Sheet 3

Exercises for March 21

Question 7. Extinction probability of Poisson BP (Level A, 3 pts)
Consider a branching process whose offspring distribution is Poisson with parameter λ. Recall that η_λ denotes its extinction probability.

(a) Show that for $\lambda > 1$

$$\frac{d}{d\lambda}\eta_\lambda = -\frac{\eta_\lambda(1-\eta_\lambda)}{1-\mu_\lambda},$$

where μ_λ is the conjugate exponent defined in the lecture (see (2.42) in the lecture notes).

(b) Show that

$$1-\eta_\lambda = 2(\lambda-1)(1+o(1)), \qquad \text{as } \lambda \downarrow 1.$$

In particular, η_λ is a continuous function of λ and its right derivative at $\lambda = 1$ equals (-2).

Hint. Recall formula (2.40) of the notes

Question 8. Factorial moments of sum of indicators (Level A, 2 pts)
Let $X = \sum_{i\in\mathcal{I}} I_i$, where $\mathcal{I}$ is a finite index set, and I_i are $\{0,1\}$-valued random variables. We write $(X_r) = X(X-1)\cdots(X-r+1)$, so that $E[(X)_r]$ is the rth *factorial moment* of X. Show that

$$E[(X)_r] = \sum_{i_1,i_2,\dots,i_r\in\mathcal{I}}^{*} P[I_{i_1} = \cdots = I_{i_r} = 1],$$

where $\sum^*$ denotes a sum over distinct indices.

Question 9. Triangles in ER graph (Level B, 4 pts)
Consider ER graph with $p = \lambda/n$, as in the lecture.

(a) What is the distribution of the number E_n of the edges of this graph?

(b) Let $m_n = \lambda n/2$. Prove that $m_n^{-1/2}(E_n - m_n)$ converges to a standard normal random variable as $n \to \infty$.

(c) We say that distinct vertices $x, y, z \in [n]$ form a triangle if edges xy, yz and zx are present in the graph. Note that (x, y, z) is the same triangle as e.g. (y, x, z). Let T_n be the number of distinct triangles in the graph. Compute ET_n

(d) Show that, as $n \to \infty$, T_n converges to a Poisson random variable.

Hint. You may use the following claim: A sequence X_n of $\mathbb{N}$-valued random variables converges in distribution to a Poisson(λ) random variable, iff

$$\lim_{n\to\infty} E[(X_n)_r] = \lambda^r, \qquad \text{for all } r \in \mathbb{N}.$$

Question 10. Thinning of binomials and neutral vertices (Level B, 4 pts)

(a) Let $N \sim \text{Bin}(n, p)$ and, conditionally on N, let $M \sim \text{Bin}(N, q)$. Show that $M \sim \text{Bin}(n, pq)$.

(b) Recall the exploration algorithm for the $\text{ER}(n, p)$ random graph from pages 17–18 of the notes. Use the previous claim and an induction argument to show that the number N_k of neutral vertices at the end of kth step of the algorithm satisfies $N_k \sim \text{Bin}(n - 1, (1 - p)^k)$.