Sheet 11

Exercises for December 6

Question 39. Squared Bessel process (Level A, 3 pts)
For $m \geq 0$ consider the SDE

$$\mathrm{d}X_t = 2\sqrt{X_t}\mathrm{d}B_t + m\,\mathrm{d}t.$$

Solution of this SDE is called m-dimensional squared Bessel process.

(a) To understand the name, consider a d-dimensional Brownian motion β, $d \geq 1$, and show that $X_t = |\beta_t|^2$ solves this equation with $m = d$

(b) Show that the process

$$M_t = \begin{cases} (X_t)^{1-\frac{m}{2}}, & \text{if } m \neq 2, \\ \log(X_t), & \text{if } m = 2 \end{cases}$$

is a continuous local martingale.

(c) For $a < x < b$ and $H_a = \inf\{s \geq 0 : X_s = a\}$ find $P[H_a < H_b]$ given $X_0 = x$.

Question 40. Ornstein-Uhlenbeck process (Level A, 4 pts)
For $\lambda > 0$ consider the SDE

$$\mathrm{d}X_t = \mathrm{d}B_t - \lambda X_t\,\mathrm{d}t,$$

where B_t is a Brownian motion.

(a) Show that this SDE is solved by

$$X_t = X_0 e^{-\lambda t} + \int_0^t e^{-\lambda(s-t)}\,\mathrm{d}B_s.$$

This solution is called Ornstein-Uhlenbeck process.

(b) Observe that the integrand of the above integral does not depend on B and use it to deduce that X_t is a Gaussian process which belongs to the Gaussian space generated by B.

(c) Compute the mean function and the covariance function of this process in the case when $X_0 = x_0$ and in the case when X_0 is distributed according to $\mathcal{N}(0, \frac{1}{2\lambda})$. What can you say about X in the latter case?

Question 41. Yamada-Watanabe uniqueness criterion (Level B–C, 4 pts) Consider one-dimensional SDE $\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t$ with $X_0 = x_0$. Assuming that

$$|b(x) - b(y)| \leq K|x - y| \qquad \text{and} \qquad |\sigma(x) - \sigma(y)| \leq K\sqrt{|x - y|},$$

for every $x, y \in \mathbb{R}$ and $K < \infty$, show that this SDE has a unique strong solution:

(a) Preparatory step: Let Z be a semimartingale with $\langle Z \rangle_t = \int_0^t h_s\,\mathrm{d}s$ for some h satisfying $0 \leq h_s \leq C|Z_s|$, with $C < \infty$. Show that

$$E\Big[\int_0^t |Z_s|^{-1}\mathbf{1}_{\{0<|Z_s|\leq 1\}}\,\mathrm{d}\langle Z \rangle_s\Big] \leq Ct < \infty,$$

and use this to deduce that

$$\lim_{n\to\infty} nE\Big[\int_0^t \mathbf{1}_{\{0<|Z_s|\leq 1/n\}}\,\mathrm{d}\langle Z \rangle_s\Big] = 0.$$

(b) For $n \in \mathbb{N}$, let $\phi_n : \mathbb{R} \to [0, \infty)$ be given by $\phi_n(0) = 2n$, $\phi_n(x) = 0$ for $|x| \geq 1/n$, ϕ_n is linear on $(0, 1/n)$ and $(-1/n, 0)$. Let F_n be the unique C^2 function on $\mathbb{R}$ such that $F_n(0) = F_n'(0) = 0$ and $F_n'' = \phi_n$. (Note that $F_n(x) \xrightarrow{n\to\infty} |x|$ and $F_n'(x) \xrightarrow{n\to\infty} \mathbf{1}_{x>0} - \mathbf{1}_{x<0} =: \operatorname{sign}(x)$.)

Let now X and X' be two solution to the above SDE on the same filtered probability space with the same Brownian motion B. Use (a) to show that

$$\lim_{n\to\infty} E\Big[\int_0^t \phi_n(X_s - X_s')\,\mathrm{d}\langle X - X' \rangle_s\Big] = 0.$$

(c) For a stopping time T making the semimartingale $X^T - X'^T$ bounded, apply Itô's formula to $F_n(X_t^T - X_t'^T)$ and show that

$$E[|X_t^T - X_t'^T|] = E\Big[\int_0^{T\wedge t} (b(X_s) - b(X_s'))\operatorname{sign}(X_s - X_s')\mathrm{d}s\Big].$$

(d) Use Gronwall's lemma to deduce the main claim of the exercise.