Sheet 1

Exercises for February 28

This series develops two important related concepts: coupling of two random variables and total variation distance. The results will be used in the lecture.

Definition. For random variables $X_1, \dots, X_n$, a random vector $(\hat{X}_1, \dots, \hat{X}_n)$ is called coupling of $X_1, \dots, X_n$, if, for every $i \in [n]$, $\hat{X}_i$ has the same distribution as X_i.

Remark. The random variables X_i do not need to be defined on the same probability space. On the other hand, all $\hat{X}_i$'s must be. Since the distribution of the random vector $(\hat{X}_1, \dots, \hat{X}_n)$ is not uniquely determined by its marginal distribution, there are typically many different couplings of $X_1, \dots, X_n$.

Definition. For two measures μ, ν on the same measurable space $(\Omega, \mathcal{A})$, we define their total variation distance by

$$d_{TV}(\mu, \nu) = \sup_{A \in \mathcal{A}} |\mu(A) - \nu(A)|.$$

Question 1. Total variation distance (Level A, 3 pts)

(a) Show that d_{TV} is a metric on the space of measures on $(\Omega, \mathcal{A})$.

(b) Assume now that Ω is countable, $\mathcal{A} = \mathcal{P}(\Omega)$, and denote $p_x = \mu(\{x\})$, $q_x = \nu(\{x\})$, $x \in \Omega$. Show that

$$d_{TV}(\mu, \nu) = \frac{1}{2} \sum_{x \in \Omega} |p_x - q_x| = \sum_{x \in \Omega : p_x > q_x} p_x - q_x,$$

that is, in this case, d_{TV} is actually a (half of) ℓ^1-distance.

Question 2. Maximal coupling and variation distance (Level B, 4 pts)
Let X, Y be two random variables with distributions μ_X, μ_Y, and set $d_{TV}(X, Y) = d_{TV}(\mu_X, \mu_Y)$. Show that there exists a coupling $(\hat{X}, \hat{Y})$ of X with Y so that

$$P[\hat{X} \neq \hat{Y}] = d_{TV}(X, Y),$$

and for any other coupling $(\hat{X}, \hat{Y})$ of X and Y, one has

$$P[\hat{X} \neq \hat{Y}] \geq d_{TV}(X, Y).$$

Question 3. (Level B, 4 pts)
Let $(I_i)_{i=1,\dots,n}$, be independent Bernoulli random variables with respective parameters p_i, $\lambda = \sum_{i=1}^n p_i$, $X = \sum_{i=1}^n I_i$, and Y a Poisson(λ) random variable.

(a) Show that $d_{TV}(X, Y) \leq \sum_{i=1}^n (p_i)^2$.

(b) If X is Binomial(n, p) random variable and Y as above with $\lambda = np$, show that $d_{TV}(X, Y) \leq \lambda^2/n$.

Hint. Construct first a coupling of I_i with J_i, where J_i is Poisson(p_i), distributed, with the property that $P(\hat{I}_i \neq \hat{J}_i) \leq p_i^2$. Use that the Poisson distribution is (sum-)stable.