Sheet 8

Exercises for May 2

Question 25. Diverging degree in PA (Level A, 3 pts)
Show that in $\mathsf{PA}_{m=1,\delta}$, for every $i \in \mathbb{N}$, $\lim_{t\to\infty} D_i(t) = \infty$ a.s. Try to find a direct proof not based on martingale argument of the lecture.

Question 26. Gamma function identities (Level A, 3 pts)
Recall that $t\Gamma(t) = \Gamma(t+1)$ and use it to show that

$$\frac{\Gamma(k+a)}{\Gamma(k+b)} = \frac{1}{b-a-1}\Big(\frac{\Gamma(k+a)}{\Gamma(k+b-1)} - \frac{\Gamma(k+a+1)}{\Gamma(k+b)}\Big).$$

Use this identity to show that p_k, $k \geq 1$, defined by

$$p_k = (2+\delta)\frac{\Gamma(k+\delta)\Gamma(3+2\delta)}{\Gamma(k+3+2\delta)\Gamma(1+\delta)},$$

is a probability distribution.

Question 27. Uniform recursive tree (Level B, 6 pts)
In uniform recursive tree, we attach each vertex to a uniformly chosen old vertex, that is, denoting $\mathsf{URT}(t)$ the uniform recursive tree after adding t vertices and using a similar notation as for the preferential attachment model,

$$P(v_{t+1} \to v_i | \mathsf{URT}(t)) = t^{-1} \quad \text{for all } i \in [t].$$

This can be viewed as $\delta \to \infty$ limit of $\mathsf{PA}_{m=1,\delta}$. Let $P_k(t) = t^{-1}\sum_{i=1}^t \mathbf{1}_{D_i(t)=k}$. Show that for some $C \in (0,\infty)$, the degree sequence of UST satisfies

$$P\big[\max_{k\geq 1} |P_k(t) - p_k| \geq C\sqrt{t^{-1}\log t}\big] \xrightarrow{t\to\infty} 0,$$

where $p_k = 2^{-k-1}$.