski construction of BM (Level C, 4 pts)
Read the Lévy-Ciesielski construction of BM on page 4/5 of handwritten lecture notes. This is an alternative construction of BM, not relying on the use of Kolmogorov extension theorem.

Show the uniform convergence stated in the last paragraph on the page. In order to prove it, observe first that for a normal r.v. ξ, $P(|\xi| \geq a) \leq e^{-a^2/2}$ for any $a > 1$. Use this inequality to bound the probability of the event

$$\{\sup\{\xi_{m,k} : 0 \leq k \leq n_0 2^m - 1\} > 2^{m/4}\}$$

for every $m \geq 1$. Use then Borel-Cantelli to deduce the stated uniform convergence.