Sheet 6

Exercises for November 1

In this sheet the stopping theorem for martingales is applied to compute various quantities related to Brownian motion.

Question 20. Hitting probabilities (Level A, 3 pts)

Let B be a Brownian motion started from 0 and a < 0 < b. With H_a and H_b as usual, show that

$$P[H_a < H_b] = \frac{b}{b-a}, \qquad P[H_b < H_a] = \frac{-a}{b-a}.$$

Hint. Consider the stopping time $T = H_a \wedge H_b$ and observe that $M_t = B_{T \wedge t}$ is UI martingale. Apply the stopping theorem on this martingale.

Question 21. First moment of exit time (Level B, 3 pts)

For a > 0, consider the stopping time $T_a = \inf\{s \ge 0 : |B_s| = a\}$. Show that $E[T_a] = a^2$.

Hint. Use the martingale $M_t = B_t^2 - t$, and apply stopping theorem in time $T_a \wedge t$.

Question 22. Laplace transform of hitting time (Level B, 3 pts)

For a > 0 show that for every $\lambda > 0$

$$E[e^{-\lambda H_a}] = e^{-a\sqrt{2\lambda}}.$$

Hint. Recall that $\exp(\lambda B_t - \lambda^2 t/2)$ is a martingale.