Sheet 8

Exercises for November 15

Question 26. '...from the last week:"' (Level B, 3 pts)
Let M be a continuous local martingale with $M_0 = 0$. Show that

$$\Big\{ \lim_{t\to\infty} M_t \text{ exists and is finite} \Big\} = \big\{ \langle M \rangle_\infty < \infty \big\}.$$

Hint. Consider stopping times $T_n = \inf\{t : |M_t| \geq n\}$ and $S_n = \inf\{t : \langle M \rangle_t \geq n\}$ and use them to prove the first set is a sub-/super-set of the second.

Question 27. Properties of stochastic integrals (Level A, 3 pts)
Consider stochastic integral w.r.t. a semimartingale X. Prove the following natural properties (*Hint:* Use the properties of the stochastic integral w.r.t. a martingale in H^2 and of the integral w.r.t. a process of finite variation):

(a) $(H, X) \mapsto (H \cdot X)$ is bilinear.

(b) $H \cdot (K \cdot X) = (HK) \cdot X$, when H, K locally bounded.

(c) For every stopping time T, $H\mathbf{1}_{[0,T]} \cdot X = H \cdot X^T = (H \cdot X)^T$.

Question 28. DCT for stochastic integral (Level B, 3 pts)
Prove the following theorem: Let $X = X_0 + M + A$ be a continuous semimartingale, and $t \geq 0$. Let $(H^n)_{n\geq 1}$ and H be locally bounded progressive processes, and K a non-negative progressive process. Assume that the following properties hold a.s.:

(i) $H^n_s \xrightarrow{n\to\infty} H_s$ for every $s \geq 0$,

(ii) $|H^n_s| \leq K_s$ for every $n \geq 1$ and $s \in [0, t]$,

(iii) $\int_0^t (K_s)^2 \, \mathrm{d}\langle M \rangle_s < \infty$, and $\int_0^t K_s \, \mathrm{d}|A|_s < \infty$.

Then,

$$\int_0^t H^n_s \, \mathrm{d}X_s \xrightarrow{n\to\infty} \int_0^t H_s \, \mathrm{d}X_s.$$

Question 29. (Level B, 3 pts)
Let X be a continuous semimartingale and H, K locally bounded processes on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{A}_t), P)$ satisfying the usual conditions. Assume that A, B are two $\mathcal{A}_0$-measurable random variables. Prove that

$$\int_0^t (AH + BK)\,\mathrm{d}X = A\int_0^t H\,\mathrm{d}X + B\int_0^t K\,\mathrm{d}X.$$