Sheet 9

Exercises for May 9

Question 28. Increasing coupling in percolation (Level A, 3 pts)
Construct, on one probability space, a family of random variables $X_p(e)$, $p \in [0,1]$, $e \in E^d$, so that

- the law of $(X_p(e))_{e\in E^d}$ is P_p,
- $X_p(e) \geq X_{p'}(e)$ for every $p \geq p'$ and $e \in E^d$.

Use it to give a formal proof of the following facts:

(a) $p \mapsto E_p(Y)$ is non-decreasing for every increasing random variable Y.

(b) $p \mapsto \theta(p)$ is non-decreasing.

Question 29. Properties of product σ-field (Level B-C, 2 pts)
Show that every event in the product σ-field $\mathcal{A}$ on $\{0,1\}^{E^d}$ can be approximated by events depending on finitely many edges only. More precisely, denoting B_n the collection of edges with both ends in the box $[-n,n]^d \cap \mathbb{Z}^d$, for every $A \in \mathcal{A}$ there is a sequence A_n of events such that A_n is $\sigma(\omega(e) : e \in B_n)$-measurable, and $\lim_{n\to\infty} P_p(A_n \Delta A) = 0$. (This question might be easy or difficult, depending on how much measure theory you know.)

Question 30. Bond and site percolation (Level A, 2 pts)
Recall from the lecture that we speak about 'site percolation' if vertices (instead of edges) of the graph are independently declared open or closed. Show that the bond percolation on $\mathbb{Z}^d$ corresponds to the site percolation on a modified graph (Z^d, E_1^d) (specify E_1^d). Use it to show that (with obvious notation)

$$p_c^{site}(\mathbb{Z}^d) \geq p_c^{bond}(\mathbb{Z}^d).$$

Question 31. . . . continued (Level B, 2 pts)
Show that

$$p_c^{site}(\mathbb{Z}^d) \leq 1 - (1 - p_c^{bond}(\mathbb{Z}^d))^{2d}.$$

Question 32. Square root trick (Level B, 2 pts)
Show that for n increasing events $A_1, \dots, A_n$

$$\max_{i \le n} P_p(A_i) \ge 1 - (1 - P_p(A_1 \cup \dots \cup A_n))^{1/n}.$$

Question 33. BK for non-local events (Level B, 3 pts)
In the lecture we proved the BK inequality only for events depending on finitely many edges. Prove that

$$P[x \leftrightarrow y \circ u \leftrightarrow v] \le P[x \leftrightarrow y] P[u \leftrightarrow v]$$

Remark that the above events do not depend on finitely many edges only. However, as, by definition, every path between x and y has a finite length, those can be approximated by local events.