Sheet 2

Exercises for March 7

Question 4. Correction to lecture (Level A, 3 pts)
There was a mistake in the argument for the upper bound for the size of the maximal connected component during the lecture on February 26. I am sorry for this. Fortunately, this mistake can easily be corrected:

In the proof we remarked that (see below (3.18) on page 21 of the lecture notes)

$$P^{ER}_{n,p}[|\mathcal{C}_{max}| \geq a \log n] \leq P^{BP}_{n,p}[T \geq a \log n],$$

and followed by approximating the binomial branching process by Poisson branching process. This approximation cannot directly be used there.

Instead of this one recalls that $P^{BP}_{n,p}[T \geq k] = P[H_0 \geq k]$, where the random variable H_0 is defined by $H_0 = \inf\{k \geq 0 : \sum_{i=1}^k X_i - (k-1) = 0\}$, and X_i's are i.i.d. binomial(n, p). Hence

$$P^{BP}_{n,p}[T \geq k] \leq P[\sum_{i=1}^{k} X_i - (k-1) \geq 0] = P[\sum_{i=1}^{k} X_i \geq k-1].$$

To complete the proof of (3.17) in the notes, it remains to recall $p = \lambda/n$, set $k = a \log n$ with $a > 1/I_\lambda$ and show that the last probability is smaller than $n^{-1-\delta(\lambda,a)}$ with $\delta(\lambda, a) > 0$. This large deviation estimate is left for exercise.

Question 5. Binary branching process (Level B, 3 pts)
We consider the branching process whose offspring distribution is given by

$$P(X = 2) = p = 1 - P(X = 0), \qquad p \in [0, 1].$$

(a) Compute the extinction probability η in this case.

(b) Give a formula for the generating function of the total progeny T. Sketch this function for $p < 1/2$ and $p > 1/2$.

(c) Give an estimate on $\mathbb{P}[T \geq k]$ for k large in the case $p = 1/2$.

Hint. What is the process S_k in this case?

Question 6. Martingales and branching processes (Level B, 4 pts)
Consider a branching process with an arbitrary offspring distribution satisfying $E[X] = \mu < \infty$. Recall that Z_n denotes the size of nth generation. Show that

(a) $M_n = \mu^{-n} Z_n$ is a non-negative martingale w.r.t. filtration $(\mathcal{F}_n)_{n \geq 0}$, $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$.

(b) Use a suitable martingale convergence theorem to show that Z_n/μ_n converges a.s. to a finite random variable W_∞.

(c) Show that $W_\infty = 0$ if $\mu \leq 1$ and that $P(W_\infty = 0) \geq \eta$.

(d) Show that the generating function of W_∞ satisfies the functional equation

$$G_{W_\infty}(s) = G_X(G_{W_\infty}(s^{1/\mu})).$$

Remark. In general, it is not true that $P[W_\infty = 0] = \eta$. Actually, as proved by Kesten and Stigum (1966), this holds true iff $E[X \log X] < \infty$. In this case M_n is uniformly integrable martingale and thus also $E[W_\infty] = 1$.