Sheet 10

Exercises for May 16

The first two exercises are from the last week, since you have no time to discuss them on Thursday ...

Question 32. Square root trick (Level B, 2 pts)
Show that for n increasing events $A_1, \dots, A_n$

$$\max_{i \le n} P_p(A_i) \ge 1 - (1 - P_p(A_1 \cup \cdots \cup A_n))^{1/n}.$$

Question 33. BK for non-local events (Level B, 3 pts)
In the lecture we proved the BK inequality only for events depending on finitely many edges. Prove that

$$P[x \leftrightarrow y \circ u \leftrightarrow v] \le P[x \leftrightarrow y] P[u \leftrightarrow v]$$

Remark that the above events do not depend on finitely many edges only. However, as, by definition, every path between x and y has a finite length, those can be approximated by local events.

... and now few new questions:

Question 34. Uniqueness of infinite cluster (Level B, 5 pts)
Instead of $(\mathbb{Z}^d, E^d)$, consider the edge percolation on an infinite graph $G = (V, E)$ which is

- *transitive*, that is for every pair $x, y \in V$ there exists a graph automorphism[1] $\phi_{x,y}$ such that $\phi_{x,y}(x) = y$.
- *amenable*, that is $\inf_{A \subset V} \frac{|\partial A|}{|A|} = 0$.

[1] Graph automorphism $\phi : V \to V$ is a bijection preserving the graph structure, i.e. $(x, y) \in E$ iff $(\phi(x), \phi(y)) \in E$.

Let $N(\omega)$ be number of infinite clusters in the percolation configuration ω. Show that, as on $\mathbb{Z}^d$, for every $p \in [0,1]$, either $P_p[N=0]=1$ or $P_p[N=1]=1$.

Does the previous claim remain true when the graph is non-amenable? Think about G being regular tree.

Question 35. One more property of θ (Level A, 3 pts)
On $\mathbb{Z}^d$, show that $p \mapsto \theta(p)$ is strictly increasing for $p > p_c$.

Question 36. Differentiation formula (Level B, 3 pts)
Let A be an arbitrary event depending on states of edges in a finite set E only. Show that

$$\frac{\mathrm{d}}{\mathrm{d}p} P_p[A] = \frac{1}{p(1-p)} \sum_{e \in E} \mathrm{Cov}(\mathbf{1}_A, \omega_e).$$

Hint. Write $P_p[A] = \sum_\omega \mathbf{1}_A(\omega) p^{|\omega|}(1-p)^{|E|-|\omega|}$, with $|\omega| = \sum_{e \in E} \omega_e$.

Bonus question. Try to use it to prove Russo's formula.