Sheet 12

Exercises for December 13

Question 42. Markov processes on a finite space (Level A, 5 pts)
Let $(Q_t)_{t\geq 0}$ be a transition semigroup on a finite space E. As remarked in the lecture, in this case Q_t might be identified with the $n\times n$ matrix $Q_t(i,j) = Q_t(i,\{j\})$. Also, $\mathcal{B}(E)$ and $C_0(E)$ coincide with $\mathbb{R}^{|E|}$.

(a) Prove that Q is Feller iff $\lim_{t\downarrow 0} Q_t(i,j) = \delta_{ij}$ for all $1 \leq i,j \leq |E|$. Assume from now on that Q is Feller.

(b) For the resolvent R_λ of this semigroup, show that $\lim_{\lambda\to\infty} \lambda R_\lambda(i,j) = \delta_{ij}$. Deduce that the range $\mathcal{R}$ of the resolvent is $\mathbb{R}^{|E|}$, and thus $t \mapsto Q_t(i,j)$ must be differentiable (you may freely use the theory developed in the lecture)

(c) Let $L(i,j) = \frac{\mathrm{d}}{\mathrm{d}t} Q_0(i,j)$ be the generator matrix of Q. Show that $L(i,j) \geq 0$ for $i \neq j$, and that $\sum_{j=1}^{|E|} L(i,j) = 0$.

(d) Show that $\frac{\mathrm{d}}{\mathrm{d}t} Q_t(i,j) = (LQ_t)(i,j)$ where LQ_t stands for matrix multiplication. Deduce that $Q_t = \mathrm{e}^{tL}$ (matrix exponential).

(e) Define now $\rho(i) = -L(i,i)$ and $P(i,j) = L(i,j)/\rho(i)$. Observe that P is a stochastic matrix. Let $(\xi_n)_{n\geq 0}$ be a discrete-time Markov chain on E with transition matrix P and given $(\xi_n)_{n\geq 0}$, let $(\tau_n)_{n\geq 0}$ be a sequence of (conditionally) independent random variables, τ_n having exponential distribution with parameter $\rho(\xi_n)$. Set $T_n = \sum_{i=0}^{n-1} \tau_i$ and define

$$X_t = \xi_n \qquad \text{for } T_n \leq t < T_{n+1}.$$

Show that $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} P[X_t = j | X_0 = i] = L(i,j)$ and deduce that X is a Markov process associated to the transition semigroup Q.

Question 43. Reflected Brownian motion (Level B, 5 pts)
Let $(B_s)_{s\geq 0}$ be a Brownian motion on some $(\Omega, \mathcal{A}, (\mathcal{A}_t), P)$ with $B_0 = a$, $a \geq 0$, and let $p_t(z) = (2\pi t)^{-1/2} \mathrm{e}^{-z^2/2t}$.

(a) Let $X_t = |B_t|$. Verify that for every $s, t \geq 0$ and every $f \in \mathcal{B}(\mathbb{R}_+)$

$$E[f(X_{t+s})|\mathcal{A}_s] = Q_t f(X_s),$$

where $Q_0 f = f$ and $Q_t f(x) = \int_0^\infty (p_t(x-y) + p_t(x+y)) f(y)\, \mathrm{d}y$.

(b) Infer that (Q_t) is a transition semigroup, and then that X_t is a Markov process associated to this semigroup.

(c) Verify that (Q_t) is Feller, and denote by L its generator.

(d) Show that $D(L) \subset \{f \in C_0(\mathbb{R}_+) : f'' \in C_0(\mathbb{R}_+) \text{ and } f'(0) = 0\}$. For such f, show that $Lf = \frac{1}{2} f''$.

Hint. Show that the function $g(y) = f(|y|)$ must be differentiable on $\mathbb{R}$.

(e) Show that if $f'(0) \neq 0$ then $f \notin D(L)$.

Question 44. Function of Markov process (Level B, 4 pts)
Let $(Q_t)_{t \geq 0}$ be a transition semigroup on E, and let π be a measurable mapping from E to some other measurable space F. Assume that for every measurable $A \subset F$, $t \geq 0$ and every $x, y \in E$ such that $\pi(x) = \pi(y)$ we have

$$Q_t(x, \pi^{-1}(A)) = Q_t(y, \pi^{-1}(A)),$$

and define, for $z \in F$ and an arbitrary $x \in \pi^{-1}(z)$.

$$Q'_t(z, A) = Q_t(x, \pi^{-1}(A))).$$

Assume that Q' satisfies all necessary measurability conditions.

(a) Show that Q' is a transition semigroup.

(b) If X is a Markov process associated to Q, show that $Y_t = \pi(X_t)$ is a Markov process associated to Q'.

(c) Let now B be a Brownian motion in $\mathbb{R}^d$, $d \geq 1$. Show that $Y_t = |B_t|$ is a Markov process and give its transition semigroup.