Sheet 7

Exercises for November 8

Question 23. Trivias about local martingales (Level A, 4 pts)
Show the following statements (cf. (4.37) in the notes):

(a) In the definition of a continuous local martingale starting from 0, one can replace "UI martingale" by "martingale".

(b) If M is a local martingale and T is a stopping time, then M^T is a local martingale (recall $M_t^T = M_{t\wedge T}$).

(c) If the sequence $(T_n)_{n\geq 0}$ of stopping times reduces a local martingale M and $(S_n)_{n\geq 0}$ is another sequence of stopping times with $S_n \to \infty$, then $(S_n \wedge T_n)_{n\geq 0}$ reduces M as well.

(d) The space of local martingales on a given filtered probability space is a vector space.

Question 24. "Gaussian martingales" (Level B, 3 pts)

(a) Assume that M is a continuous martingale with $M_0 = 0$ which is also a Gaussian process. Show that for every $s < t$, the increment $M_t - M_s$ is independent of $\sigma(M_u : u \leq s)$.

(b) For M as in (a), prove that there exists a (deterministic) continuous non-decreasing function $f : [0,\infty) \to [0,\infty)$ such that $\langle M\rangle_t = f(t)$ for every $t \geq 0$.

Hint: Recall the limit expression for $\langle M\rangle$ and use it to determine the value of $E[\langle M\rangle_t]$. Then show that $\langle M\rangle_t$ concentrates around this value.

Question 25. (Level B, 3 pts)

(a) Let M, N be two independent local martingales on $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, P)$. Show that $\langle M, N\rangle = 0$. (In particular, show that if $B = (B^1, \dots, B^d)$ is a BM in $\mathbb{R}^d$, then $\langle B^i, B^j\rangle_t = \delta_{ij} t$.)

(b) Show that the converse to (a) does not hold. To this end consider a BM B_t and a stopping time T and examine the processes B^T and $B - B^T$.

(c) (Level C) Let M, N be two continuous local martingales with $M_0 = N_0 = 0$ which are not identically zero (to exclude the trivial case). Assume that $\langle M, N\rangle^2 = \langle M\rangle\langle N\rangle$, and set $T_\varepsilon = \inf\{t \geq 0 : \langle M\rangle_t \vee \langle N\rangle_t > \varepsilon\}$. Show that $M = \gamma N$ for some random variable γ which is measurable with respect to $\cap_{\varepsilon>0}\mathcal{A}_{T_\varepsilon}$.

Question 26. (Level B, 3 pts)
Let M be a continuous local martingale with $M_0 = 0$. Show that

$$\Big\{\lim_{t\to\infty} M_t \text{ exists and is finite}\Big\} = \big\{\langle M\rangle_\infty < \infty\big\}.$$

Hint. Consider stopping times $T_n = \inf\{t : |M_t| \geq n\}$ and $S_n = \inf\{t : \langle M\rangle_t \geq n\}$ and use them to prove the first set is a sub-/super-set of the second.