# Percolation
# ETH Zürich, HS 09

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# Contents

CHAPTER 1

# Introduction

The intention of this course is to give an introduction to the percolation theory and prove some of its most important results. These notes contain the material discussed in the lectures only. Interested reader should consult, e.g., the books of Grimmett [**Gri99**] or Bollobás and Riordan [**BR06a**] and the references therein.

## 1. Bond percolation on $\mathbb{Z}^d$

Of central interest will be so called *bond percolation on* $\mathbb{Z}^d$, $d \geq 2$ mostly. This model is very simply stated. Namely, one considers $\mathbb{Z}^d$ and the edge set

$$E_d = \big\{\{x,y\} : x,y \in \mathbb{Z}^d \text{ with } |x-y| = 1\big\}, \tag{1.1}$$

so called set of *nearest-neighbour edges* in $\mathbb{Z}^d$. One fixes $0 \leq p \leq 1$, and declares each edge $e \in E_d$ to be open with probability $p$ or closed with probability $(1-p)$, in an i.i.d. fashion. That is we have

$$\begin{aligned} &\Omega = \{0,1\}^{E_d} \\ &\mathcal{A} = \text{`the canonical product } \sigma\text{-algebra}^1\text{'} \\ &\mathbb{P} = \mathbb{P}_p = \mu^{\otimes E_d} \text{ with } \mu \text{ Bernoulli law with success parameter } p. \end{aligned} \tag{1.2}$$

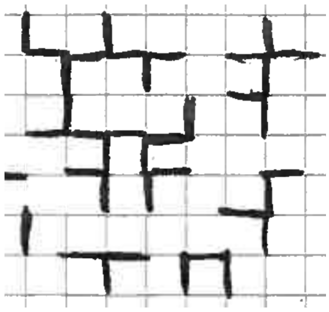

FIGURE 1.1. Illustration of a percolation configuration.

This model goes back to Broadbent and Hammersley [**BH57**]. They introduced it as a model of disordered porous medium trough which a fluid or gas was supposed to flow. Since then thousands of papers and many books have been devoted to the subject.

One of the most important question of the theory is: "Is there, for a $\mathbb{P}$-typical configuration, an infinite cluster of open bonds?[2]" We will see later that the existence of such infinite cluster has probability either 0 or 1, and this probability is 1 exactly when

$$\theta(p) = \mathbb{P}_p[|\mathcal{C}_0| = \infty] > 0, \tag{1.3}$$

[1] the $\sigma$-algebra generated by cylinder events $N_{\omega,A} = \{\omega' \in \Omega : \omega'(e) = \omega(e) \text{ for every } e \in A\}$, where $A \subset E_d$ finite and $\omega \in \Omega$.

[2] The standard terminology of percolation theory differs from that of graph theory: vertices and edges ale called sites and bonds, components are called clusters. We will use the both terminologies interchangeably.

where

$$\mathcal{C}_0 = \text{'the connected cluster of open bonds containing the origin'}. \tag{1.4}$$

The striking feature of the model is the existence of a *critical probability* $p_c \in (0,1)$ such that

$$\theta(p) \begin{cases} = 0, & \text{if } p < p_c, \\ > 0, & \text{if } p > p_c. \end{cases} \tag{1.5}$$

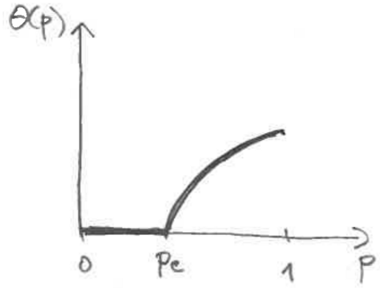


FIGURE 1.2. Believed shape of $\theta(p)$.

Hence, if $p$ is small there is no infinite cluster $\mathbb{P}$-a.s., but for $p$ large enough there is an infinite cluster $\mathbb{P}$-a.s. (as we will see later, this cluster is $\mathbb{P}$-a.s. unique).

The shape of the graph of $\theta$ as on Figure 1.2 is proved for $d = 2$ and $d \geq 19$ only. For $d$ between 3 and 19, we still miss the proof of the left-continuity of $\theta$ in the point $p = p_c$.

The exact value of $p_c$ is in general unknown. In the special case of $\mathbb{Z}^2$ Kesten [**Kes80**] showed that

$$p_c(d = 2) = \tfrac{1}{2}. \tag{1.6}$$

We will see the proof of this result later. There are other few two-dimensional lattices where $p_c$ is known exactly, cf. [**Gri99**, page 53]. For three- and more-dimensional lattices we have numerical estimates only.

Resolving the question of existence/non-existence of the infinite cluster, one is further interested to study the geometric properties of clusters in various *phases* of this model, namely in *sub-critical* ($p < p_c$) and *super-critical* ($p > p_c$) phase. Special interest has also the behaviour of the model near the criticality, that is $p$ is close or equal to $p_c$. In this lecture we will see proofs of some of the following results.

**Sub-critical phase ($p < p_c$).** In this phase $\mathbb{P}$-a.s. clusters are finite. One is interested in the tail of the distribution of the size of the cluster $\mathcal{C}_0$. We will show that

$$\mathbb{P}[|\mathcal{C}_0| = n] = \exp\{-n(\psi(p) + o(1))\} \qquad \text{as } n \to \infty, \tag{1.7}$$

where $\psi(p) = \psi_d(p) > 0$, when $p < p_c(d)$.

**Super-critical phase ($p > p_c$).** As mentioned above, an infinite cluster exists $\mathbb{P}$-a.s. in this phase. We will use methods of [**BK89**] to show that

$$\text{There is a } \textbf{unique} \text{ infinite cluster, } \mathbb{P}_p\text{-a.s.} \tag{1.8}$$

One is then mainly interested in the geometry of this cluster. We will see that the geometry of this cluster is "not far" from the geometry of the lattice $\mathbb{Z}^d$ itself: E.g. Grimmett and Marstrand [**GM90**] proved that if $p > p_c$, $d \geq 3$, then for $k$ large

$$\mathbb{P}\begin{bmatrix} \text{there is an infinite open cluster in the two-dimensional} \\ \text{slab } \{x \in \mathbb{Z}^d, -k \leq x_j \leq k, \text{ for } 3 \leq j \leq d\} \end{bmatrix} = 1. \tag{1.9}$$

As many of important result in the super-critical phase, this result is shown using renormalization techniques.

Another way how to see that the cluster 'looks like the full lattice' is the following. For a site $x$ in the cluster $\mathcal{C}_0$ define its *distance in the cluster to the origin* as

$$d(0,x) = \text{'minimal length of a path in } \mathcal{C}_0 \text{ connecting } x \text{ to 0'}. \tag{1.10}$$

We will show that

$$\mathbb{P}\text{-a.s., on } \{|\mathcal{C}_0| = \infty\}, \qquad \limsup_{x\to\infty, x\in\mathcal{C}_0} \frac{d(0,x)}{|x|} < \infty. \tag{1.11}$$

Since the inequality $\liminf_{x\to\infty} d(0,x)/|x| > 0$ is trivial, we find that, at large scales, the distance $d(\cdot,\cdot)$ is equivalent with the Euclidean one. This is result of Antal and Pisztora [**AP96**].

One might also be interested in the distribution of the size of $\mathcal{C}_0$ if it is finite. It can be proved that this distribution decays sub-exponentially. Namely,

$$\exp(-\beta_1 n^{(d-1)/d}) \le \mathbb{P}[|\mathcal{C}_0| = n] \le \exp(-\beta_2 n^{(d-1)/d}), \qquad \text{for all } n \ge 0. \tag{1.12}$$

(A rough idea how to see it is as follows. To separate the finite cluster $\mathcal{C}_0$ of volume $n$ from the rest of the lattice we should close all the bonds on the 'surface of $\mathcal{C}_0$'. The size of the surface of such cluster is typically volume$^{(d-1)/d} = n^{(d-1)/d}$, hence the cost to close all the bonds on the surface is very approximately $(1-p)^{n^{(d-1)/d}}$.)

**Critical point and near-critical phase ($p \to p_c$).** This is the domain where major open problems can be found, even if great progress has been achieved for $d = 2$ in the last decade. As we have already remarked, even the continuity of $\theta$ is not proved for $3 \le d < 19$. For other open problems consult page 22 and Section 10.3 of [**Gri99**].

The striking feature of the critical two-dimensional percolation is its *conformal invariance*. I was conjectured in the physics literature many years ago and proved first time by Smirnov [**Smi01**], however for a slightly different model, the site percolation on the triangular lattice $\mathbb{T}$, see Figure 1.3. The critical parameter of this percolation $p_c(\mathbb{T}) = \frac{1}{2}$ too.

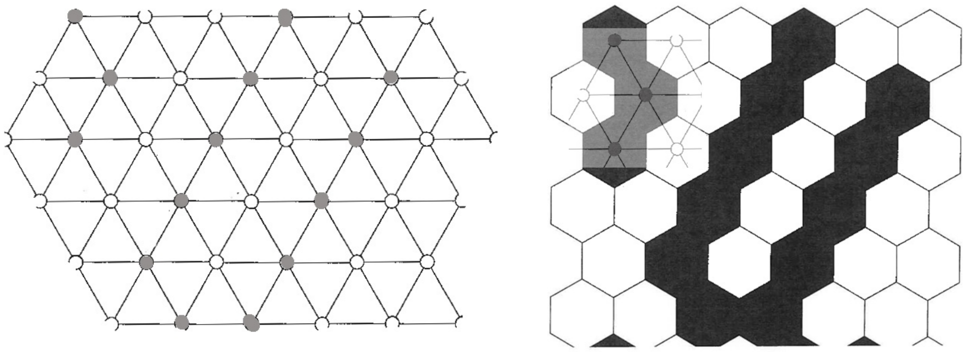

FIGURE 1.3. The site percolation on the triangular lattice and its representation as coloring of the hexagonal lattice.

Let us state briefly the result of Smirnov. Consider a *conformal rectangle* $D \subset \mathbb{R}^2 \simeq \mathbb{C}$, that is a domain bounded by a Jordan curve equipped with (in counterclockwise order) four points $a, b, c, d$ on its boundary. Let $P_\delta(D)$ be the probability that the arcs $(b,c)$ and $(d,a)$ are connected by a path contained in $D$ and passing through open sites (or open=white hexagons) only, in the site percolation on the triangular lattice with mesh size $\delta$, see Figure 1.4.

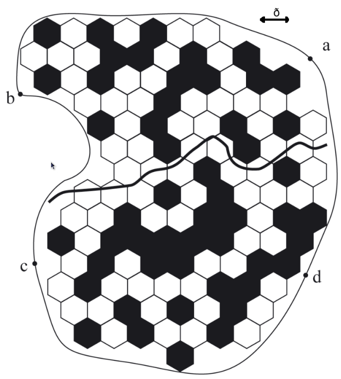


FIGURE 1.4. Event with probability $P_\delta(D)$.

Consider another conformal rectangle $(D', a', b', c', d')$ which is *conformally equivalent* to $D$, that is there exists a conformal mapping[3] $\Phi : D \to D'$ which extends continuously to the boundary, such that $a' = \Phi(a), \ldots, d' = \Phi(d)$. Smirnov Theorem claims that

$$\lim_{\delta\to 0} P_\varepsilon(D) = \lim_{\delta\to 0} P_\varepsilon(D'). \tag{1.13}$$

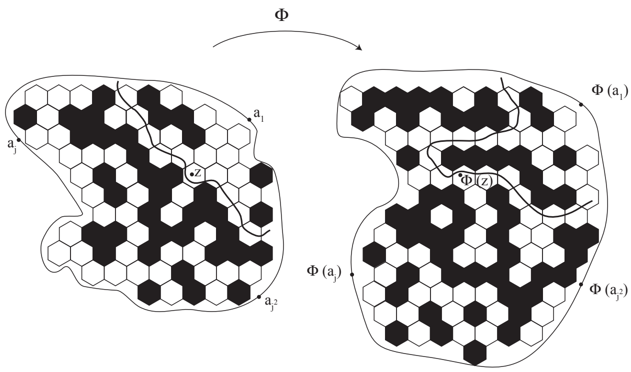


FIGURE 1.5. Illustration of Smirnov Theorem (Remark, that the triangular/hexagonal lattice is not transformed by the mapping $\Phi$).

## 2. Other related models.

The bond percolation on $\mathbb{Z}^d$ is only one, even if prominent, example of percolation models. Let us present the most important ones.

We first generalise the bond percolation slightly. There is no special reason to consider $\mathbb{Z}^d$ only. In fact, the bond percolation can be defined on any (infinite) non-oriented graph $\Lambda$ with vertex set $V(\Lambda)$ and edge set $E(\Lambda)$. Given parameter $p$ one then declares the edges in $E(\Lambda)$ open with probability $p$ and closed otherwise, independently.

The *site percolation* on $\Lambda$ is obtained by a similar construction, however instead of opening/closing edges ones do the same with vertices. Clusters of the site percolation are then connected subgraphs of $\Lambda$ induced by open sites/vertices.

[3] that is bijective holomorphic with non-vanishing derivative

If one takes $\Lambda$ to be oriented graph, one obtains *oriented percolation* by opening/closing vertices or edges of this graph. Of course, the paths defining the clusters should respect the orientation of the edges.

Slightly different flavour has the *first-passage percolation*, see e.g. [**Kes86**]. Let us define it on $\mathbb{Z}^d$. Let $(t_e : e \in E_d)$ be a collection of non-negative i.i.d. random variables with some given marginal distribution. $t_e$ is interpreted as a time needed to traverse the edge $e$. One is, e.g., interested into the set of sites that can be reached from the origin within certain time $T$, that is in the set $C_T = \{d(0,x) \leq T\}$, where the distance $d$ is defined as

$$d(0,x) = \inf\{\sum_{e\in\gamma} t_e : \gamma \text{ a path connecting } 0 \text{ to } x\}. \tag{1.14}$$

It turns out that, under suitable moment condition on the distribution of $t(e)$, $C_T$ grows approximately linearly with time. More precisely, it was shown that there exists a non-random set $L \subset \mathbb{R}^d$ such that for every $\varepsilon > 0$

$$(1-\varepsilon)TL \cap \mathbb{Z}^d \subset C_T \subset (1+\varepsilon)TL \qquad \text{eventually a.s.} \tag{1.15}$$

Leaving the discrete setting of the graph theory one obtains so called *continuum percolation* defined briefly as follows. Let $\Pi$ be the Poisson process in $\mathbb{R}^d$ with intensity $\lambda$[4] Let $O$ be a collection of balls in $\mathbb{R}^d$ with radius one centred on the points of $\Pi$. One is then interested in percolative properties of the set of these balls. One can e.g. show that, similarly as in the discrete case, there exists a critical intensity $\lambda_c$ such that $O$ contains an unique infinite component a.s. iff $\lambda > \lambda_c$. Standard reference on Continuum Percolation is the book of Meester and Roy [**MR96**].

Many other percolation related models (together with references) are given in Chapter 12 of [**Gri99**].

## 3. Random-cluster measure

The random-cluster measure is related to the percolation measure, however they lack its independence. Namely, for $B$ a finite box in $\mathbb{Z}^d$, $E_B$ the set of edges of nearest-neighbour sites in $B$, we consider an integer $q \geq 1$, $p \in [0,1]$ and the finite set

$$\Gamma_B = \{1,\dots,q\}^B \times \{0,1\}^{E_B} =: \Sigma_B \times \Omega_B. \tag{1.16}$$

One then defines a probability on $\Gamma_B$ by

$$\mu(\gamma) = Z^{-1} \prod_{e=\{x,y\}\in E_B} \{(1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_{\sigma(x),\sigma(y)}\}, \tag{1.17}$$

where $\gamma = (\sigma,\omega) \in \Gamma_B$, $Z$ is a normalising constant and $\delta$ the Kronecker symbol. This, so called Edwards-Sokal [**ES88**] measure, has quite interesting marginals. Its marginal on $\Sigma_B$ is

$$\mu_1(\sigma) := \sum_{\omega\in\Omega_B} \mu(\sigma,\omega) = Z_1^{-1} \exp\Big\{\beta \sum_{\{x,y\}\in E_B} \delta_{\sigma(x),\sigma(y)}\Big\}, \tag{1.18}$$

with $p = 1 - e^{-\beta}$. This is so called Potts model, or in special case $q = 2$ the Ising model, which is the most important model of statistical mechanics, and which was studied probably even more than the percolation.

[4]That is a random set of points in $R^d$ satisfying: (i) For any measurable $A \subset \mathbb{R}^d$, the number of points in $A$, $N(A)$ has Poisson distribution with parameter $\lambda|A|$, the Lebesgue measure of $A$. (ii) If $A, B \subset R^d$ are disjoint, then $N(A)$ and $N(B)$ are independent.

Its marginal on $\Omega_B$ is

$$\mu_2(\omega) := \sum_{\sigma\in\Sigma_B} \mu(\sigma,\omega) = Z_2^{-1}\Big\{ \prod_{e\in E_B} p^{\omega(e)}(1-p)^{1-\omega(e)}\Big\} q^{k(\omega)}, \tag{1.19}$$

with

$$k(\omega) = \begin{array}{l}\text{‘number of connected components of the graph with vertex}\\ \text{set } B \text{ and edge set } \{e \in E_B, \omega(e)=1\} \text{ (open edges)’}\end{array}. \tag{1.20}$$

This marginal is closely related to the percolation measure (in fact, if $q = 1$, this is exactly the percolation measure in $B$). It is usually called *random cluster model.* The coupling $\mu$ of the random cluster model with the Potts and Ising models allows to apply percolation techniques in statistical mechanics. The up-to-date reference for the random cluster model is the book of Grimmett [**Gri06**].

## 4. Random interlacement and related problems

The random cluster model above can be viewed as a percolation model with a dependency. There are, however, many other ways how to define ‘dependent percolation model’. One of them is of interest of the probability group at ETH. Without going into details, I will try to give you a flavour of the studied problems.

The model I will discuss relates to site percolation in a finite box, say $B = [0,n]^d \cap \mathbb{Z}^d$ of the lattice. Instead of turning the sites open/closed independently, we use a random walker to do it. Originally all sites are open. The walker starts at point $(0,0)$ and performs a simple random walk (with reflection on the boundary of the rectangle, say). When it visits a site, it changes its state to closed. The question is how the typical picture of open/closed clusters looks after $T$ steps of the walk.

One fact is obvious, the closed component is always connected (since the trajectory of the walk is). We will therefore look at the open set. If $T$ is small, then the walk have not visit many sites yet, and one can guess that the open set contains one large component of size of the box. On the other hand, if $T$ is large, random walk visited almost all sites, and the open set has only small components. This resembles changing $p$ from 1 to 0 in the Bernoulli case.

One believes that there is a critical value of $u_c$, such that if $T = un^d$, and $u < u_c$, that is the number of steps is small, then with high probability the open set has a unique component of with volume of order $n^d$, and if $u > u_c$ than the open set has only small components. The complete proof of such claim is however missing at present.

CHAPTER 2

# Basic concepts and results

## 1. Critical probability

We now consider the bond percolation on $\mathbb{Z}^d$ and keep the notation of Chapter 1. Let $\mathcal{C}_x$ be the connected open component containing the vertex $x$, $x \in \mathbb{Z}^d$. Define the events

$$J_x = \{|\mathcal{C}_x| = \infty\},$$
$$I = \{\omega \in \Omega : \text{there exists an infinite cluster in } \omega\} = \bigcup_{x \in \mathbb{Z}^d} J_x. \tag{2.1}$$

It is easy to see that $I$ and $J_x$ are events in $\mathcal{A}$. Indeed, let $B(n) = [-n, n]^d \cap \mathbb{Z}^d$ be the box in $\mathbb{Z}^d$ and let $A_n = \{0 \text{ is connected by an open path}^1 \text{ to the set } B^c = \mathbb{Z}^d \setminus B\}$. $A_n$ is an event, since it depends on the state of finitely many edges of $E_{B(n)}$[2] only. Moreover, since infinite cluster cannot exist in a finite box, $J_0 = \cap_{n \geq 0} A_n$, and hence $J_0 \in \mathcal{A}$. Analogically, $J_x$ are events for all $x \in \mathbb{Z}^d$. Since, $I = \cup_{x \in \mathbb{Z}^d} J_x$, $I$ is an event too.

**Proposition 2.1.** *The probability $\mathbb{P}_p[I]$ equals 0 or 1. It has value 0 exactly when $\theta(p) = 0$.*

Proof. We first show that

$$I \text{ is } \sigma(\omega(e) : e \in E_d \setminus E_{B(n)}) \text{ measurable for all } n \geq 1. \tag{2.2}$$

Indeed, let $I_n$ be the event 'the restriction of $\omega$ to $E_d \setminus E_{B(n)}$ contains an infinite cluster'. Of course, $I_n \in \sigma(\omega(e) : e \in E_d \setminus E_{B(n)})$, by the same proof as for (2.1). The claim (2.2) then directly follows from the following equality:

$$I = I_n. \tag{2.3}$$

To check this observe first that $I_n \subset I$. Conversely, if $\omega \in I$, let $\mathcal{C}$ be an infinite cluster of $\omega$. Consider connected components of $\mathcal{C} \setminus B(n)$ induced by $\omega$. If they are at least two such components, any of them must contain at least one vertex neighbouring with $B(n)$ (since it must be connected by an open path in $B(n)$ to the remaining components). Hence, $\mathcal{C} \setminus B(n)$ has finitely many connected components, and thus one of them should be infinite. Hence $\omega \in I_n$. This shows $I \subset I_n$ and consequently (2.3) and (2.2).

From (2.2) if follows that $I$ is tail-measurable, that is

$$I \in A_\infty := \cap_{E \subset E_d, E \text{ finite}} \sigma(\omega(e) : e \in E_d \setminus E). \tag{2.4}$$

From the Kolmogorov 0-1 law (see [**Szn07**, pp. 27–28]) we then deduce that $\mathbb{P}_p(I) = 0$ or 1.

Note also that $J_0 \subset I \subset \cup_{x \in \mathbb{Z}^d} J_x$ and thus,

$$\theta(p) = \mathbb{P}_p(J_0) \leq \mathbb{P}_p(I) \leq \sum_{x \in \mathbb{Z}^d} \mathbb{P}_p[J_x] = \sum_{x \in \mathbb{Z}^d} \theta(p). \tag{2.5}$$

(the last equality follows from the translation invariance of $\mathbb{Z}^d$) and second claim of the proposition follows. ☐

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[1] path is a *finite* sequence of neighbouring vertices

[2] $E_{B(n)}$ is the set of nearest-neighbour bonds with both endpoints in $B(n)$.

We now show the existence of the critical probability.

**Theorem 2.2.** *If $d \geq 2$, then there exists a $p_c(d) \in (0,1)$ such that*

$$\theta(p) = \mathbb{P}_p[|\mathcal{C}_0| = \infty] = \mathbb{P}_p[J_0] \begin{cases} = 0, & \text{if } p < p_c, \\ > 0, & \text{if } p > p_c. \end{cases} \tag{2.6}$$

**Remark 2.3.** It is easy to see that $p_c(d = 1) = 1$. (Exercise!)

PROOF. We prove the theorem in 4 steps.

**Step 1.** $\mathbf{p_c \geq 1/(2d)}$, which is equivalent to

$$\theta(p) = 0 \quad \text{for } p < \frac{1}{2d}. \tag{2.7}$$

To see this consider the set $\mathcal{P}_n$ of self-avoiding paths[3] starting at the origin and having length[4] $n$. Since on $J_0$, at least one $\pi \in \mathcal{P}_n$ should be open,

$$\theta(p) \leq \sum_{\pi \in \mathcal{P}_n} \mathbb{P}_p[\text{for all } e \in \pi,\, \omega(e) = 1] = |\mathcal{P}_n| p^n. \tag{2.8}$$

To construct an element of $\mathcal{P}_n$ we start at the origin. At the first step we can choose one of $2d$ edges incident to the orgin. On every next step, we can choose at most $(2d - 1)$ edges that are not yet used by the path. Hence $|\mathcal{P}_n| \leq 2d(2d - 1)^{n-1} \leq (2d)^n$ and

$$\theta(p) \leq (2dp)^n \xrightarrow{n\to\infty} 0, \quad \text{if } p < 1/(2d). \tag{2.9}$$

This completes the proof Step 1.

**Step 2. When $\mathbf{d = 2}$, then $\mathbf{p_c(2) \leq 3/4}$**, which is equivalent to

$$\theta(p) > 0 \quad \text{for } p > \frac{3}{4}, \qquad \text{in } \mathbb{Z}^2. \tag{2.10}$$

We prove this claim using so-called *Peierl's* argument (going back to Peierl's article about the Ising model from 1936). To this end we consider the dual lattice of $(\mathbb{Z}^2, E_2)$ (see Figure 2.1). The vertices of the dual lattice are the points of the shifted lattice $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$. Two dual vertices are joined by a dual edge if there are at distance one or, equivalently, if they are contained in the two neighbouring squares of the original lattice. Given $\omega \in \Omega$, we declare a dual edge open (resp. closed) if the edge it crosses is open (resp. closed). Note that this construction induces at the dual lattice again percolation on $\mathbb{Z}^2$ with succes

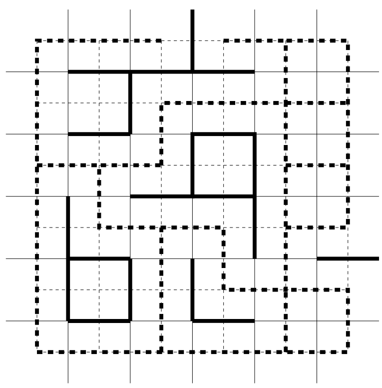

FIGURE 2.1. Part of$(\mathbb{Z}^2, E_2)$ (solid lines), and its dual lattice (dashed lines). The open edges appear bold and closed dual edges bold-dashed.

[3]self-avoiding path = no vertex of the path is traversed more then ones

[4]length = the number of traversed edges

probability $p$ .

*Important Observation.* If $\mathcal{C}_0$ is finite, then we can find a closed dual circuit (or cycle) containing $\mathcal{C}_0$ in its interior (see Figure 2.1 again). The proof of this intuitively obvious observation is slighly complicated and we will encounter similar claims later. So we let it without proof, which can be found e.g. in [**BR06a**], pp. 14–15.

In particular, if $\mathcal{C}_0$ is finite, then there is a dual closed circuit containing the origin it its interior; let $\mathcal{P}'_n$ be the set of such circuits of length $n$. Observe that $n$ should be even, since $\mathbb{Z}^2$ is bipartite, and $n \geq 4$. Therefore,

$$1 - \theta(p) = \mathbb{P}[|\mathcal{C}_0| < \infty] \leq \sum_{k \geq 2} (1-p)^{2k} |\mathcal{P}'_{2k}|. \tag{2.11}$$

To estimate the size of $\mathcal{P}'_{2k}$ observe that any circuit in this set should intersect the segment connecting $(0,0)$ to $(0,k)$. Hence, to construct an element of $\mathcal{P}'_{2k}$ we should pick a dual edge crossing this segment ($k$-choices). For placing every next edge we have then $(2d-1) = 3$ possibilities. Ignoring the fact that the circuit should be closed we get the upperbound $|\mathcal{P}'_{2k}| \leq k(2d-1)^{2k-1}$ and thus

$$1 - \theta(p) \leq \sum_{k \geq 2} (1-p)^{2k} k (2d-1)^{2k-1} < 1, \text{ when } p \geq 3/4. \tag{2.12}$$

This implies (2.10).

**Step 3.** $p_c(d) \leq p_c(2)$ **for every** $d \geq 2$**.** We embed canonically $(\mathbb{Z}^2, E_2)$ into $(\mathbb{Z}^d, E_d)$. Observe that percolation[5] in dimension two then implies percolation in dimension in $d \geq 2$. In particualar, if $p > p_c(2)$, then with a positive probability the origin is contained in an infinite component contained in $\mathbb{Z}^2$. Hence $\theta_p(d) \geq \theta_p(2)$, which implies the claim.

**Step 4.** $\theta(p)$ **is non-decreasing.** The event $J_0 = \{|\mathcal{C}_0| = \infty\}$ is so-called *increasing event*. That is, if $\omega(e) \leq \omega'(e)$ for all $e \in E_d$ (denoted by $\omega \leq \omega'$), then $\omega \in J_0 \implies \omega' \in J_0$.

We now prove the claim by a *coupling argument*. Consider the probability space $\Sigma = [0,1]^{E_d}$ equipped with the canonical product $\sigma$-algebra $\mathcal{B}$ and the probability measure $\mathbb{Q}$, under which are the canonical coordinates $X_e$, $e \in E_d$, are i.i.d. uniform random variables.

We couple each $P_p$, $p \in [0,1]$ with $\mathbb{Q}$ by the following construction. Let

$$\Phi_p : \sigma \in \Sigma \mapsto \omega = (\omega(e) = \mathbf{1}_{X_e(\sigma) \leq p})_{e \in E_d} \in \Omega. \tag{2.13}$$

It is easy to see that $\Phi_p$ is measurable and that it sends $\mathbb{Q}$ to $\mathbb{P}_p$. Moreover, $\Phi_p$ is increasing in $p$, that is if $p \leq p'$ then $\Phi_p(\sigma) \leq \Phi_{p'}(\sigma)$, for every $\sigma \in \Sigma$. Therefore, for $p \leq p'$, since $J_0$ is increasing

$$\theta(p) = \mathbb{P}_p[J_0] = \mathbb{Q}[\sigma \in \Sigma, \Phi_p(\sigma) \in J_0] \leq \mathbb{Q}[\sigma \in \Sigma, \Phi_{p'}(\sigma) \in J_0] = \mathbb{P}_{p'}[J_0] = \theta(p'). \tag{2.14}$$

This completes the proof of Step 4. The theorem then follows. $\square$

## 2. FKG inequality

We now prove three inequalities that are very important in the percolation theory. We start with the FKG inequality, which shows that "increasing events are positively correlated" (in the sense of (2.16)). This inequality goes back to Harris [**Har60**]. It is named after Fortuin, Kesteleyn and Ginibre [**FKG71**].

**Theorem 2.4** (FKG inequality)**.** *Let $X, Y \in L^2(\mathbb{P})$ be increasing random variables (that is if $\omega \leq \omega'$ then $X(\omega) \leq X(\omega')$ and $Y(\omega) \leq Y(\omega')$). Then*

$$\mathbb{E}[XY] \geq \mathbb{E}[X]\mathbb{E}[Y]. \tag{2.15}$$

[5]sometimes the event 'there exists an infinite cluster' is referred itself to as 'percolation'

*In particular, if $A$, $B$ are increasing events, then*

$$\mathbb{P}[A \cap B] \geq \mathbb{P}[A]\mathbb{P}[B]. \tag{2.16}$$

**Example 2.5.** This is the typical application of the FKG inequality. Let, for $x, y \in \mathbb{Z}^d$, $\{x \leftrightarrow y\}$ be the event $\{\omega \in \Omega : x \text{ is connected to } y \text{ by an open path in } \omega\}$. This even is clearly increasing. Therefore

$$\mathbb{P}[\{x \leftrightarrow y\} \cap \{u \leftrightarrow v\}] \geq \mathbb{P}[\{x \leftrightarrow y\}]\mathbb{P}[\{u \leftrightarrow v\}]. \tag{2.17}$$

This formalises the intuitively obvious fact that the existence of an open path from $x$ to $y$ helps to build a path from $u$ to $v$ (see Figure 2.2).

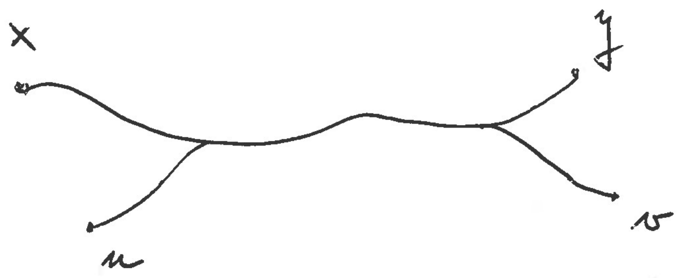


FIGURE 2.2. Illustration of Example 2.5

**Remark 2.6.** By replacing $X$, $Y$ by $-X$, $-Y$, it is easy to see that the FKG inequality (2.16) also holds for decreasing random variables (or events).

**Remark 2.7.** We will see that the proof of the FKG inequality does not use the geometry of $\mathbb{Z}^d$. In fact, the inequality holds for any product measure $\mathbb{P} = \bigotimes_{e \in E} \mu_{p_e}$ on $\{0,1\}^E$, where $E$ is a at most countable index set and $\mu_{p_e}$ are Bernoulli distributions with success $p_e \in [0,1]$, which may depend on $e \in E$.

PROOF OF THEOREM 2.4. We only need to show (2.15), which we prove in two steps.

**Step 1. FKG on finite sets.** We consider random variables $X$, $Y$, which depend on the state of $n$ edges $e_1, \dots, e_n$ only. We proceed by induction.

Suppose first that $n = 1$, that is $X$, $Y$ are functions of the state $\omega(e_1)$ of the edge $e_1$. As $X$, $Y$ are increasing, for any pair $\omega_1, \omega_2 \in \{0,1\}$,

$$\{X(\omega_1) - X(\omega_2)\}\{Y(\omega_1) - Y(\omega_2)\} \geq 0. \tag{2.18}$$

Multiplying the last inequality by $\mathbb{P}_p[\omega(e_1) = \omega_1]\mathbb{P}_p[\omega(e_1) = \omega_2]$ and summing over all $\omega_1$, $\omega_2$ we get

$$\begin{aligned} 0 &\leq \sum_{\omega_1,\omega_2} \{X(\omega_1) - X(\omega_2)\}\{Y(\omega_1) - Y(\omega_2)\}\mathbb{P}_p[\omega(e_1) = \omega_1]\mathbb{P}_p[\omega(e_1) = \omega_2] \\ &= 2\{\mathbb{E}_p[XY] - \mathbb{E}_p[X]\mathbb{E}_p[Y]\} \end{aligned} \tag{2.19}$$

as required.

Suppose now that (2.15) is valid for all $n < k$ and consider $X$, $Y$ depending on $\omega(e_1), \dots, \omega(e_k)$. Observe that fixing the states $\omega(e_1), \dots, \omega(e_{k-1})$ of the first $k-1$ edges, $X$ and $Y$ are increasing functions of the last state $\omega(e_k)$. Setting $\mathcal{F} = \sigma(\omega(e_1), \dots, \omega(e_{k-1}))$, we thus obtain

$$\mathbb{E}_p[XY] = \mathbb{E}_p[\mathbb{E}_p(XY|\mathcal{F})] \geq \mathbb{E}_p[\mathbb{E}_p(X|\mathcal{F})\mathbb{E}_p(Y|\mathcal{F})] \tag{2.20}$$

by the FKG inequality for $n = 1$. Moreover, as $X$ and $Y$ are increasing, $\mathbb{E}_p(X|\mathcal{F})$ and $\mathbb{E}_p(Y|\mathcal{F})$ are increasing functions of $k-1$ variables $\omega(e_1), \dots, \omega(e_{k-1})$. Hence, by the induction hypothesis,

$$\mathbb{E}_p[XY] \geq \mathbb{E}_p[\mathbb{E}_p(X|\mathcal{F})\mathbb{E}_p(Y|\mathcal{F})] \geq \mathbb{E}_p[\mathbb{E}_p(X|\mathcal{F})]\mathbb{E}_p[\mathbb{E}_p(Y|\mathcal{F})] = \mathbb{E}_p[X]\mathbb{E}_p[Y], \tag{2.21}$$

completing the induction step.

**Step 2. Lift to general random variables.** We now consider two increasing random variables $X, Y \in L^2(\mathbb{P}_p)$. We fix $e_1, e_2, \dots$ an enumeration of $E_d$ and define a filtration $\mathcal{F}_n = \sigma(\omega(e_1), \dots, \omega(e_n))$, $n \geq 1$. The random variables $X_n := \mathbb{E}_p[X|\mathcal{F}_n]$, $Y_n := \mathbb{E}_p[Y|\mathcal{F}_n]$ are increasing and depend only on the state of $n$ first edges. Hence by Step 1,

$$\mathbb{E}_p[X_nY_n] \geq \mathbb{E}_p[X_n]\mathbb{E}_p[Y_n]. \tag{2.22}$$

Moreover, the process $X_n, n \geq 1$ is an $L^2(\mathbb{P}_p)$ martingale. Hence, by Martingale Convergence Theorem, $X_n \to X$, $Y_n \to Y$ in $L^2(\mathbb{P}_p)$. So that

$$\mathbb{E}_p[XY] = \lim_{n\to\infty} \mathbb{E}_p[X_nY_n] \geq \lim_{n\to\infty} \mathbb{E}_p[X_n]\mathbb{E}_p[Y_n] = \mathbb{E}_p[X]\mathbb{E}_p[Y]. \tag{2.23}$$

This completes the proof. □

**Exercise 2.8.** Consider bond percolation on a general connected non-oriented graph $G = (V, E)$ with both $V$, $E$ countable, that is the measure $\mathbb{P}_p = \bigotimes_{e\in E} \mu_p$ on $\{0,1\}^E$. Define

$$\theta_x(p) = \mathbb{P}_p[|C_x(\omega)| = \infty], \qquad p_c(x) = \inf\{p \in [0,1], \theta_x(p) > 0\}. \tag{2.24}$$

We do not assume any symmetry of the graph so in general, $\theta_x(p) \neq \theta_y(p)$ for $x \neq y \in V$. None the less, $p_c(x)$ is independent of $x$. Proof this claim using the FKG inequality.

## 3. BK inequality

In various situation the FKG inequality is useless since it goes in the wrong direction. We thus need a complementary opposite inequality, which of course cannot hold for the intersection $A \cap B$ of two increasing events $A$, $B$. We are, thus, going to define a new event, so called *disjoint occurrence* of $A$, $B$, denoted by $A \circ B$. It is useful to start by an example.

**Example 2.9.** Consider four vertices $x, y, u, v$ and consider the event "$\{x \leftrightarrow y\}$ and $\{u \leftrightarrow v\}$ by edge disjoint open paths" (see Figure 2.3). It seems plausible that the fact that we cannot use the same edges for connecting $x$,$y$ and $u$,$v$ makes the probability of this event smaller than $\mathbb{P}_p[x \leftrightarrow y]\mathbb{P}_p[u \leftrightarrow v]$.

Figure 2.3. Illustration of Example 2.9

We are now going to formally define the disjoint occurrence. To this end we set

$$O_H = \{\omega \in \Omega : \omega(e) = 1 \text{ for all } e \in H\}, \qquad H \subset E_d. \tag{2.25}$$

**Definition 2.10.** Let $A$, $B$ be two increasing events depending only on the state of edges in a finite set $E \subset E_d$. We define the disjoint occurrence of $A$ and $B$ as

$$\begin{aligned} A \circ B = \{\omega \in \Omega : \exists H_1, H_2 \subset E \text{ with } H_1 \cap H_2 = \emptyset, O_{H_1} \subset A, O_{H_2} \subset B,\\ \text{such that } \omega(e) = 1 \text{ for all } e \in H_1 \cup H_2\}. \end{aligned} \tag{2.26}$$

Note that $A \circ B$ is also an increasing event. The following theorem was proved by Kesten and van den Berg [**vdBK85**].

**Theorem 2.11** (BK inequality)**.** *Let $A$, $B$ be as in Definition 2.10. Then*

$$\mathbb{P}_p[A \circ B] \le \mathbb{P}_p[A]\mathbb{P}_p[B]. \tag{2.27}$$

The typical application of Theorem 2.11 is the next.

**Corollary 2.12.**

$$\mathbb{P}_p[\exists \text{ edge disjoint open paths joining } x_1 \text{ to } y_1 \text{ and } x_2 \text{ to } y_2] \le \prod_{i=1}^{2} \mathbb{P}_p[x_i \leftrightarrow y_i]. \tag{2.28}$$

Proof of the corollary. Consider a sequence of finite sets $E_N$ such that $E_N \uparrow E$ as $N \to \infty$. For $i = 1, 2$ define

$$A_{N,i} = \{\exists \text{ open path joining } x_i \text{ to } y_i \text{ in } E_N\}. \tag{2.29}$$

Then (since paths have by definition finite length), the left-hand side of (2.28) is simply the increasing limit $\lim_{N\to\infty} \mathbb{P}_p[A_{N,1} \circ A_{N,2}]$. Inequality (2.28) then follows from (2.27), since $\lim_{N\to\infty} \mathbb{P}_p[A_{N,i}] = \mathbb{P}_p[x_i \leftrightarrow y_i]$. □

Proof of Theorem 2.11. We follow the proof by Bollobás and Leader. Let $e_1, \dots, e_N$ be a fixed enumeration of $E$ and set $\mathcal{F}_k = \sigma(\omega(e_1), \dots, \omega(e_k))$, $0 \le k \le N$. For any event $A \in \mathcal{F}_k$ we define two events $A_0$ and $A_1 \in \mathcal{F}_{k-1}$ such that

$$A = \cup_{i=0,1} \{\omega : \omega(e_k) = i, (\omega(e_1), \dots, \omega(e_{k-1})) \in A_i\}. \tag{2.30}$$

Obviously,

$$\mathbb{P}_p[A] = (1-p)\mathbb{P}_p[A_0] + p\mathbb{P}_p[A_1]. \tag{2.31}$$

$$\text{If } A \text{ is increasing, then } A_0 \subset A_1 \text{ and } A_0,\ A_1 \text{ are increasing.} \tag{2.32}$$

We now proceed by induction. It is easy to check that if $A, B \in \mathcal{F}_0$ or $A, B \in \mathcal{F}_1$, then (2.27) holds. Assume now that (2.27) holds for all $A, B \in \mathcal{F}_{k-1}$. Take $A, B \in \mathcal{F}_k$ and define $C = A \circ B$. It is easy to see that

$$\begin{aligned} C_0 &= A_0 \circ B_0,\\ C_1 &= (A_1 \circ B_0) \cup (A_0 \circ B_1). \end{aligned} \tag{2.33}$$

As $A$, $B$ are increasing, it follows from (2.32) that

$$\begin{aligned} C_0 &\subset (A_1 \circ B_0) \cap (A_0 \circ B_1),\\ C_1 &\subset (A_1 \circ B_1). \end{aligned} \tag{2.34}$$

By induction hypothesis

$$\begin{aligned} \mathbb{P}_p[C_0] &\le \mathbb{P}_p[A_0]\mathbb{P}_p[B_0],\\ \mathbb{P}_p[C_1] &\le \mathbb{P}_p[A_1]\mathbb{P}_p[B_1]. \end{aligned} \tag{2.35}$$

Further, by (2.34)

$$\begin{aligned}\mathbb{P}_p[C_0] + \mathbb{P}_p[C_1] &\le \mathbb{P}_p[(A_1 \circ B_0) \cap (A_0 \circ B_1)] + \mathbb{P}_p[(A_1 \circ B_0) \cup (A_0 \circ B_1)] \\ &= \mathbb{P}_p(A_1 \circ B_0) + \mathbb{P}_p(A_0 \circ B_1).\end{aligned} \tag{2.36}$$

Multiplying the last three inequalities by $(1-p)^2$, $p^2$ and $p(1-p)$ and summing them together we obtain

$$(1-p)\mathbb{P}_p[C_0] + p\mathbb{P}_p[C_1] \le \{(1-p)\mathbb{P}_p[A_0] + p\mathbb{P}_p[A_1]\}\{(1-p)\mathbb{P}_p[B_0] + p\mathbb{P}_p[B_1]\}. \tag{2.37}$$

Applying three times (2.31) we complete the induction step. □

**Exercise 2.13.** Prove that the operation '$\circ$' is associative. That is for $A$, $B$, $C$ as in Definition 2.10, $(A \circ B) \circ C = A \circ (B \circ C) =: A \circ B \circ C$. Extend the BK inequality to more than two events.

**Remark 2.14.** There is a generalisation of the BK inequality for general (not increasing) events depending only on the state of edges in finite $E \subset E_d$ proved by Reimer [**Rei00**]. The disjoint occurrence of such events is defined by

$$A \square B = \{\omega \in \Omega : \text{ for some } H \subset E, N_{H,\omega} \subset A, N_{E \setminus H,\omega} \subset B\}, \tag{2.38}$$

where

$$N_{I,\omega} = \{\omega' \in \Omega : \omega'(e) = \omega(e) \text{ for all } e \in I\}. \tag{2.39}$$

Reimer's theorem then states

$$\mathbb{P}_p[A \square B] \le \mathbb{P}_p[A]\mathbb{P}_p[B]. \tag{2.40}$$

## 4. Russo's Formula

Using the coupling as in Step 4 of the proof of Theorem 2.2, it can be proved that for any increasing event $A$, the probability $\mathbb{P}_p[A]$ is a non-decreasing function of $p$ (Exercise!). In this section we show Russo's formula which gives a control of the derivative of this function in the case when $A$ depends on the state of finitely many edges only.

We will need the following definition.

**Definition 2.15.** Let $A \in \mathcal{A}$ (that is $A$ is an event) and let $\omega \in \Omega$, $e \in E_d$. We say that $e$ *is pivotal for* $(A, \omega)$ if $\mathbf{1}_A(\omega) \neq \mathbf{1}_A(\omega^e)$, where

$$\omega^e(f) = \begin{cases} \omega(f), & \text{for all } f \in E_d \setminus \{e\}, \\ 1 - \omega(e), & \text{if } f = e. \end{cases} \tag{2.41}$$

**Example 2.16.** Let $A$ be the event that the origin lies in an infinite open cluster. The edge $e$ is pivotal for $A$ if, when $e$ is removed from the lattice, one end-vertex of $e$ is in a finite open component containing the origin, and the second end-vertex of $e$ is in an infinite cluster (see Figure 2.4).

**Remark 2.17.** Note that the event '$e$ is pivotal for $A$'$= \{\omega : e \text{ is pivotal for } (A, \omega)\}$ does not depend on the state of the edge $e$. In other words it is included in the $\sigma$-algebra $\sigma(\omega(f) : f \in E_d \setminus e)$.

We can now state the Russo's formula.

**Theorem 2.18** (Russo's formula [**Rus81**])**.** *Let $A$ be an increasing event depending on the state of finitely many edges only. Then*

$$\frac{d}{dp}\mathbb{P}_p[A] = \sum_{e \in E_d} \mathbb{P}_p[e \textit{ is pivotal for } A] = \mathbb{E}_p[N(A)], \tag{2.42}$$

*where $N(A)$ is the number of edges that are pivotal for $A$.*

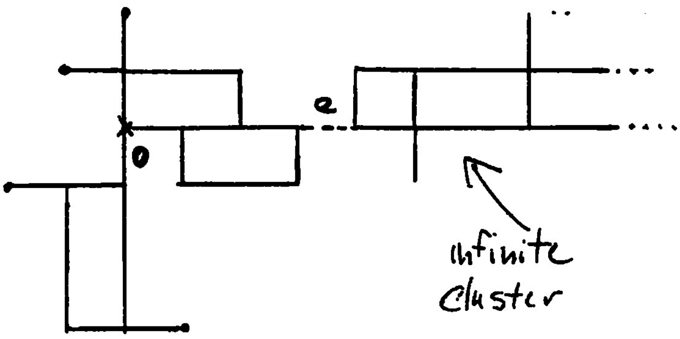


FIGURE 2.4. Illustration of Example 2.16

PROOF OF THEOREM 2.18. Since $A$ depends only on the state of finitely many edges, its probability is a polynomial function in $p$, and therefore its derivative exists. Let $E = \{e_1, \dots, e_n\}$ be the set of the edges on whose states $A$ depends. Similarly as in Step 4 of proof of Theorem 2.2, consider a family of independent random variables $(X_e : e \in E)$ on a probability space $(\Sigma, \mathbb{Q})$ having uniform distribution on $[0,1]$. Let $\boldsymbol{p} = (p_e : e \in E)$ be a collection of numbers, $0 \le p_e \le 1$. We then define the mapping $\Phi_{\boldsymbol{p}}$ by

$$\Phi_{\boldsymbol{p}} : \sigma \in \Sigma \mapsto \omega = (\omega(e) = \mathbf{1}_{X_e(\sigma) \le p_e})_{e \in E} \in \{0,1\}^E. \tag{2.43}$$

Under this mapping the measure $\mathbb{Q}$ maps to a probability measure $\mathbb{P}_{\boldsymbol{p}}$ on $\{0,1\}^E$ under which every edge $e$ is open with probability $p_e$, independently of all others edges. This provides a natural coupling between $\mathbb{P}_{\boldsymbol{p}}$'s for various values of $\boldsymbol{p}$.

Let now observe how $\mathbb{P}_{\boldsymbol{p}}[A]$ changes if we change the parameter $p_f$ for a given edge while all others $p_e$, $e \neq f$ are kept constant. That is, we consider a fixed edge $f$ and two collections $\boldsymbol{p}$, $\boldsymbol{p}'$ such that $p_e = p'_e$ for all $e \neq f$, $p_f < p'_f$. By the coupling construction, since $A$ is increasing

$$\begin{aligned} \mathbb{P}_{\boldsymbol{p}'}[A] - \mathbb{P}_{\boldsymbol{p}}[A] &= \mathbb{Q}[\sigma : \Phi_{\boldsymbol{p}}(\sigma) \notin A, \Phi_{\boldsymbol{p}'}(\sigma) \in A] \\ &= \mathbb{Q}[\sigma : (\Phi_{\boldsymbol{p}'}(\sigma))_f = 1, (\Phi_{\boldsymbol{p}}(\sigma))_f = 0, \text{ and } f \text{ is pivotal for } (A, \Phi_{\boldsymbol{p}}(\sigma))] \\ &= (p'_f - p_f)\mathbb{P}_{\boldsymbol{p}}[f \text{ is pivotal for } A]. \end{aligned} \tag{2.44}$$

In the last equality we used the observation that $f$ being pivotal does not depend on the state of $f$, and the fact that the first two events on the second line occur exactly when $X_f \in (p_f, p'_f]$.

The relation (2.44) implies directly that

$$\frac{\partial}{\partial p_f}\mathbb{P}_{\boldsymbol{p}}[A] = \lim_{p'_f \downarrow p_f} \frac{\mathbb{P}_{\boldsymbol{p}'}[A] - \mathbb{P}_{\boldsymbol{p}}[A]}{p'_f - p_f} = \mathbb{P}_{\boldsymbol{p}}[f \text{ is pivotal for } A]. \tag{2.45}$$

Since $A$ depends on finitely many edges only, we obtain, by the chain rule,

$$\frac{d}{dp}\mathbb{P}_p[A] = \sum_{f \in E} \frac{\partial}{\partial p_f}\mathbb{P}_{\boldsymbol{p}}[A]\Big|_{\boldsymbol{p}=(p,\dots,p)} = \sum_{f \in E_d} \mathbb{P}_p[f \text{ is pivotal for } A] \tag{2.46}$$

which completes the proof. □

**Exercise 2.19.** If $A$ does not depend on finitely many edges, $\mathbb{P}_p[A]$ need not to be differentiable (take for instance $\theta(p) = \mathbb{P}_p[0 \text{ is in an infinite open cluster}]$, cf. Theorem 3.11).

On the other hand, the argument above can be extended to show a lower bound on the right-hand side derivative: For any increasing event $A$

$$\liminf_{\delta \to 0} \frac{\mathbb{P}_{p+\delta}[A] - \mathbb{P}_p[A]}{\delta} \geq \mathbb{E}_p[N(A)]. \tag{2.47}$$

The precise argument can be found in [**Gri99**, p.45].

**Application of Russo's formula.** Let $A$ be increasing event depending on edges in $E$ only. Russo's formula can be rewritten as

$$\begin{aligned} \frac{d}{dp}\mathbb{P}_p[A] &= \sum_{f \in E} \mathbb{P}_p[f \text{ is pivotal for A}] \\ &= \sum_{f \in E} \frac{1}{p}\mathbb{P}_p[f \text{ is open and pivotal for } A] \\ &= \sum_{f \in E} \frac{1}{p}\mathbb{P}_p[A \cap f \text{ and pivotal for } A] \\ &= \sum_{f \in E} \frac{1}{p}\mathbb{P}_p[f \text{ and pivotal for } A|A]\mathbb{P}_p[A] = \frac{1}{p}\mathbb{E}_p[N(A)|A]\mathbb{P}_p[A]. \end{aligned} \tag{2.48}$$

Integrating this differential equality over $[p_1, p_2]$ we obtain

$$\mathbb{P}_{p_2}[A] = \mathbb{P}_{p_1}[A] \exp\Big\{ \int_{p_1}^{p_2} p^{-1}\mathbb{E}_p[N(A)|A] \, \mathrm{d}p \Big\}. \tag{2.49}$$

If we use the trivial bound $\mathbb{E}_p[N(A)|A] \leq |E|$, we obtain integrating (2.49)

$$\mathbb{P}_{p_2}[A] \leq (p_2/p_1)^{|E|}\mathbb{P}_{p_1}[A]. \tag{2.50}$$

This bounds the possible growth of $\mathbb{P}_p[A]$.

CHAPTER 3

# Sub-critical phase

We have seen in the previous chapter that $p_c$ is non-trivial for all $d \geq 2$. In this chapter we describe how the percolation configuration looks like in the sub-critical phase, $p < p_c$.

## 1. Menshikov's theorem

We first show that the distribution of the radius of the cluster containing the origin decays at least exponentially.

We use $\|x\|$ to denote the $L^1$ norm of $x$, that is $\|x\| = |x_1| + \cdots + |x_d|$. We use $S_n$ to denote the ball in this norm, $S_n = \{x : \|x\| \leq n\}$, $\partial S_n = \{x : \|x\| = n\}$ stays for its surface, and $A_n = \{\omega : 0 \leftrightarrow \partial S_n\}$.

**Theorem 3.1.** *($d \geq 2$) If $0 < p < p_c$, then there exist constants $C(p,d), c(p,d) \in (0,\infty)$ such that*

$$\mathbb{P}_p[A_n] \leq C(p,d)e^{-c(p,d)n}, \qquad \textit{for all } n. \tag{3.1}$$

Proof. The strategy of the proof is based on Menshikov [**Men86**]. Let $g_n(p) = \mathbb{P}_p[A_n]$. Observe that $A_n$ is increasing and depends on the state of finitely many edges only. Hence we can use Russo's formula as in (2.48) to obtain, for any $0 \leq p_1 < p_2 \leq 1$,

$$\begin{aligned} g_n(p_1) &= g_n(p_2)\exp\Big\{-\int_{p_1}^{p_2} \frac{1}{p}\mathbb{E}_p[N(A_n)|A_n]\mathrm{d}p\Big\} \\ &\leq g_n(p_2)\exp\Big\{-\int_{p_1}^{p_2} \mathbb{E}_p[N(A_n)|A_n]\mathrm{d}p\Big\}. \end{aligned} \tag{3.2}$$

We will see that if $p < p_c$, then the expectation $\mathbb{E}_p[N(A_n)|A_n]$ increases roughly linearly with $n$ when $p < p_c$ and we will use (3.2) to prove the theorem.

Let $n \geq 1$ and consider a configuration $\omega \in A_n$. Let

$$e_1, \dots, e_N(A_n) \text{ be the (random) edges which are pivotal for } (A_n, \omega), \tag{3.3}$$

see Figure 3.1. Since $A_n$ is increasing, all edges $e_1, \dots, e_{N(A_n)}$ are open in $\omega$. Moreover, any open path $\pi$ from the origin to $\partial S_n$ should traverse $e_j$ for every $j$, otherwise $e_j$ would be not pivotal. We assume that $e_j$'s are enumerated in the order in which they are traversed by $\pi$ (this ordering is independent of $\pi$). We write, for all $1 \leq j \leq N(A_n)$, $e_j = \{x_j, y_j\}$ where $x_j$ is the end-vertex of $e_j$ that is encountered first by $\pi$. Further we set, $\rho_1 = \|x_1\|$, and $\rho_j = \|x_j - y_{j-1}\|$, $2 \leq j \leq N(A_n)$. For $j > N$ we fix $\rho_j = \infty$. We are going to compare the random variables $\rho_i$ under $\mathbb{P}_p[\cdot|A_n]$ to a sum of i.i.d. random variables. Define a new random variable $M$ by

$$M = \sup\{\|z\| : z \in \mathcal{C}_0\}. \tag{3.4}$$

Note that if $p < p_c$, then $\mathbb{P}_p[M < \infty] = 1$. Let $M_1, M_2, \dots$ be an i.i.d. sequence distributed as $M$. It is useful to define $M_n$ on the same probability space $(\Omega, \mathbb{P}_p)$ as the percolation (by enlarging this space), so that $M_i$'s are independent of $(\omega(e), e \in E_d)$. We will see later that for any $p \in (0,1)$ and $k \geq 1$,

$$\mathbb{P}_p[\rho_1 + \cdots + \rho_k \leq n - k | A_n] \geq \mathbb{P}_p[M_1 + \cdots + M_k \leq n - k]. \tag{3.5}$$

The basic step for proving (3.5) is the following lemma.

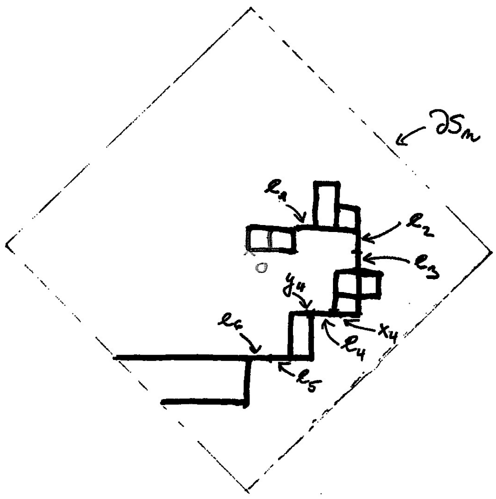


FIGURE 3.1. Sequence of pivotal edges for the event $A_n$.

**Lemma 3.2.** *Let $k$ be a positive integer, $r_1, \ldots, r_k$ be non-negative integers such that*[1] $\sum_{i=1}^k r_i \le n-k$ *and* $p \in (0,1)$. *Then*

$$\mathbb{P}_p[\rho_k \le r_k, \rho_i = r_i, \forall 1 \le i < k | A_n] \ge \mathbb{P}_p[M \le r_k]\mathbb{P}_p[\rho_i = r_i, \forall 1 \le i < k | A_n]. \tag{3.6}$$

□

PROOF. We will first explain the case $k = 1$, $0 \le r_1 < n$. Observe that,

$$\{\rho_1 > r_1\} \cap A_n \subset A_{r_1+1} \circ A_n, \tag{3.7}$$

see also Figure 3.2. Indeed, by Menger's theorem, on $\{\rho_1 > r_1\}$, there are at least two

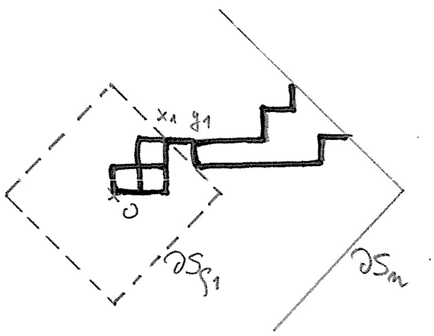


FIGURE 3.2. Sequence of pivotal edges for the event $A_n$.

edge-disjoint path from 0 to $x_1$, and $\|x\|_1 = \rho_1 > r_1$, hence $x_1$ lies either outside $S_{r_1+1}$ or at its surface. Both events on the right-hand side depend on the state of finitely many edges, hence by BK inequality

$$\mathbb{P}_p[\{\rho_1 > r_1\} \cap A_n] \le \mathbb{P}_p[A_{r_1+1}]\mathbb{P}_p[A_n]. \tag{3.8}$$

and thus

$$\mathbb{P}_p[\rho_1 > r_1 | A_n] \le g_{r_1+1}(p). \tag{3.9}$$

Since $\mathbb{P}_p[M \ge m] = g_m(p)$, we have obtained (3.6) for $k = 1$.

[1]Observe that the condition $\sum_{i=1}^k r_i \le n-k$ assures that $y_k \in S_n$.

We not turn to the general case. For any edge $e = \{u, v\}$, let $D_e$ be the open component of $\omega$ when $e$ is removed from $\omega$. Let $B_e$ be the event that the following holds

- exactly one of $u$, or $v$ lies in $D_e$, say $u$;
- $e$ is open;
- $D_e$ contains no vertex of $\partial S_n$;
- there is at least $k-1$ pivotal edges for the event $\{0 \leftrightarrow v\}$. They are, in order, $\{x_1, y_1\}, \ldots, \{x_{k-2}, y_{k-2}\}$, $\{x_{k-1}, y_{k-1}\} = e$, where $\|y_{i-1} - x_i\| = r_i$, for $1 \le i < k$, and $y_0 = 0$.

We define $B = \bigcup_e B_e$. For all $\omega \in A_n \cap B$, there exists a unique edge $e = e(\omega)$ such that $B_e$ occurs. (Indeed, $e$ should be $(k-1)^{\text{th}}$ pivotal edge on a path from 0 to $\partial S_n$.) Therefore,

$$\mathbb{P}_p[A_n \cap B] = \sum_{e=\{u,v\},G} \mathbb{P}_p[B_e, D_e = G, y_{k-1} = v, A_n], \tag{3.10}$$

where the sum runs over all edges in $S_n$ and all connected subgraphs of $G$ of $S_n$. This can be further written as

$$\begin{aligned} \mathbb{P}_p[A_n \cap B] &= \sum_{e=\{u,v\},G} \mathbb{P}_p[B_e, D_e = G, y_{k-1} = v, v \leftrightarrow \partial S_n \text{ off } G \text{ and } \{u,v\}] \\ &= \sum_{e=\{u,v\},G} \mathbb{P}_p[B_e, D_e = G, y_{k-1} = v]\mathbb{P}_p[v \leftrightarrow \partial S_n \text{ off } G \text{ and } \{u,v\}], \end{aligned} \tag{3.11}$$

where in the last equality we used the fact that given $G$ and $\{u, v\}$ the events on the right-hand side depend on disjoint set of edges and are therefore independent. Similarly as in the case $k = 1$, using the same reasoning as in (3.11), we obtain

$$\begin{aligned} \mathbb{P}_p[\{\rho_k > r_k\} \cap A_n \cap B] &= \sum_{e=\{u,v\},G} \mathbb{P}_p[B_e, D_e = G, y_{k-1} = v] \\ &\times \mathbb{P}_p\big[\{v \leftrightarrow \partial S_{r_k+1}(v) \text{ off } G,\, \{u,v\}\} \circ \{v \leftrightarrow \partial S_n \text{ off } G,\, \{u,v\}\}\big], \end{aligned} \tag{3.12}$$

where $S_r(v) = \{z : \|v - z\| \le r\}$. Using the BK inequality,

$$\begin{aligned} \mathbb{P}_p[\{\rho_k > r_k\} \cap A_n \cap B] &\le \sum_{e=\{u,v\},G} \mathbb{P}_p[B_e, D_e = G, y_{k-1} = v] \\ &\times \mathbb{P}_p\big[v \leftrightarrow \partial S_{r_k+1}(v) \text{ off } G,\, \{u,v\}\big]\mathbb{P}_p\big[v \leftrightarrow \partial S_n \text{ off } G,\, \{u,v\}\big]. \\ &\le g_{r_k+1}(p)\mathbb{P}_p[A_n \cap B], \end{aligned} \tag{3.13}$$

where in the last inequality we used (3.11) and

$$\mathbb{P}_p\big[v \leftrightarrow \partial S_{r_k+1}(v) \text{ off } G,\, \{u,v\}\big] \le \mathbb{P}_p\big[v \leftrightarrow \partial S_{r_k+1}(v)\big] = g_{r_k+1}(p). \tag{3.14}$$

Hence, by dividing by $\mathbb{P}_p[A_n \cap B]$, we get $\mathbb{P}_p[\rho_k \le r_k | A_n \cap B] \ge 1 - g_p(r_k + 1)$ and the result of the lemma follows by multiplying by $\mathbb{P}_p[B|A_n]$. □

We now use the lemma to prove (3.5). We have

$$\begin{aligned} \mathbb{P}_p[\rho_1 + \cdots + \rho_k \le n-k | A_n] &= \sum_{i=0}^{n-k} \mathbb{P}_p[\rho_1 + \cdots + \rho_{k-1} = i, \rho_k \le n-k-i | A_n] \\ &\ge \sum_{i=0}^{n-k} \mathbb{P}_p[\rho_1 + \cdots + \rho_{k-1} = i | A_n]\mathbb{P}_p[M \le n-k-i] \qquad \text{by (3.6)} \\ &= \mathbb{P}_p[\rho_1 + \cdots + \rho_{k-1} + M_k \le n-k | A_n], \end{aligned} \tag{3.15}$$

where in the last equality we used the fact that $M_k$ is independent of $(\omega(e), e \in E_d)$ and thus of all $\rho_i$'s. Iterating the argument finishes the proof of (3.5).

We now derive a lower bound on $\mathbb{E}_p[N(A_n)|A_n]$

**Lemma 3.3.** *For $p \in (0,1)$*

$$\mathbb{E}_p[N(A_n)|A_n] \geq \frac{n}{\sum_{i=0}^n \mathbb{P}_p[M \geq i]} - 1 = \frac{n}{\sum_{i=0}^n g_i(p)} - 1. \tag{3.16}$$

Proof. We introduce a truncation of $M_i$:

$$M_i' = 1 + M_1 \wedge n, \tag{3.17}$$

to avoid problems with possible defectiveness of the distribution of $M_i$ for $p > p_c$. Observe that $\rho_1 + \cdots + \rho_k \leq n - k$ implies $N(A_n) \geq k$, and therefore, using (3.5),

$$\mathbb{P}_p[N(A_n) \geq k|A_n] \geq \mathbb{P}_p[M_1 + \cdots + M_k \leq n-k] = \mathbb{P}_p[M_1' + \cdots + M_k' \leq n]. \tag{3.18}$$

Summing over $k$ we obtain

$$\mathbb{E}_p[N(A_n)|A_n] \geq \sum_{k=1}^{\infty} \mathbb{P}_p[M_1' + \cdots + M_k' \leq n] = \sum_{k\geq 1} \mathbb{P}_p[T > k] = \mathbb{E}_p[T] - 1, \tag{3.19}$$

where $T = \inf\{k \geq 0, M_1' + \dots M_k' > n\}$ (observe that $T \leq n+1$). The Wald equation gives in this setting

$$n < \mathbb{E}_p[M_1' + \cdots + M_T'] = \mathbb{E}_p[T]\mathbb{E}_p[M_1'] \tag{3.20}$$

and thus

$$\mathbb{E}_p[T] \geq \frac{n}{\mathbb{E}_p[M_1']} = \frac{n}{1 + \mathbb{E}_p[M \wedge n]} = \frac{n}{\sum_{i=0}^n \mathbb{P}_p[M \geq i]}. \tag{3.21}$$

The claim of the lemma then follows from (3.19) and (3.21). □

We will inject (3.16) into (3.2). To this end we need to obtain an upper bound on $g_i(p) = \mathbb{P}_p[A_i]$ or, more precisely, on $\sum_{i=0}^n g_i(p) = 1 + \mathbb{E}_p[M \wedge n]$ (cf. (3.21)). It will be sufficient to show that

$$\mathbb{E}_p[M] = \sum_{i=1}^{\infty} g_i(p) < \infty \qquad \text{for all } p < p_c. \tag{3.22}$$

We will show later

**Lemma 3.4.** *For $p < p_c$ there exists a $\delta(p) < \infty$ such that*

$$g_n(p) \leq \delta(p) n^{-1/2}, \qquad \textit{for } n \geq 1. \tag{3.23}$$

On the first sight, this lemma is not sufficient to prove (3.22). It can be however combined with (3.2) to show faster decay of $g_n(p)$. Indeed, the last lemma implies that for $p < p_c$, for some $\Delta(p) \in (0, \infty)$,

$$\sum_{i=0}^{\infty} g_n(p) \leq \Delta(p) n^{1/2}. \tag{3.24}$$

Take $p_1 < p_c$ and choose $p_2$ such that $p_1 < p_2 < p_c$. Then from (3.2) and (3.16) it follows that

$$\begin{aligned} g_n(p_1) &\leq g_n(p_2) \exp\Big( - \int_{p_1}^{p_2} \Big[\frac{n}{\sum_{i=0}^n g_i(p)} - 1\Big] \mathrm{d}p\Big) \\ &\leq g_n(p_2) \exp\Big( - (p_2 - p_1)\Big[\frac{n}{\sum_{i=0}^n g_i(p_2)} - 1\Big]\Big), \end{aligned} \tag{3.25}$$

since $g_i(p_2) \ge g_i(p_1)$ for all $i \ge 0$. Inserting (3.24),

$$g_n(p_1) \le g_n(p_2) \exp\Big( - (p_2 - p_1)\Big[\frac{n^{1/2}}{\Delta(p_2)} - 1\Big]\Big), \tag{3.26}$$

and thus $\sum_{i=1}^{\infty} g_i(p_1) = C(p_1) < \infty$ for all $p_1 < p_c$.

Fix now $p < p_c$ and fix $p'$ such that $p < p' < p_c$. Using (3.25) with $p, p'$ in the place of $p_1, p_2$, and using the last claim of the previous paragraph for $p'$ in the place of $p_1$ we get

$$g_n(p) \le g_n(p') \exp\Big( - (p' - p)\Big[\frac{n}{C(p')} - 1\Big]\Big) \le C(p,d)e^{-c(p,d)n}. \tag{3.27}$$

This finishes the proof of the theorem, given we show Lemma 3.4.[2]

Proof of Lemma 3.4. We proceed in three steps in which we first find a sequence $n_1, n_2, \dots$ along which $g_n(p)$ decreases rather quickly, and then we fill the gaps in this sequence.

**First step.** We show that for any $0 < \beta < p_c$, if for $n \ge 1$ and $n' := n\lfloor g_n(\beta)^{-1}\rfloor$,

$$\alpha := \beta - 3g_n(\beta)(1 - \log g_n(\beta)) > 0, \tag{3.28}$$

then

$$0 < \alpha < \beta \quad \text{and} \quad g_{n'}(\alpha) \le g_n(\beta)^2. \tag{3.29}$$

Note that since $\beta < p_c$ we have $\lim_{n\to\infty} g_n(\beta) = 0$, hence $\alpha > 0$ holds for all $n \ge n_0(\beta)$.

To prove (3.29) we have from (3.2), (3.25) that for $0 \le u < \beta$, $m \ge 1$

$$g_m(u) \le g_m(\beta) \exp\Big((\beta - u)\Big(1 - \frac{m}{\sum_{i=0}^m g_i(\beta)}\Big)\Big) \le g_m(\beta) \exp\Big(1 - \frac{m(\beta - u)}{\sum_{i=0}^m g_i(\beta)}\Big). \tag{3.30}$$

For $m \ge n$ we have, since $g_\cdot(\beta)$ is decreasing,

$$\frac{1}{m}\sum_{i=0}^m g_i(\beta) \le \frac{1}{m}(ng_0(\beta) + mg_n(\beta)) \le \frac{1}{m}(n + mg_n(\beta)). \tag{3.31}$$

Setting $m = n'$ and using $\lfloor x\rfloor \ge x/2$ for all $x > 0$ we obtain

$$\frac{1}{n'}\sum_{i=0}^{n'} g_i(\beta) \le 3g_n(\beta). \tag{3.32}$$

Inserting this into (3.30) and using monotonicity of $g_n(p)$ we get

$$g_{n'}(u) \le g_n(\beta) \exp\Big(1 - \frac{(\beta - u)}{3g_n(\beta)}\Big). \tag{3.33}$$

For $n \ge n_0(\beta)$ we can replace $u$ by $\alpha$ and use the definition of $\alpha$ to obtain (3.29).

**Second step. Iteration of the first step.** Problem of the iteration relation in the first step is that it allows us to control $g_n(p)$ as $n$ increases at the price that we change $p$ in the same time. We now show that this iteration relation can be useful even for $p$ fixed.

We consider $p \in (0, p_c)$, $\pi \in (p, p_c)$, and we construct sequences $(p_i : i \ge 0)$ and $(n_i : i \ge 0)$ such that $p_0 = \pi$, $n_0$ to be fixed later, and for $i > 0$

$$g_i = g_{n_i}(p_i), \qquad n_{i+1} = n_i\lfloor g_i\rfloor, \qquad p_{i+1} = p_i - 3g_i(1 - \log g_i), \tag{3.34}$$

such that $p_i > p$ for all $i \ge 0$. This will allow us "to iterate the first step starting from $\pi$ while staying above $p$".

[2] Expecting more carefully (3.25), the last inequality in (3.27) can be proved with $C(p,d) = 1$.

To see that such construction is possible, define for $x_0 \in (0,1)$, the sequence $(x_i : i > 0)$ by $x_{i+1} = x_i^2$, that is $x_j = x_0^{2^j}$, and set

$$S(x_0) = \sum_{j=0}^{\infty} 3x_j(1 - \log x_j). \tag{3.35}$$

Obviously $S(x_0) < \infty$ and $S(x_0) \to 0$ as $x_0 \to 0$. Hence we can choose $x_0$ such that $S(x_0) < \pi - p$, and $n_0$ such that $g_{n_0}(\pi) < x_0$.

It is easy to show by induction that $p_i$, $n_i$, $g_i$ as defined above satisfy $p_i \geq \pi - \sum_{j=0}^{i-1} 3x_j(1 - \log x_j)$ and $g_i < x_i$ for all $i \geq 0$. Indeed, it is true for $i = 0$ and assuming that it holds up to $i$, then, using twice the induction assumption,

$$p_{i+1} = p_i - 3g_i(1 - \log g_i) \geq p_i - 3x_i(1 - \log x_i) \geq \pi - \sum_{j=0}^{i} 3x_j(1 - \log x_j)(> p) \tag{3.36}$$

and, by the first step, $g_{i+1} \leq g_i^2 < x_i^2 = x_{i+1}$.

**Third step. Filling the gaps.** For $n \geq n_0$ we find $k$ such that $n_{k-1} \leq n < n_k$. This is always possible since $n_i$ is strictly increasing and diverging. Since $p_k > p$ we have

$$g_n(p) \leq g_n(p_{k-1}) \leq g_{n_{k-1}}(p_{k-1}) = g_{k-1}. \tag{3.37}$$

But, by the first step we have

$$g_{k-1}^2 \leq g_{k-1} g_{k-2}^2 \leq g_{k-1} \dots g_1 n_0^{-1} n_0 g_{n_0}(\pi)^2 \leq \delta^2 / n_k, \tag{3.38}$$

since by definition of $n_i$'s, $n_k \leq n_0 g_1^{-1} \dots g_{k-1}^{-1}$, and since $n_0 g_{n_0}(\pi)$ is a constant. Combining last two compuations we get,

$$g_n(p) \leq \delta / \sqrt{n_k} \leq \delta / \sqrt{n}, \tag{3.39}$$

for $n > n_0$. By adjusting the constants, the claim of the lemma holds for all $n \geq 0$. This completes the proof of Menshikov's Theorem. □

## 2. Equivalence of critical points

We have defined the critical probability $p_c$ as $\inf\{p \in [0,1] : \theta(p) > 0\}$. There are, however, other reasonable definitions of the critical point. We can, for example, replace the function $\theta(p) = \inf\{p : \mathbb{P}_p[0 \leftrightarrow \infty] > 0\}$ used in the definition of $p_c$ by by another functions of $\mathcal{C}_0$: Considering the expected size of $\mathcal{C}_0$,

$$\chi(p) = \mathbb{E}_p[|C_0|], \tag{3.40}$$

we can define $\bar{p}_c$ as

$$\bar{p}_c(d) = \sup\{p \in [0,1] : \chi(p) < \infty\}. \tag{3.41}$$

The question if $\bar{p}_c = p_c$ was one of the major problems of the percolation theory for nearly twenty years. Menshikov's theorem gives the affirmative answer to this question as can be seen from the following two arguments.

**Corollary 3.5** ($d \geq 2$)**.** *For $0 \leq p < p_c$,*

$$\mathbb{E}_p[|\mathcal{C}_0|] < \infty. \tag{3.42}$$

*and thus $\bar{p}_c \geq p_c$.*

PROOF. Since $|S_n| \leq c(d)(n+1)^d$, for $n \geq 0$, we have

$$\begin{aligned} \mathbb{E}_p[|\mathcal{C}_0|] &= \sum_{n\geq 0} \mathbb{E}_p[|\mathcal{C}_0|, M = n] \leq \sum_{n\geq 0} c(d)(n+1)^d \mathbb{P}_p[M = n] \\ &\leq \sum_{n\geq 0} c(d)(n+1)^d g_n(p) \leq \sum_{n\geq 0} c(d)(n+1)^d e^{-c(d,p)n} < \infty, \end{aligned} \tag{3.43}$$

and the claim follows. □

**Remark 3.6.** The inequality $\bar{p}_c \leq p_c$ is trivial, since for $p > p_c$ one has $\mathbb{P}_p[|\mathcal{C}_0| = \infty] > 0$ and thus $\mathbb{E}_p[|\mathcal{C}_0|] = \infty$.

**Remark 3.7.** If one assumes $\mathbb{E}_p[|\mathcal{C}_0|] < \infty$ instead of $p < p_c$ in Theorem 3.1, then the proof is much simpler, see Chapter 6 of [**Gri99**]. In this case, however, the equality of $p_c$ and $\bar{p}_c$ does not follow. This is the main reason why we presented the theorem with the weaker assumption.

## 3. Exponential-tail of the radius

Menshikov's theorem implies that the tail of the radius of $\mathcal{C}_0$ is at least exponential. We now show that the exponential decay is the right one.

**Theorem 3.8** ($d \geq 2$, $p \in (0,1)$)**.** *The limit*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_p[0 \leftrightarrow \partial S_n] = -\phi(p) \tag{3.44}$$

*exists, and $\phi(p) > 0$ iff $p < p_c$.*

PROOF. For $p \neq p_c$, the second claim of the theorem follows easily from Theorem 3.1 and the definition of $p_c$. The case $p = p_c$ is slightly more complicated, see [**Gri99**, Theorem 6.14].[3]

We will now show the existence of the limit. To this end we will use the following very useful lemma (for its proof see e.g. [**DZ98**], p. 255)

**Lemma 3.9** (Subadditive limit theorem)**.** *Let $(x_r : r \geq 1)$ be a sequence of real numbers which is subadditive, that is it satisfies the inequality*

$$x_{m+n} \leq x_m + x_n, \qquad \text{for all } m,n \geq 1. \tag{3.45}$$

*Then the limit*

$$\lambda = \lim_{r\to\infty} \frac{x_r}{r} \tag{3.46}$$

*exists and satisfies $-\infty \leq \lambda < \infty$. Moreover,*

$$\lambda = \inf\Big\{\frac{x_r}{r}, r \geq 1\Big\}. \tag{3.47}$$

To apply this lemma we fix $m, n \geq 1$. Obviously, on $A_{m+n} = \{\omega : 0 \leftrightarrow \partial S_{m+n}\}$ there exists a vertex $x \in \partial S_m$ which is connected by edge disjoint path to 0 and to $\partial S_{m+n}$, see Figure 3.3. Hence, using the BK inequality,

$$\mathbb{P}_p[A_{m+n}] \leq \sum_{x\in\partial S_m} \mathbb{P}_p[0 \leftrightarrow x \circ x \leftrightarrow \partial S_{m+n}] \leq \sum_{x\in\partial S_m} \mathbb{P}_p[0 \leftrightarrow x]\mathbb{P}_p[x \leftrightarrow \partial S_{m+n}]. \tag{3.48}$$

[3] In fact, it can be proved that $\phi$ is continuous on $(0,1]$.

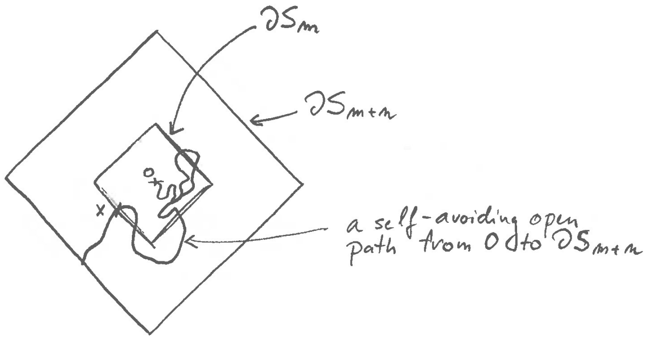


FIGURE 3.3. Towards the sub-multiplicativity.

It is easy to see that $\mathbb{P}_p[0 \leftrightarrow x] \le \mathbb{P}_p[A_m]$ and $\mathbb{P}_p[x \leftrightarrow \partial S_{m+n}] \le \mathbb{P}_p[A_n]$, by the translation invariance and the fact that $\partial S_n(x) \subset S_{m+n}$, see Figure 3.3 again. Therefore,

$$\mathbb{P}_p[A_{m+n}] \le \sum_{x\in\partial S_m} \mathbb{P}_p[A_m]\mathbb{P}_p[A_n] \le |\partial S_m|\mathbb{P}_p[A_m]\mathbb{P}_p[A_n] \le c_d m^{d-1}\mathbb{P}_p[A_m]\mathbb{P}_p[A_n]. \tag{3.49}$$

(Without the prefactor $c_d m^{d-1}$, we would obtain subaditivity for the sequence $\log \mathbb{P}_p[A_n]$. The presence of this prefactor needs some additional treatment.) We consider function $h(n) = \log(c_d n^{d-1}) = \log c_d + (d-1)\log n$. This function satisfies for $m \le n$

$$h(m+n) - h(n) = (d-1)\log(1+\tfrac{m}{n}) \le (d-1)\log 2. \tag{3.50}$$

So, if we define $a_n = \log \mathbb{P}_p[A_n] + h(n) + (d-1)\log 2$, we obtain using (3.49), (3.50), for $m \le n$,

$$\begin{aligned} a_{m+n} &= \log \mathbb{P}_p[A_{m+n}] + h(m+n) + (d-1)\log 2 \\ &\le h(m) + \log \mathbb{P}_p[A_m] + \log \mathbb{P}_p[A_n] + h(n) + 2(d-1)\log 2 = a_m + a_n, \end{aligned} \tag{3.51}$$

and thus the sequence $(a_k)$ is subadditive. By Lemma 3.9, $a_n/n$ converges. Moreover, since obviously $P[A_n] \ge p^n$ (consider one path joining 0 and $\partial S_n$) and thus $a_n \ge n\log p$, this limit must be finite. It is also non-positive. The claim of the theorem the follows from the fact that $\lim h(n)/n = 0$. □

**Remark 3.10.** Observe that from the last proof one can obtain

$$\mathbb{P}_p[A_n] \ge c_d n^{1-d} e^{-n\phi(p)}, \qquad n \ge 1. \tag{3.52}$$

It is believed that that for some $\kappa$, $P[A_n] = cn^{\kappa}e^{-n\phi(p)}(1+o(1))$. See Chapter 6.2 of [**Gri99**] for more comments.

## 4. Critical behaviour of $\theta(p)$

We conclude this chapter by one consequence of the proof of Menshikov's theorem for the super-critical behaviour of the function $\theta(p)$.

**Theorem 3.11.** *There exist $a, b > 0$ such that*

$$\theta(p) - \theta(p_c) \ge a(p-p_c), \qquad \text{for } 0 \le p - p_c \le b. \tag{3.53}$$

**Remark 3.12.** By work of Hara and Slade [**HS94**], it is known that for $d \ge 19$ there exist $c_1, c_2 \in (0,\infty)$ such that

$$c_1(p-p_c) \le \theta(p) - \theta(p_c) \le c_2(p-p_c), \qquad p > p_c. \tag{3.54}$$

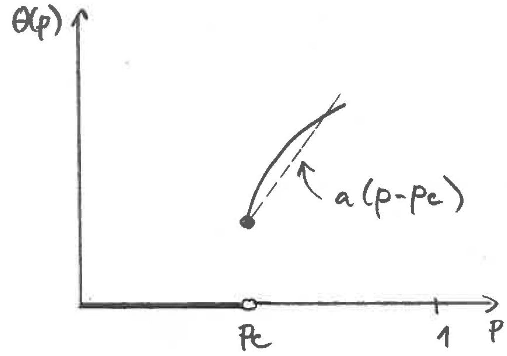


FIGURE 3.4. Illustration of Theorem 3.11.

This estimates are believed to hold for $d \geq 7$. On the other hand, in $d = 2$ it was proved by Kesten and Zhang [**KZ87**] that

$$c_1(p - p_c)^b \leq \theta(p) - \theta(p_c), \qquad \text{for } p > p_c \text{ and } b < 1. \tag{3.55}$$

Hence the (so-called mean-field) behaviour of (3.54) fails for $d \geq 2$. It is believed that $\theta(p) \sim (p - p_c)^{\beta(d)}$ as $p \downarrow p_c$ with $\beta(d) = 1$ iff $d \geq 7$ (with possible logarithmic corrections for $d = 6$). The constant $\beta$ is one of so-called critical exponents.

PROOF OF THEOREM 3.11. Observe that $\theta(p)$ is right-continuous. Indeed, $\theta(p)$ is a decreasing limit of continuous functions, $\theta(p) = \lim_{n\to\infty} \mathbb{P}_p[A_n]$, hence $\theta(p)$ is upper-semicontinuous. Since it is increasing, it is right-continuous.

Recall that, for $p \in (0, 1)$ with (3.2) and Lemma 3.3, we have

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p[A_n] \geq \mathbb{P}_p[A_n]\Big\{\frac{n}{\sum_{i=0}^n \mathbb{P}_p[M_n \geq i]} - 1\Big\}, \qquad n \geq 1. \tag{3.56}$$

Assume now that $p_c < p < 1$ and pick an $\varepsilon \in \big(0, \frac{1}{2}(1-\theta(p))\big)$. Using the right-continuity of $\theta$, for any $\alpha \in (p_c, p)$ we can choose $\beta \in (\alpha, p]$ such that

$$\theta(\beta) \leq (1 + \varepsilon)\theta(\alpha). \tag{3.57}$$

Since $\mathbb{P}_\beta[A_n] \xrightarrow{n\to\infty} \theta(\beta)$, we can fix $N$ such that

$$\frac{1}{n}\sum_{i=0}^{n} \mathbb{P}_\beta[A_i] \leq \frac{1}{1-\varepsilon}\theta(\beta) \qquad \text{if } n \geq N. \tag{3.58}$$

For $\gamma \in [\alpha, \beta]$ we have $\theta(\alpha) \leq \mathbb{P}_\alpha[A_i] \leq \mathbb{P}_\gamma[A_i] \leq \mathbb{P}_\beta[A_i]$. Using this inequality in (3.56) with $p = \gamma$, we get

$$\begin{aligned}
\Big(\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p[A_n]\Big)_{p=\gamma} &\geq \theta(\alpha)\Big(\frac{n}{\sum_{i=0}^n \mathbb{P}_\beta[A_i]} - 1\Big) \\
&\geq \theta(\alpha)\Big(\frac{1-\varepsilon}{\theta(\beta)} - 1\Big) \qquad \text{by (3.58) for } n \geq N \\
&\geq \frac{1}{1+\varepsilon}\big(1 - \varepsilon - \theta(\beta)\big) \qquad \text{by (3.57)} \\
&\geq \frac{\varepsilon}{1+\varepsilon}, \qquad \text{since } \beta \leq p \text{ and } 1 - \theta(p) \geq 2\varepsilon.
\end{aligned} \tag{3.59}$$

Integrating over $\gamma$ and letting $n \to \infty$ we find

$$\theta(\gamma) - \theta(\alpha) \geq (\gamma - \alpha)\frac{\varepsilon}{1+\varepsilon}, \qquad \alpha \leq \gamma < \beta. \tag{3.60}$$

We now want to send $\alpha \downarrow p_c$ but since $\beta$ depends on $\alpha$ in (3.60), see (3.57), a little care is necessary. We will now prove that one can choose $\beta = p$. Indeed, define

$$\mu(\alpha) = \sup\{\beta \in (\alpha, p], \text{such that (3.60) holds for } \alpha \le \gamma \le \beta\}. \tag{3.61}$$

Assume that $\alpha < \mu(\alpha) < p$. We can apply the above construction for $\mu(\alpha)$ on the place of $\alpha$ and find a $\xi \in (\mu(\alpha), p]$ such that

$$\theta(\rho) - \theta(\mu(\alpha)) \ge (\rho - \mu(\alpha))\frac{\varepsilon}{1+\varepsilon} \qquad \text{for } \rho \in [\mu(\alpha), \xi). \tag{3.62}$$

Moreover, by upper-semicontinuity of $\theta$ and (3.60),

$$\theta(\mu(\alpha)) - \theta(\alpha) \ge \lim_{q \uparrow \mu(\alpha)} \theta(q) - \theta(\alpha) \ge \lim_{q \uparrow \mu(\alpha)} (q - \alpha)\frac{\varepsilon}{1+\varepsilon} = (\mu(\alpha) - \alpha)\frac{\varepsilon}{1+\varepsilon}. \tag{3.63}$$

Combining last two claims we thus obtain

$$\theta(\rho) - \theta(\alpha) \ge (\rho - \alpha)\frac{\varepsilon}{1+\varepsilon} \qquad \text{for } \rho \in [\mu(\alpha), \xi) \tag{3.64}$$

which contradicts with definition of $\mu(\alpha)$.

As a result, we can choose $\beta = p$ in (3.60) and by letting $\alpha \downarrow p_c$ we obtain

$$\theta(\gamma) - \theta(p_c) \ge (\gamma - p_c)\frac{\varepsilon}{1+\varepsilon}, \qquad p_c \le \gamma < p, \tag{3.65}$$

which proves the theorem. □

CHAPTER 4

# Super-critical phase

## 1. Uniqueness of the infinite cluster

We now consider the situation when $p > p_c$ that is $\theta(p) > 0$. We have seen in Proposition 2.1 that an infinite open cluster exists a.s. We will now discuss its *uniqueness*.

**Theorem 4.1** ($d \geq 2$)**.** *If* $\theta(p) > 0$*, then*

$$\mathbb{P}_p[\text{there exists exactly one infinite open cluster}] = 1. \tag{4.1}$$

PROOF. We will use the argument of Burton and Keane [**BK89**]. We set

$$N(\omega) = \#\text{ of distinct infinite open clusters in } \omega, \tag{4.2}$$

that is $0 \leq N(\omega) \leq \infty$.

**Exercise 4.2.** Show that $N$ is a random variable. *Hint.* Prove that $\{N \geq k\}$ is an event. To this end approximate this event by an event depending only on finitely many edges.

For a finite set of vertices $B \subset \mathbb{Z}^d$ we introduce

$$\begin{aligned} &M_B(\omega) = \#\text{ of infinite clusters of } \omega \text{ touching the set } B,\\ &N_{B,0}(\omega) = N(\omega_B^0), \qquad N_{B,1}(\omega) = N(\omega_B^1), \end{aligned} \tag{4.3}$$

where the configuration $\omega_B^i$, $i = 0, 1$, is defined by

$$\omega_B^1 = \begin{cases} i, & \text{on } E_B, \qquad (E_B \text{ is the set of edges with both vertices in } B)\\ \omega, & \text{on } E_d \setminus E_B. \end{cases} \tag{4.4}$$

In words, $N_{B,0}(\omega)$ is the number of infinite clusters in $\omega$ if all edges in $E_B$ are erased (set to be closed), and $N_{B,1}$ is the number of infinite clusters in $\omega$ if all edges in $E_B$ are set to be open. Obviously

$$N_{B,0}(\omega) \geq N(\omega) \geq N_{B,1}(\omega) \qquad \text{for every } \omega \text{ and } B \text{ finite.} \tag{4.5}$$

Denote by $t_x$, $x \in \mathbb{Z}^d$, the translations of $\omega$ given by

$$(t_x\omega)(e) = \omega(e + x), \qquad \text{where for } e = \langle y, z\rangle,\ e + x = \langle x + y, x + z\rangle. \tag{4.6}$$

Observe that $N$ is invariant by translations:

$$N \circ t_x = N, \qquad \text{for all } x \in \mathbb{Z}^d. \tag{4.7}$$

It follows by the ergodicity (see e.g. book [**Kre85**]) of the measure $\mathbb{P}_p$ that $N$ is $\mathbb{P}_p$-a.s. constant, that is

$$\mathbb{P}_p[N = k] = 0 \text{ or } 1, \qquad \text{for } k \in \mathbb{N} \cup \infty. \tag{4.8}$$

**Exercise 4.3.** The general argument of ergodicity is not necessary to prove (4.8). Prove (4.8) by mimicking the proof of Kolmogorov's 0-1 law. *Hint.* Approximate the event $\{N = k\}$ by an event depending only on finitely many edges, and use the translation invariance together with the independence of $\mathbb{P}_p$.

We now restrict the possible set of values of $N$.

**Lemma 4.4** ($d \geq 2$, $p \in (0,1)$).

$$\mathbb{P}_p[N = k] = 1, \qquad \textit{for some } k \in \{0, 1, \infty\}. \tag{4.9}$$

PROOF. Consider $2 \leq k < \infty$ and assume that $\mathbb{P}_p[N = k] = 1$. Take $B = S_n = \{x : \|x\|_1 \leq n\}$ and observe that since $\mathbb{P}_p$ is a product measure and $S_n$ is finite

$$\mathbb{P}_p[\omega|_{E_B} \equiv 1] > 0 \qquad \text{and} \qquad \mathbb{P}_p[\omega|_{E_B} \equiv 0] > 0. \tag{4.10}$$

Moreover, the law of $\omega_B^0$ and $\omega_B^1$ is simply

$$\mathbb{P}_p[\cdot|\omega|_{E_B} \equiv 0] \qquad \text{and} \qquad \mathbb{P}_p[\cdot|\omega|_{E_B} \equiv 1], \quad \text{respectively.} \tag{4.11}$$

Therefore,

$$\mathbb{P}_p[N_{B,0} = k] = \mathbb{P}_p[N_{B,1} = k] = 1, \quad \text{and thus} \quad \mathbb{P}_p[N_{B,0} = N_{B,1}] = 1. \tag{4.12}$$

Observe that $N_{B,0} = N_{B,1} < \infty$(!) implies that $B$ is intersected by *at most one* infinite cluster, since otherwise, by switching the edges of $E_B$ on and off, we would find $N_{B,0} > N_{B,1}$. Therefore, by (4.12), we have

$$\mathbb{P}_p[M_B \geq 2] = 0. \tag{4.13}$$

But, $M_B$ is non-decreasing in $B$, and as $B \uparrow \mathbb{Z}^d$, $M_B \uparrow N$. We thus find

$$\mathbb{P}_p[N \geq 2] = 0, \tag{4.14}$$

which contradicts with the assumption that $\mathbb{P}_p[N = k]$ with $2 \leq k < \infty$. □

To prove Theorem 4.1 we should now exclude the possibility $N = \infty$. To this end we introduce the notion of trifurcation.

**Definition 4.5.** We say that $x \in \mathbb{Z}^d$ is a trifurcation for $\omega \in \Omega$, if

(a) $x$ belongs to an infinite open cluster,
(b) there are exactly three open edges containing $x$,
(c) the deletion of these three open edges splits the infinite open cluster containing $x$ into three distinct infinite open clusters.


FIGURE 4.1. A trifurcation at $x$.

We denote by $T_x$ the event 'there is a trifurcation at $x$'. By the translation invariance of the measure $\mathbb{P}_p$,

$$\mathbb{P}_p[T_x] = \mathbb{P}_p[T_0], \qquad \text{for all } x \in \mathbb{Z}^d. \tag{4.15}$$

We now show that

$$\mathbb{P}_p[N = \infty] = 1 \text{ implies } \mathbb{P}_p[T_0] > 0. \tag{4.16}$$

To see this assume that $N = \infty$ with probability one, take $B = S_n$ and define $M_{B,0}(\omega) = M(\omega_B^0)$ to be the number of infinite open clusters touching $B$ if all edges of $E_B$ are removed. Note that $N_{B,0} \neq M_{B,0}$ and $M_{B,0} \geq M_B$. Therefore,

$$\mathbb{P}_p[M_{S_n,0} \geq 3] \geq \mathbb{P}_p[M_{S_n} \geq 3] \xrightarrow{n \to \infty} \mathbb{P}_p[N \geq 3] = 1. \tag{4.17}$$

We can thus choose $n$ such that $\mathbb{P}_p[M_{S_n,0} \geq 3] > \frac{1}{2}$. Observe that the event $M_{S_n,0} \geq 3$ does not depend on the state of edges in $E_B$.

For $\omega \in \{M_{S_n,0} \geq 3\}$ we can find three vertices $x = x(\omega^0_{S_n}), y = y(\omega^0_{S_n}), z = z(\omega^0_{S_n}) \in \partial S_n$ so that they belong to distinct infinite open clusters, see Figure 4.2

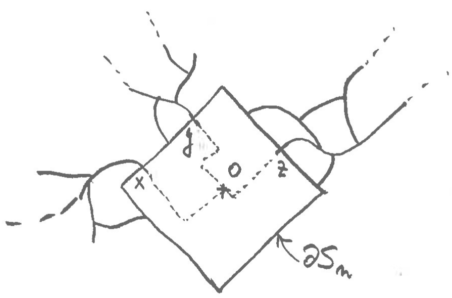


FIGURE 4.2. Construction of a trifurcation at 0.

Moreover, for any such $x$, $y$, $z$ we can construct three path joining 0 to $x$, $y$, $z$ respectively, so that

(1) each path touches exactly one vertex of $\partial S_n$,
(2) 0 is the only common vertex of these paths.

We denote by $A_{x,y,z}$ the event 'all edges of these three paths are open and all remaining edges in $E_{S_n}$ are closed'. Clearly, $A_{x,y,z}$ depends only on the state of edges in $E_B$ and

$$\mathbb{P}_p[A_{x,y,z}] \geq (p \wedge (1-p))^{|E_{S_n}|}. \tag{4.18}$$

As a result we obtain

$$\mathbb{P}_p[T_0] \geq \mathbb{P}_p[M_{S_n,0} \geq 3, A_{x(\omega^0_{S_n}),y(\omega^0_{S_n}),z(\omega^0_{S_n})}]. \tag{4.19}$$

Using the uniform estimate (4.18) and the independence of $M_{S_n,0}$ of the state of edges in $E_{S_n}$, we obtain

$$\mathbb{P}_p[T_0] \geq (p \wedge (1-p))^{|E_{S_n}|}\mathbb{P}_p[M_{S_n,0} \geq 3] \geq \tfrac{1}{2}(p \wedge (1-p))^{|E_{S_n}|} > 0, \tag{4.20}$$

where we used our choice of $n$. This proves (4.16)

We have just seen that on $N = \infty$ we can find trifurcations. Actually, due to (4.15), we can expect that the number of trifurcations in a given set is proportional to the volume of this set. The trifurcations create a tree like structure in this set, see Figure 4.3. By looking at traces of this tree at the boundary of a large box $S_n$, we will find that the boundary of $S_n$ is not large enough to accommodate all leaves of the tree. We now present the rigorous argument

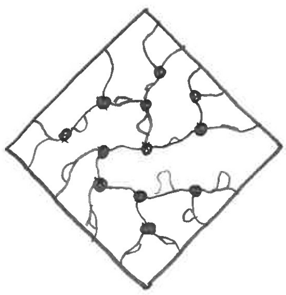

FIGURE 4.3. The tree-like structure of the open cluster coming from its trifurcations.

We first introduce several 'set-theoretic' objects.

**Definition 4.6.** Let $Y$ be a finite set $|Y| \geq 3$. A 3-partition of $Y$ is a partition $\pi = (\pi_1, \pi_2, \pi_3)$ of $Y$ into three *non-empty* subsets.

Two 3-partitions $\pi$, $\pi'$ are said to be compatible when for a suitable ordering of these 3-partitions $\pi_1 \supset \pi'_2 \cup \pi'_3$ (or equivalently by taking complements $\pi'_1 \supset \pi_2 \cup \pi_3$).

A collection $\mathcal{P}$ of 3-partitions is compatible when any two distinct 3-partition of $\mathcal{P}$ are compatible.

We will apply these concepts for the trace left on $\partial S_n$ by an open infinite cluster $C$ intersecting $S_n$. Note that any trifurcation in $S_{n-1}$ of $C$ induces a 3-partition of $\partial S_n \cap C$. We will need the following lemma.

**Lemma 4.7.** *If $\mathcal{P}$ is a compatible collection of 3-partitions of $Y$, then*

$$|\mathcal{P}| \leq |Y| - 2. \tag{4.21}$$

PROOF. We proof the lemma by induction over $|Y|$. The claim is obviously satisfied for $|Y| = 3$.

Assume now that (4.21) holds for all $3 \leq k \leq n$. Let $|Y| = n + 1$. Pick a $y \in Y$ and define $Y' = Y \setminus \{y\}$. Let $\mathcal{P}$ be a family of compatible 3-partitions of $Y$. Any $\pi \in \mathcal{P}$ can be written as $\pi = \{\pi_1 \cup \{y\}, \pi_2, \pi_3\}$, with $\pi_1, \pi_2, \pi_3$ disjoint and $\pi_2$, $\pi_3$ non-empty.

Split $\mathcal{P}$ into two sub-collections $\mathcal{P}' = \{\pi \in \mathcal{P} : \pi_1 \neq \emptyset\}$, $\mathcal{P}'' = \mathcal{P} \setminus \mathcal{P}'$. If $\pi \in \mathcal{P}'$, then $\pi_1$, $\pi_2$, $\pi_3$ form a 3-partition of $Y'$. Moreover, if $\pi, \tilde{\pi} \in \mathcal{P}'$ are distinct and compatible on $Y$, then $(\pi_1, \pi_2, \pi_3)$ and $(\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3)$ are distinct and compatible (on $Y'$). By the induction hypothesis we have thus

$$|\mathcal{P}'| \leq |Y'| - 2 = |Y| - 3. \tag{4.22}$$

Next, we will now show that $|\mathcal{P}''| \leq 1$. Indeed, otherwise we can find two distinct 3-partitions of $Y$, $(\{y\}, A_2, A_3)$ and $(\{y\}, B_2, B_3)$ which are compatible. However, then either $A_2$ or $A_3$ contains either $\{y\} \cup B_2$, or $\{y\} \cup B_3$, or $B_2 \cup B_3$. But, $A_2 \supset B_2 \cup B_3$ or $A_3 \supset B_2 \cup B_3$ is impossible because then $A_3$ or $A_2$ are empty. Similarly, $A_2 \supset \{y\} \cup B_2$, or $A_2 \supset \{y\} \cup B_3$, or $A_3 \supset \{y\} \cup B_2$, or $A_3 \supset \{y\} \cup B_3$ is impossible since $y \notin A_2$ and $y \notin A_3$. As a result $|P''| \leq 1$ which together with (4.22) implies the induction step. □

We can now finish the proof of Theorem 4.1. Assume $\mathbb{P}_p[N = \infty] = 1$. Pick $\omega \in \Omega$ and $n \geq 2$. Let $C$ be an infinite cluster intersecting $S_n$. If $x \in C \cap S_{n-1}$ is a trifurcation then the deletion of $x$ partitions $C \cap \partial S_n$ into three non-empty sets (those points of $C \cap S_n$ joined to $x$ with an open self-avoiding path in $C$ using one of the three open edges incident to $x$, respectively). Hence, each trifurcation in $S_{n-1}$ induces a 3-partition of $\partial S_n \cap C$.

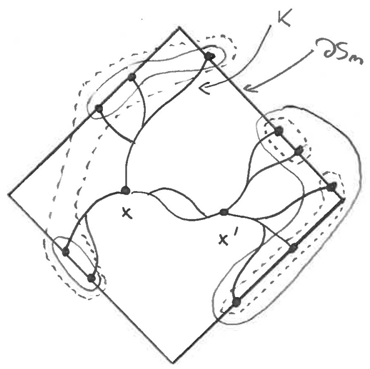


FIGURE 4.4. 3-partitions of $\partial S_n \cap K$ coming from trifurcations at $x$ (solid lines) and at $x'$ (dashed lines).

Moreover, if $x$, $x'$ are different trifurcation in $S_{n-1}$ the corresponding 3-partitions are compatible and distinct, see Figure 4.4. As a result we find

$$\sum_{x\in K\cap S_{n-1}} \mathbf{1}_{T_x} \le |\partial S_n \cap K| - 2, \tag{4.23}$$

and summing over all infinite clusters intersecting $S_n$

$$\sum_{x\in S_{n-1}} \mathbf{1}_{T_x} \le |\partial S_n|. \tag{4.24}$$

Taking expectations, we obtain from (4.15), (4.16)

$$|S_{n-1}|\mathbb{P}_p[T_0] \le |\partial S_n|, \qquad \text{for all } n \ge 2. \tag{4.25}$$

However this is impossible since $\mathbb{P}_p[T_0] \ge 0$, $|S_{n-1}| \ge cn^d$ and $\partial S_n \le c'n^{d-1}$. This completes the proof. □

## 2. Renormalization

We are now explore the geometry of the unique open infinite cluster in the super-critical phase. To this end we will use the technique called *renormalization*. Informally said, this technique serves to 'increase the occupation probability $p$' of the percolation. This then allows to prove many claims only for $p$ large enough (i.e. close to 1), which is easier, and then propagate them to all $p > p_c$. The technique is based on the fact that certain well-chosen properties of big blocks are more percolative than the original model.

Let us start with the formal construction. Let $B_n$ be the box of size $2n$,

$$B_n = [-n, n]^d \cap \mathbb{Z}^d, \tag{4.26}$$

and let $M_n = M_n(\omega)$ be an open cluster in $B_n$ (when we view all edges in $E_d \setminus E_{B_n}$ as closed). We say that $M_n$ is a *crossing cluster of* $B_n$ if $M_n$ crosses $B_n$ in every direction $i$, $1 \le i \le d$ (that is there exist $x, y \in M_n$ such that their $i$th coordinates satisfy $x_i = -n$, $y_i = n$).

For $A \subset Z^d$ let $\operatorname{diam} A = \max_{1\le i\le d} \sum_{x,y\in A} |x_i - y_i|$. Observe that if $M_n$ is crossing, then $\operatorname{diam} M_n = 2n$. We now define an important concept of $\varepsilon$-good block.

**Definition 4.8** ($0 < \varepsilon < 1$)**.** Given $\omega \in \Omega$ we say that $B_n$ is $\varepsilon$-good for $\omega$ if there is an open cluster $M_n$ of $B_n$ such that

(i) $M_n$ is crossing.
(ii) All clusters $C$ of $B_n$ distinct from $M_n$ satisfy $\operatorname{diam} C < n$.
(iii) $|M_n| \ge (1-\varepsilon)\theta(p)|B_n|$.

Observe that $B_n$ being $\varepsilon$-good depend only on the state of edges of $E_{B_n}$. The third condition intuitively correspond to the fact that $M_n$ should be the trace of the unique infinite cluster in the box $B_n$, hence its density should be not much lower than $\theta(p)$ which is the density of this infinite cluster.

To explain the role of the second condition we first need to define blocks

$$B_{x,n} = nx + B_n, \qquad x \in \mathbb{Z}^d, \tag{4.27}$$

and a family of random variables $(X^{\varepsilon}_{x,n})_{x\in\mathbb{Z}^d} = (X_x)_{x\in\mathbb{Z}^d}$ given by

$$X_x = \mathbf{1}\{B_{x,n} \text{ is } \varepsilon\text{-good}\}, \tag{4.28}$$

where the 'being good' for $B_{x,n}$ is defined similarly as for $B_n$, requiring that $M_{x,n}$, a cluster in $B_{x,n}$, satisfies "translated versions" of the above three conditions.

The implication of the condition (ii) is the following geometrical property. Let $x$, $y$ be neighbours on $\mathbb{Z}^d$, for sake of concreteness $y = x + (0, \dots, 0, 1, 0, \dots, 0)$ where 1 is at $i$th position. Suppose that $X_x = X_y = 1$. Then $M_{y,n}$ crosses $B_{y,n}$ in $i$th direction and thus $M_{y,n}$ contains an open path having diameter $n$ lying entirely in the intersection $B_{x,n} \cap B_{y,n}$. Since we assumed that $X_x = 1$ this path must be contained in $M_{x,n}$, otherwise the condition (ii) would be violated. In particular, we get that $M_{x,n} \cap M_{y,n} \neq \emptyset$. See Figure 4.5 for an illustration of this reasoning.

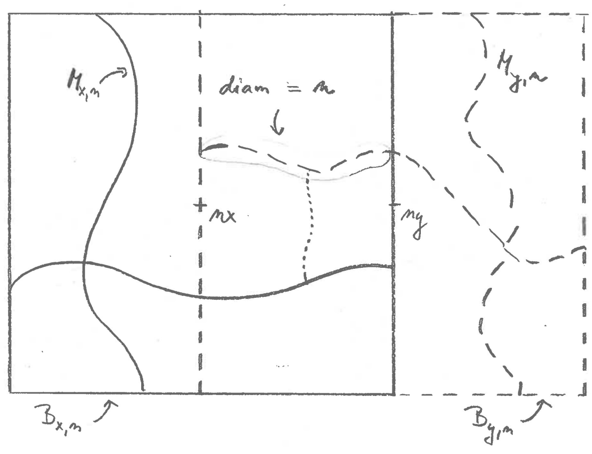


FIGURE 4.5. Illustration of the renormalization construction. The dotted connection should be present since the intersection of the cluster $M_{y,n}$ (dashed cluster) with $B_{x,n}$ has diameter at least $n$.

In particular, if $s_1, \ldots, s_k$ be a nearest-neighbour path in $\mathbb{Z}^d$ and $X_{s_1} = \cdots = X_{s_k} = 1$, then for any $x \in M_{s_1,n}$ and $y \in M_{s_k,n}$ there exists an open path linking $x$ and $y$ lying entirely in $\cup_{i=1}^{k} M_{s_k,n}$.

Let us now look at the distribution of the family $(X_x)$. First, this distribution is stationary with respect to the shifts of $\mathbb{Z}^d$, as follows easily from the translation invariance of $\mathbb{P}_p$. The family $(X_x)$ is not independent, but satisfies the following weaker property.

$$\text{The family } (X_x)_{x\in\mathbb{Z}^d} \text{ is } 3d\text{-dependent,} \tag{4.29}$$

which means that for all $A, B \subset \mathbb{Z}^d$ such that $\operatorname{dist}(A,B) = \inf_{x\in A, y\in B} \|x-y\|_1 \geq 3d$ the families $(X_x)_{x\in A}$ and $(X_x)_{x\in B}$ are independent. Indeed, the first family depends only on the edges of $\cup_{x\in A} B_{x,n}$, the second one on the edges of $\cup_{y\in B} B_{y,n}$ and these two edge sets are disjoint since $\operatorname{dist}(A,B) \geq 3d$, implying the independence.

The next two theorems will be at the heart of the renormalization construction. The first one will show that by taking large blocks, the probability that $X_x = 1$ can be made arbitrarily large. The second will then show that the 'dependent site percolation' $(X_x)_{x\in\mathbb{Z}^d}$ dominates an independent Bernoulli site percolation $(Z_x^\rho)_{x\in\mathbb{Z}^d}$ with success probability $\rho$ arbitrarily close to 1. These construction goes back to Pisztora [**Pis96**].

**Theorem 4.9** ($d \geq 2$, $p > p_c$, $\varepsilon \in (0,1)$)**.** *Good blocks are typical, more precisely*

$$\lim_{n\to\infty} \mathbb{P}_p[X_{0,n}^\varepsilon = 1] = 1. \tag{4.30}$$

To state the second theorem we need the following definition.

**Definition 4.10.** Let $(Y_x)_{x\in\mathbb{Z}^d}$ and $(Z_x)_{x\in\mathbb{Z}^d}$ be two collections of (not necessarily independent) Bernoulli variables. We say that $Y$ dominates $Z$, writing $Y \succ Z$, if for every increasing bounded measurable function $f : \Omega \to \mathbb{R}$ one has

$$E[f(Y)] \geq E[f(Z)]. \tag{4.31}$$

It is known fact, see e.g. [**Xxx1**], that

$$Y \succ Z \text{ iff there exists a coupling}^1 Q \text{ of } Y \text{ and } Z \text{ such that } Q[Y_x \geq Z_x \forall x \in \mathbb{Z}^d] = 1. \quad (4.32)$$

We can now state the second theorem which allows us to pass from dependent to independent site percolation.

**Theorem 4.11** ($d \geq 1$, $k \geq 1$)**.** *There exists a non-decreasing function $\phi : [0,1] \to [0,1]$ satisfying $\lim_{u\to 1} \phi(u) = 1$ such that if $(Y_x)_{x\in\mathbb{Z}^d}$ is a $k$-dependent family satisfying*

$$\mathbb{P}[Y_x = 1] \geq \delta \qquad \text{for all } x \in \mathbb{Z}^d, \quad (4.33)$$

*then*

$$Y \succ Z^{\phi(\delta)}, \quad (4.34)$$

*where $(Z^\rho_x)_{x\in\mathbb{Z}^d}$ is i.i.d. Bernoulli family with success parameter $\rho$.*

Proof. The proof follows the strategy of Liggett, Schonmann and Stacey [**LSS97**]. For a given $\delta$ we choose $\alpha, \rho \in (0,1)$ such that

$$\begin{aligned} (1-\alpha)(1-\rho)^{|S(k)|} &\geq 1-\delta \\ (1-\alpha)\alpha^{|S(k)|} &\geq 1-\delta. \end{aligned} \quad (4.35)$$

(Such choice is possible for $\delta$ large enough. For $\delta$ small we can set $\phi(\delta) = 0$ and therefore $Y \succ Z^{\phi(\delta)}$ automatically holds.)

The idea of the proof is to dilute $(Y_x)$ a little bit, and show that the diluted percolation dominates an independent one. We dilute $Y$ using an independent Bernoulli family $(W_x)$ with parameter $\rho$ independent of $(Y_x)$. The diluted version of $Y$ is the family $YW = (Y_xW_x)_{x\in\mathbb{Z}^d}$. We will show that

$$YW \succ Z^{\alpha\rho}. \quad (4.36)$$

Since obviously $Y \succ YW$, we obtain $Y \succ Z^{\alpha\rho}$. Now, as $\delta \to 1$ we may allow $\alpha$ and $\rho$ to approach one too, whence we may find $\alpha(\delta)$ and $\rho(\delta)$ satisfying (4.35) such that

$$\alpha(\delta)\rho(\delta) \to 1 \qquad \text{as } \delta \to 1. \quad (4.37)$$

This will imply the theorem.

In order to obtain (4.36) we will prove the following statement by induction: Let $j \geq 0$ and let $x_1, \dots, x_{j+1}$ be distinct points in $\mathbb{Z}^d$ and $z_1, \dots, z_j \in \{0,1\}$. Then,

$$\mathbb{P}[Y_{x_i}W_{x_i} = z_i \forall 1 \leq i \leq j] > 0, \quad (4.38)$$

implies

$$\mathbb{P}[Y_{x_{j+1}} = 1 | Y_{x_i}W_{x_i} = z_i \forall 1 \leq i \leq j] \geq \alpha. \quad (4.39)$$

Consider first the case $j = 0$. Then, by the assumption of the theorem and (4.35),

$$\mathbb{P}[Y_{x_1} = 1] \geq \delta \geq \alpha, \quad (4.40)$$

whence the claim holds for $j = 0$.

Suppose now that the claim holds for all $j < J$, where $J \geq 1$, and set $j = J$. Let $z_1, \dots, z_J$ satisfy (4.38) and partition $\{x_1, \dots, x_J\}$ into three sets $N_0$, $N_1$ and $M$ where

$$\begin{aligned} N_0 &= \{x_i : 1 \leq i \leq J, \|x_{J+1} - x_i\|_1 \leq k, z_i = 0\}, \\ N_1 &= \{x_i : 1 \leq i \leq J, \|x_{J+1} - x_i\|_1 \leq k, z_i = 1\}, \\ M &= \{x_1, \dots, x_J\} \setminus (N_1 \cup N_0). \end{aligned} \quad (4.41)$$

[1] A coupling of $Y$ and $Z$ is a probability $Q$ on $\Omega^2 = \{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d}$ such that, when $(X^1_x)_{x\in\mathbb{Z}^d}$ and $(X^2_x)_{x\in\mathbb{Z}^d}$ denote the canonical coordinate on this space, under $Q$, $(X^1_x)_{x\in\mathbb{Z}^d}$ has the same law as $(Y_x)_{x\in\mathbb{Z}^d}$ and $(X^2_x)_{x\in\mathbb{Z}^d}$ has the same law as $(Z_x)_{x\in\mathbb{Z}^d}$.

Since $(W_x)$ is i.i.d and independent of $(Y_x)$,

$$\begin{aligned}
&\mathbb{P}[Y_{x_i}W_{x_i} = z_i \forall 1 \le i \le J] \\
&\quad = \mathbb{P}[Y_{x_i}W_{x_i} = z_i \forall x_i \in M, Y_{x_i}W_{x_i} = 1 \forall x_i \in N_1, Y_{x_i}W_{x_i} = 0 \forall x_i \in N_0] \\
&\quad = \mathbb{P}[Y_{x_i}W_{x_i} = z_i \forall x_i \in M, Y_{x_i} = 1 \forall x_i \in N_1, Y_{x_i}W_{x_i} = 0 \forall x_i \in N_0]\rho^{|N_1|},
\end{aligned} \tag{4.42}$$

and

$$\begin{aligned}
&\mathbb{P}[Y_{x_{J+1}} = 1, Y_{x_i}W_{x_i} = z_i \forall 1 \le i \le J] \\
&\quad = \mathbb{P}[Y_{x_{J+1}} = 1, Y_{x_i}W_{x_i} = z_i \forall x_i \in M, Y_{x_i}W_{x_i} = 1 \forall x_i \in N_1, Y_{x_i}W_{x_i} = 0 \forall x_i \in N_0] \\
&\quad = \mathbb{P}[Y_{x_{J+1}} = 1, Y_{x_i}W_{x_i} = z_i \forall x_i \in M, Y_{x_i} = 1 \forall x_i \in N_1, Y_{x_i}W_{x_i} = 0 \forall x_i \in N_0]\rho^{|N_1|},
\end{aligned} \tag{4.43}$$

the conditional probability of (4.39) can be written as

$$\mathbb{P}[Y_{x_{J+1}} = 1 | Y_{x_i}W_{x_i} = z_i \forall 1 \le i \le J] = \mathbb{P}[Y_{x_{J+1}} = 1 | A_0, A_1, A], \tag{4.44}$$

where

$$\begin{aligned}
A_0 &= \{Y_{x_i}W_{x_i} = 0 \forall x_i \in N_0\}, \\
A_1 &= \{Y_{x_i} = 1 \forall x_i \in N_1\}, \\
A &= \{Y_{x_i}W_{x_i} = z_i \forall x_i \in M\}.
\end{aligned} \tag{4.45}$$

Now,

$$\mathbb{P}[Y_{x_{J+1}} = 1 | A_0, A_1, A] \ge 1 - \frac{\mathbb{P}[Y_{x_{J+1}} = 0 \cap A]}{\mathbb{P}[B_0, A_1, A]}, \tag{4.46}$$

where $B_0 = \{W_{x_i} = 0 \forall x_i \in N_0\}$. Since $Y$ is $k$-dependent and $M$ does not contain any vertex within distance $k$ of $x_{J+1}$, we have that

$$\mathbb{P}[Y_{x_{J+1}} = 0 \cap A] = \mathbb{P}[Y_{x_{J+1}} = 0]\mathbb{P}[A] \le (1-\delta)\mathbb{P}[A], \tag{4.47}$$

by the hypothesis of the theorem. Further, using that $(W)$ and $(Y)$ are independent we get

$$\mathbb{P}[B_0, A_1, A] = (1-\rho)^{|N_0|}\mathbb{P}[A_1, A]. \tag{4.48}$$

Therefore,

$$\mathbb{P}[Y_{x_{J+1}} = 1 | Y_{x_i}W_{x_i} = z_i \forall 1 \le i \le J] \ge 1 - \frac{1-\delta}{(1-\rho)^{|N_0|}\mathbb{P}[A_1|A]}. \tag{4.49}$$

To obtain a lower bound on $\mathbb{P}[A_1|A]$ we use the induction hypothesis. Assume first that $N_1$ is non-empty and write $N_1 = \{y_1, \ldots, y_n\}$ for some $n \ge 1$. We have,

$$\mathbb{P}[A_1|A] = \prod_{l=1}^{n} \mathbb{P}[Y_{y_l} = 1 | A, Y_{y_i} = 1 \forall i < l]. \tag{4.50}$$

Using again the same reasoning as in (4.42)–(4.44)

$$\mathbb{P}[Y_{y_l} = 1 | A, Y_{y_i} = 1 \forall i < l] = \mathbb{P}[Y_{y_l} = 1 | A, Y_{y_i}W_{y_i} = 1 \forall i < l] \ge \alpha, \tag{4.51}$$

For the last inequality we used $|M| + (l-1) < J$ and applied the induction hypothesis. Therefore,

$$\mathbb{P}[A_1|A] \ge \alpha^{|N_1|} \qquad \text{if } |N_1| \ge 1. \tag{4.52}$$

The same inequality holds trivially when $N_1 = \emptyset$.

Inserting (4.52) into (4.49) we get

$$\mathbb{P}[Y_{x_{J+1}} = 1 | Y_{x_i}W_{x_i} = z_i \forall 1 \le i \le J] \ge 1 - \frac{1-\delta}{(1-\rho)^{|N_0|}\alpha^{|N_1|}}. \tag{4.53}$$

Now, $|N_0| + |N_1| \le |S_k|$, by definition (4.41) of $N_0$ and $N_1$. Hence using (4.35), we obtain

$$1 - \frac{1-\delta}{(1-\rho)^{|N_0|}\alpha^{|N_1|}} \ge 1 - \frac{1-\delta}{(1-\delta)/(1-\alpha)} = \alpha \tag{4.54}$$

which completes the induction step and proves (4.39).

It remains to show that (4.39) implies (4.36). To this end we construct a coupling of $YW$ and $Z^{\alpha\rho}$. To this end consider an enumeration $x_1, x_2, \dots$ of $\mathbb{Z}^d$ and observe that (4.39) can be rewritten as

$$g_j(z_1, \dots, z_j) := \mathbb{P}[Y_{x_{j+1}} W_{x_{j+1}} = 1 | Y_{x_i} W_{x_i} = z_i \forall 1 \le i \le j] \ge \rho\alpha. \tag{4.55}$$

Consider now $(U_i)_{i\ge1}$ a family of i.i.d. random variables which are uniform on $[0,1]$. Define, for all $i \ge 1$

$$V^2_{x_i} = \mathbf{1}\{U_i \le \alpha\rho\}, \tag{4.56}$$

that is $(V^2)$ has the same law as $(Z^{\alpha\rho})$. Further set

$$V^1_{x_1} = \mathbf{1}\{U_1 \le g_0\}, \tag{4.57}$$

where $g_0 = \mathbb{P}[W_{x_1} Y_{x_1} = 1] \ge \delta\rho \ge \alpha\rho$, and inductively define

$$V^1_{x_{j+1}} = \mathbf{1}\{U_{x_{j+1}} \le g_j(V^1_{x_1}, \dots, V^1_{x_j})\}. \tag{4.58}$$

With this definition $(V^1)$ has the same law as $(YW)$. Moreover the condition (4.55) implies that $V^1_x \ge V^2_x$ for all $x \in \mathbb{Z}^d$, $\mathbb{P}$-a.s. Whence, $V^1, V^2$ is a coupling of $(YW)$ and $(Z^{\alpha\rho)}$ satisfying (4.32). This implies (4.36) and completes the proof of Theorem 4.11. □

To complete the renormalisation construction, we should now show Theorem 4.9, that is to show that if $n$ is large then the probability that a block is $\varepsilon$-good is close to one.

Proof of Theorem 4.9. We prove this theorem only for $d \ge 3$. The proof for $d = 2$ is simpler (see [**Gri99**], pp. 191–193) and uses ideas that we will see in the next chapter. The proof is based on Theorem (7.2) of Grimmett's book [**Gri99**], whose proof is, in turns, based on the technique called dynamic renormalisation. From temporal reasons we do not present this technique in this lecture.

**Theorem 4.12** ([**Gri99**],Theorem (7.2)). *Let $\Sigma(L) = \mathbb{Z}_+ \times \mathbb{Z}_+ \times [0,L]^{d-2}$ and define*[2] *$p_c(\Sigma(L)) = \inf\{p \in [0,1], \mathbb{P}_p[0 \overset{\Sigma(L)}{\leftrightarrow} \infty] > 0\}$. Then for every $p > p_c = p_c(\mathbb{Z}^d)$, there exists $L$ large enough such that*

$$p > p(\Sigma(L)). \tag{4.59}$$

We proceed with the proof of Theorem 4.9 which will be split into four lemmas. For the first one, we define

$$\begin{aligned} S_n(L) &= [-n,n]^2 \times [0,L]^{d-2}, \\ U_n(L) &= [0,n]^2 \times [0,L]^{d-2}, \\ T_n(L) &= [0,n]^{d-1} \times [0,L]. \end{aligned} \tag{4.60}$$

(We will typically take $n \gg L$ so $S_n(L)$ and $U_n(L)$ are essentially 'two-dimensional', while $T_n(L)$ is '$d-1$-dimensional'.)

**Lemma 4.13** ($d \ge 3$, $p > p_c$)**.** *There exist $L \ge 1$ and $\delta(p,L) > 0$ such that*

$$\mathbb{P}_p[x \overset{S_n(L)}{\leftrightarrow} y] \ge \delta \qquad \text{for all } x,y \in S_n(L) \text{ and all } n \ge 1, \tag{4.61}$$

$$\mathbb{P}_p[x \overset{T_n(L)}{\leftrightarrow} y] \ge \delta \qquad \text{for all } x,y \in T_n(L) \text{ and all } n \ge 1. \tag{4.62}$$

[2] $A \overset{C}{\leftrightarrow} B$ denotes the event '$A$ is connected to $B$ in $C$.

PROOF. Using Theorem 4.12, fix $L$ such that $p > p_c(\Sigma(L))$, and choose $p'$ such that $p' \in (p_c(\Sigma(L)), p)$. Define

$$\theta = \mathbb{P}_{p'}[0 \overset{\Sigma(L)}{\leftrightarrow} \infty] > 0. \tag{4.63}$$

For $m \geq 1$ we consider four facets of the box $U_m(L)$,

$$\begin{aligned} H_1(m) &= [0,m] \times \{0\} \times [0,L]^{d-2}, \qquad & H_2(m) &= \{m\} \times [0,m] \times [0,L]^{d-2}, \\ H_3(m) &= [0,m] \times \{m\} \times [0,L]^{d-2}, \qquad & H_4(m) &= \{0\} \times [0,m] \times [0,L]^{d-2}. \end{aligned} \tag{4.64}$$

Let $x_{ij}$ be the intersection of facets $H_i(m)$, $H_j(m)$ with the plane $x_3 = \cdots = x_d = 0$, here $(ij) \in \{(12), (23), (34), (41)\} = \mathcal{I}$.

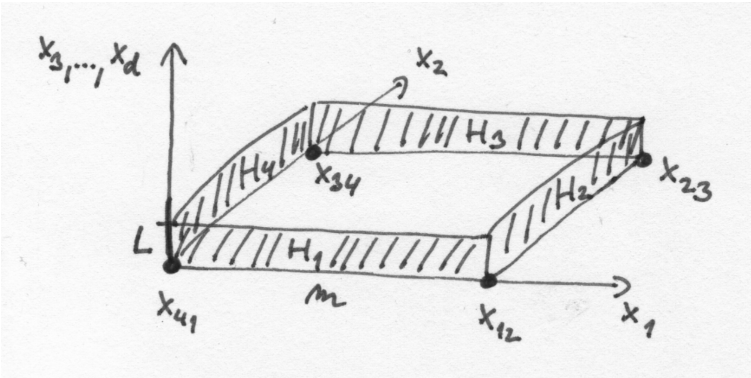


FIGURE 4.6. Notation for the slab $U_m(L)$.

We first show that all $x_{ij}$, $(ij) \in \mathcal{I}$, are connected with a positive probability, uniformly in $m$. By the definition of $\theta$, and using the symmetry, we have

$$\begin{aligned} \theta &\leq \mathbb{P}_{p'}[x_{41} \overset{U_m(L)}{\leftrightarrow} H_2(m) \cup H_3(m)] \\ &\leq \mathbb{P}_{p'}[x_{41} \overset{U_m(L)}{\leftrightarrow} H_2(m)] + \mathbb{P}_{p'}[x_{41} \overset{U_m(L)}{\leftrightarrow} H_3(m)] \\ &= 2\mathbb{P}_{p'}[x_{41} \overset{U_m(L)}{\leftrightarrow} H_2(m)]. \end{aligned} \tag{4.65}$$

Define the events $A_{ij}$, $(ij) \in \mathcal{I}$ by $A_{ij} = \{x_{ij} \overset{U_m(L)}{\leftrightarrow} H_{j+2}\}$, where $j+2$ should be taken cyclically in the set $\{1,2,3,4\}$, see Figure 4.65. By the FKG inequality and using (4.65),

$$\mathbb{P}_{p'}[\bigcap_{(ij) \in \mathcal{I}} A_{ij}] \geq (\theta/2)^4. \tag{4.66}$$

The projection of the four paths $\gamma_{ij}$ realising the events $A_{ij}$,$(ij) \in \mathcal{I}$, to the plane $x_1x_2$ is depicted on Figure 4.7. The problem is, however, that in reality these path do not intersect, since the points on the picture corresponds to cubes $[0,L]^{d-2}$. To connect those path we use the technique called *sprinkling*. Heuristically, we increase the percolation density from $p'$ to $p$ by 'sprinkling' additional open edges into $S_n(L)$ and we estimate the probability that these new open edges join the paths $\gamma_{ij}$.

More precisely, let $(Y_e)_{e \in E_d}$ be an i.i.d. collection of Bernoulli variables with success $(p-p')/(1-p')$ on an auxiliary probability space $(\Omega', \mathbb{Q})$, representing the additional edges that we add to the lattice. It is easy to see that the law of $(\max(Y_e, \omega_e))_{e \in E_d}$ under $\mathbb{Q} \times \mathbb{P}_{p'}$ is the same as the law of $(\omega_e)_{e \in E_d}$ under $\mathbb{P}_p$.

To connect, e.g., $\gamma_{12}$ with $\gamma_{23}$ we consider an arbitrary path $\gamma'$ connecting these two paths whose projection stays within point where $\gamma_{12}$ and $\gamma_{23}$ 'intersect' on Figure 4.7. The path $\gamma'$ can be chosen in such way that it contains at most $(d-2)L$ edges. The probability

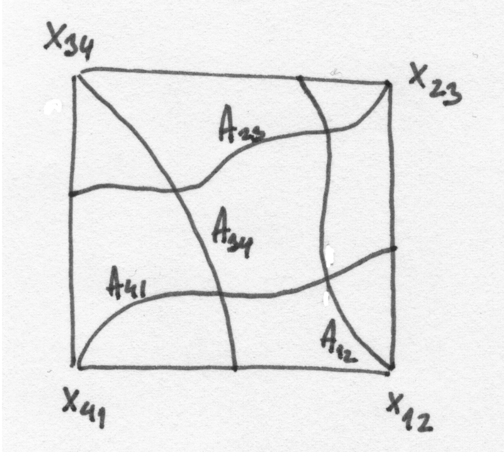


FIGURE 4.7. A realisation of events $A_{ij}$.

that all edges of $\gamma'$ are opened by the sprinkling is thus larger than $((p-p')/(1-p'))^{L(d-2)}$. Therefore,

$$\mathbb{P}_p[\gamma_{12} \leftrightarrow \gamma_{23}] = \mathbb{Q} \times \mathbb{P}_{p'}[\gamma_{12} \leftrightarrow \gamma_{23}] \geq \left(\frac{p-p'}{1-p'}\right)^{(d-2)L}. \tag{4.67}$$

Using the independence, we then obtain

$$\mathbb{P}_p[x_{41} \overset{U_m(L)}{\leftrightarrow} x_{12} \dots \overset{U_m(L)}{\leftrightarrow} x_{34}] \geq (\theta/2)^4 \left(\frac{p-p'}{1-p'}\right)^{4(d-2)L} = \delta_1, \qquad \text{for all } m \geq 1. \tag{4.68}$$

We now let $z \in S_n(L)$. We now show that $\mathbb{P}_p[0 \overset{S_n(L)}{\leftrightarrow} z] \geq \delta_2 > 0$. The claim (4.61) then follows by the FKG inequality. Without loss of generality we assume $0 \leq z_1 \leq z_2$. Let $u = (0, z_2 - z_1, 0, \dots, 0)$ and let $v = (z_1, z_2, 0, \dots, 0)$ (see Figure 4.8). Then, by FKG

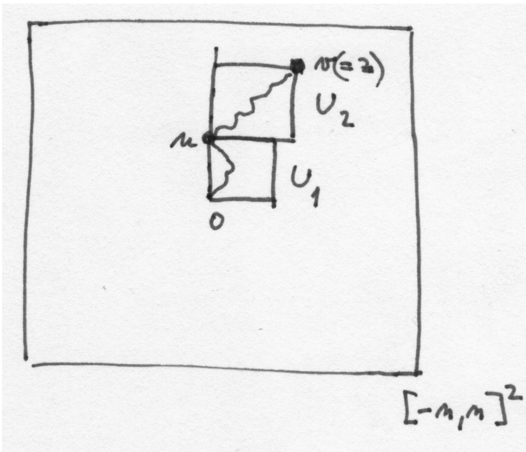


FIGURE 4.8. Construction of $u$ and $v$ (Projections of $v$ and $z$ are the same).

inequality, and by restricting the events to smaller sets,

$$\mathbb{P}_p[0 \overset{S_n(L)}{\leftrightarrow} z] \geq \mathbb{P}_p[0 \overset{S_n(L)}{\leftrightarrow} u \overset{S_n(L)}{\leftrightarrow} v \overset{S_n(L)}{\leftrightarrow} z] \geq \mathbb{P}_p[0 \overset{U_1}{\leftrightarrow} u]\mathbb{P}_p[u \overset{U_2}{\leftrightarrow} v]\mathbb{P}_p[v \overset{S_n(L)}{\leftrightarrow} z], \tag{4.69}$$

where $U_1$ and $U_2$ are two slabs whose projections are given on Figure 4.8. By (4.68), the first two probabilities are lager than $\delta_1$. The last one is larger than $p^{(d-2)L}$. Hence,

$$\mathbb{P}_p[0 \overset{S_n(L)}{\leftrightarrow} z] \geq \delta_1^2 p^{(d-2)} L = \delta_2. \tag{4.70}$$

This completes the proof of (4.61).

We now show (4.62). We fix $L$, $\delta$ so that (4.61) holds and assume $n \geq L$ first. Without loss of generality we assume that $x \in T_n(L)$ is such that $x_1 \leq 0$ for all $i = 1, \dots, d-1$. Consider the sequence of points

$$\begin{aligned} &s(0) = x, \qquad s(1) = (0, x_2, \dots, x_d), \qquad s(2) = (0, 0, x_3, \dots, x_d), \dots \\ &s(d-3) = (0, \dots, 0, x_{d-2}, x_{d-1}, x_d), \qquad s(d-2) = 0. \end{aligned} \tag{4.71}$$

For all $0 \leq j < d-2$,

$$s(j), s(j+1) \in ((0, \dots, 0, x_{j+3}, \dots, x_{d-1}, 0) + [0, L]^j) \times [-n, n]^2 \times [0, L]^{d-j-2}. \tag{4.72}$$

The last set is contained in $T_n(L)$ by assumption on $x$, and is isomorphic to $S_n(L)$. Hence $\mathbb{P}_p[s(j) \overset{T_n(L)}{\leftrightarrow} s(j+1)] \geq \delta$. By the FKG inequality again we have $\mathbb{P}_p[x \overset{T_n(L)}{\leftrightarrow} 0] \geq \delta^{d-2}$, which is (4.62) for given $x$ and $n \geq L$. Decreasing the constant $\delta$ in order to take care of $n < L$ and applying rotational symmetry completes the proof. □

Let $\partial B_n$ be the interiour boundary of $B_n$. For $x, y \in B_n$, $\alpha > 1$, $n \geq 1$, let $E_{\alpha,n}(x, y)$ be the event (see Figure 4.9)

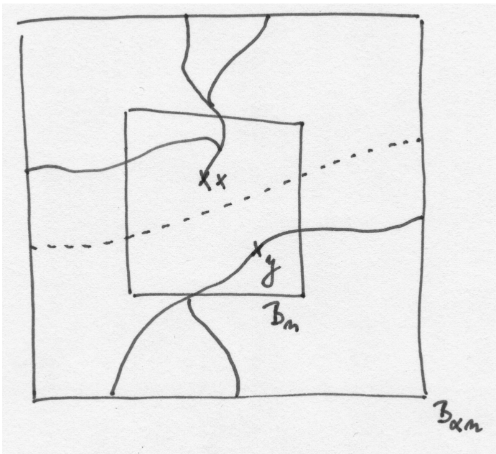


FIGURE 4.9. Event $E_{\alpha,n}(x, y)$.

$$E_{\alpha,n}(x, y) = \{x \leftrightarrow \partial B_{\alpha n}, y \leftrightarrow \partial B_{\alpha n}, x \not\leftrightarrow y \text{ in } B_{\alpha n}\}. \tag{4.73}$$

**Lemma 4.14.** *There exists $\xi > 0$ such that for all $n \geq 1$, $\alpha > 1$ and for all $x, y \in B_n$*

$$\mathbb{P}_p[E_{\alpha,n}(x, y) \leq e^{-n(\alpha-1)\xi}]. \tag{4.74}$$

PROOF. Choose $L, \delta$ as in Lemma 4.13(4.62). We are going to peel the cube $B_{\alpha n}$ by taking off layers of thickness $M = L + 1$ succesively. Let $\lceil \alpha n \rceil = n + KM + r$ where $r, K \in \mathbb{N}$, $r < M$. For $k < K$ define,

$$A_k(x, y) = \{x \leftrightarrow \partial B_{n+kM}, y \leftrightarrow \partial B_{n+kM}, x \not\leftrightarrow y \text{ in } B_{n+kM-1}\}. \tag{4.75}$$

Now,

$$E_{n,\alpha}(x, y) \subset A_K(x, y) \subset \dots \subset A_1(x, y), \tag{4.76}$$

and thus

$$\mathbb{P}_p[E_{\alpha,n}(x, y)] \leq \prod_{k=1}^{K} \mathbb{P}_p[A_k(x, y) | A_{k-1}(x, y)]. \tag{4.77}$$

We claim that

$$\mathbb{P}_p[A_k(x, y) | A_{k-1}(x, y)] \geq 1 - \delta^{d+2}, \qquad \text{for } k \geq 1, \tag{4.78}$$

which implies the lemma since

$$\mathbb{P}_p[E_{\alpha n}(x,y)] \le (1-\delta^{d+2})^K \le e^{-n\xi(\alpha-1)}. \tag{4.79}$$

To prove (4.78) define $V_k(x)$ to be the set of all vertices $u \in \partial B_{n+kM}$ which are connected to $x$ by a path whose all vertices except $u$ are in $B_{n+kM-1}$. Observe that on $A_k(x,y)$ the sets $V_k(x)$ and $V_k(y)$ are non-empty and disjoint. Obvioiusly, for $D_k = B_{n+kM-1} \setminus B_{n+(k-1)M-1}$,

$$A_k(x,y) \subset A_{k-1}(x,y) \cap \big\{u \not\leftrightarrow v \text{ in } D_k \forall u \in V_{k-1}(x), v \in V_{k-1}(y)\big\}. \tag{4.80}$$

By independence of the two events on the right-hand side (they depend on disjoint set of edges),

$$\mathbb{P}_p[A_k(x,y)|A_{k-1}(x,y)] \ge \sup\big\{\mathbb{P}_p[u \not\leftrightarrow v \text{ in } D_k] : u,v \in \partial B_{n+(k-1)M}\big\}. \tag{4.81}$$

We now sketch how to use (4.62) to prove

$$\mathbb{P}_p[u \overset{D_k}{\leftrightarrow} v] \ge \delta^{d+2} \tag{4.82}$$

which implies (4.78). The region $D_k$ may be thought as a $d$-dimensional 'shell', in rought terms comprising of $2d$ overlapping 'slices' each being isomorphic to $T_{r_k}(L)$, where $r_k = n + kM = 1$. For every $u,v \in D_k$ one can construct a sequence $T_1,\dots,T_b$ of such 'slices' such that $T_1 \ni u$, $T_b \ni v$, $T_i$ and $T_{i+1}$ overlap, and $b \le d+2$. Using then (4.62) for every slice $T_i$ and the FKG inequality it, the claim (4.82) follows. □

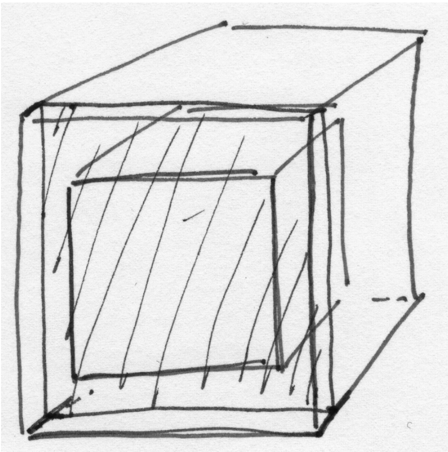

FIGURE 4.10. Illustration on a slice forming the shell $D_k$.

**Exercise 4.15.** Make yourself certain that $b \le d+2$ for any $u,v \in D_k$.

**Lemma 4.16** ($\varepsilon > 0, p > p_c$)**.**

$$\mathbb{P}_p[B_n \textit{ has a crossing cluster } C \textit{ with } |C| \ge (1-\varepsilon)\theta(p)|B_n|] \xrightarrow{n\to\infty} 1. \tag{4.83}$$

PROOF. Let $\xi$ be as in Lemma 4.14. Fix $\nu$ such that $\nu\xi > 2d$. Let

$$\begin{aligned} K_n &= \mathcal{C}_\infty \cap B_{n-\nu\log n}, \\ I_n &= \{x \in B_n : x \overset{B_n}{\leftrightarrow} K_n\}. \end{aligned} \tag{4.84}$$

We claim that with probability tending to one, $I_n$ is connected. Indeed, if $I_n$ is disconnected, then there exist $u,v \in K_n$ such that $u \not\leftrightarrow v$ in $B_n$. Therefore,

$$\mathbb{P}_p[I_n \text{ disconnected}] \le |B_n|^2 \exp\Big(-\frac{n-\nu\log n}{n}\xi\nu\log n\Big) \le cn^{2d-\nu\xi} \xrightarrow{n\to\infty} 0. \tag{4.85}$$

Furhter,

$$|I_n| \geq |K_n| = \sum_{x\in B_{n-\nu\log n}} \mathbf{1}\{x \leftrightarrow \infty\}. \tag{4.86}$$

By the ergodic theorem (see e.g. [**Dur96**, p. 341]),

$$\mathbb{P}_p[|K_n| \geq (1-\tfrac{\varepsilon}{2})\theta(p)|B_{n-\nu\log n}|] \xrightarrow{n\to\infty} 1. \tag{4.87}$$

Since also $|B_n|/|B_{n-\nu\log n}| \xrightarrow{n\to\infty} 1$, we have

$$\mathbb{P}_p[|I_n| \geq (1-\varepsilon)\theta(p)|B_n|] \xrightarrow{n\to\infty} 1. \tag{4.88}$$

It remains to show that $I_n$ is crossing. For $i \in \{1,\dots,d\}$ and $\sigma \in \{+1,-1\}$, let $F_{i,\sigma}$ be the face

$$F_{i,\sigma} = \{x \in B_n : x_i = \sigma n\}. \tag{4.89}$$

Define events $E_{i,\sigma} = \{I_n \cap F_{i,\sigma} \neq \emptyset\}$. Obviously, by FKG,

$$\mathbb{P}_p[K_n = \emptyset] \geq \mathbb{P}_p[\bigcup_{i,\sigma} E^c_{i,\sigma}] \geq \mathbb{P}_p[E^c_{1,1}]^{2d}. \tag{4.90}$$

Therefore, the probability that $I_n$ is crossing satisfies

$$\mathbb{P}_p[\bigcup_{i,\sigma} E_{i,\sigma}] \geq \prod_{i,\sigma} \mathbb{P}_p[E_{i,\sigma}] = (1-\mathbb{P}_p[E^c_{1,1}])^{2d} \geq (1-\mathbb{P}_p[K_n=\emptyset]^{1/2d})^{2d} \xrightarrow{n\to\infty} 1. \tag{4.91}$$

This completes the proof. ☐

The last lemma verifies the two of the three conditions of the definition of the good block. We now sketch how to verify the last one.

**Lemma 4.17.** *Let $T_{m,n}$ be the event that in $B_n$ there is a crossing cluster $C$ and in addition $B_n$ contains another open cluster $D$ with diameter at least $m$. Then*

$$\mathbb{P}_p[T_{m,n}] \leq d(2n+1)^{2d}e^{-\mu m}, \qquad \text{for all } m,n \geq 1. \tag{4.92}$$

Proof. The argument is similar to the peeling construction of Lemma 4.13 so we sketch it only. Let $H_{i,r} = \{x \in B_n, x_i = r\}$. Suppose that $T_{m,n}$ is realised. Then, there must be a direction $i \in \{1,\dots,d\}$, $r \in [-n, n-m]$ and two points $x, y$ such that $x_i = y_i = r$, and $x \leftrightarrow H_{i,r+m}$, $y \leftrightarrow H_{i,r+m}$ and $x \not\leftrightarrow y$, all in the slab $S = B_n \cup \{x : x_i \in [r, r+m]\}$. The slab $S$ can be pealed off as before, succesively taking off layers of thickness $M$ (for the same $M$ as in Lemma 4.13), proving that for a fixed $x$, $y$, $i$, $r$, the probability of such event is at most $e^{-\mu m}$. The prefactor in (4.92) comes from the $d$ choices of direction, and at most $|B(n)|^2$ choices of $x$, $y$ and $r$. ☐

The proof of Theorem 4.9 follows directly from the previous four lemmas. ☐

CHAPTER 5

# Critical percolation

In the last part of this lecture we study the percolation at the percolation threshold. We will prove two important results: The theorem of Kesten [**Kes80**] which states that the critical value for the bond percolation on two-dimensional square lattice equals 1/2, and the theorem of Smirnov [**Smi01**] proving the conformal invariance of two-dimensional critical site percolation on the triangular lattice.

## 1. Kesten's theorem

When studying the percolation on the critical threshold $p_c$, the first natural question to ask is 'What is the value of $p_c$ for a particular lattice?' This questions can be, however, answered in few rather special cases only, including some two-dimensional lattices. (see [**Gri99**], p. ??, and [**BR06a**], Chapter 6 for some examples and deeper discussion).

In this section we concentrate on the edge percolation on the two-dimensional square lattice $\mathbb{Z}^2$. We present the celebrated result of Kesten [**Kes80**].

**Theorem 5.1.** *The critical value $p_c(\mathbb{Z}^2)$ of the bond percolation on $\mathbb{Z}^2$ satisfies*

$$p_c(\mathbb{Z}^2) = 1/2. \tag{5.1}$$

There are many proofs of this theorem, we follow closely the one presented in [**BR06a**]. The fact that $p_c(\mathbb{Z}^2)$ can be exactly computed is largely based on the concept called duality, more precisely on the fact that $\mathbb{Z}^2$ is self-dual[1] (cf. Step 2 of the proof of Theorem 2.2).

We first explore the self-duality on a 'finite level' to prove some elementary statements on 'crossing of rectangles'. Let $R$ be a $m \times n$ rectangle in $\mathbb{Z}^2$, $R = [1, m] \times [1, n] \cap \mathbb{Z}^2$. We define its dual $R'$ to be $m - 1 \times n + 1$ rectangle $R'$ as at Figure 5.1 (Observe that $R'$ is not the dual graph of $R$ in sense of the planar graph duality)

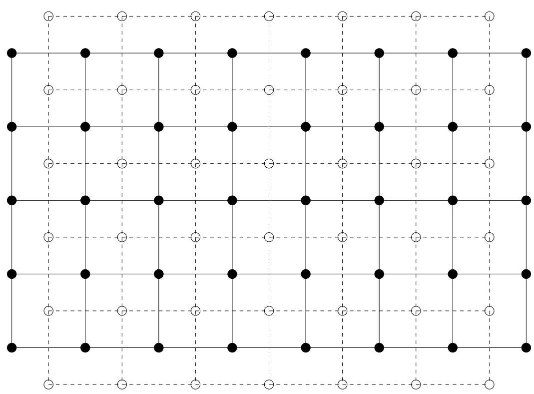

FIGURE 5.1. The construction of the dual rectangle $R'$ (dashed lines) of rectangle $R$ (solid lines)

[1]Let $G = (V, E)$ be a (possibly infinite) planar graph. The dual graph $G^\star = (V^\star, E^\star)$ of $G$ is given by: $V^\star$ = 'set of faces of $G$', $(x, y) \in E^\star$ if the faces of $G$ corresponding to $x$, $y$ share an edge. It is easy to see that $(\mathbb{Z}^2, E_2)$ and its dual are equal (in sense of graph isomorphism).

We declare an edge $e^\star$ of $R'$ open iff the (unique) edge $e$ of $R$ which crosses $e^\star$ is closed. (This does not define the state of the edges at the top and bottom side of $R'$, but these edges will be irrelevant for our considerations.) This gives us a coupling of percolation with parameter $p$ on $R$ with the percolation with parameter $1-p$ on $R'$.

We let $H(R)$ to stand for the event 'there is an open horizontal crossing of $R$', that is 'there exists a path joining left and right side of $R$ consisting only of open edges in $R$'. Note that $H(R)$ does not depend on the state of edges in the left and right side of $R$. We define $V(R)$ to be event 'there is an open vertical crossing of $R$'. We use the similar notation $H(R')$, $V(R')$ for the rectangle $R'$.

**Lemma 5.2.** *Whatever the state of edges in $R$, exactly one of the events $H(R)$, $V(R')$ occurs.*

PROOF. **Step 1. $H(R)$ and $V(R')$ cannot occur at the same time.** We explore the known fact from the graph theory stating that $K_5$, the complete graph on 5 vertices, is not planar.[2] We proof the claim of this step by contradiction. Assume that $H(R)$ and $V(R')$ both occur. Since open edges of $R$ are crossed only by closed edges of $R'$ (by definition), the crossing $\gamma$ realising $H(R)$ cannot intersect the crossing $\gamma'$ realising $V(R')$. Moreover, both $\gamma$ and $\gamma'$ lie in the convex hull of $R \cup R'$ Let $a, b$ (resp. $c, d$) be the endpoints of $\gamma$ (resp. $\gamma'$), and let $e$ be an arbitrary point in the exterior of the convex hull of $R \cup R'$. If we connect $a, b, c, d, e$ by edges (that are outside of convex hull of $R \cup R'$ and thus do not cross $\gamma$ and $\gamma'$) as on Figure 5.2 we obtain a planar drawing of $K_5$, but such drawing cannot exist.

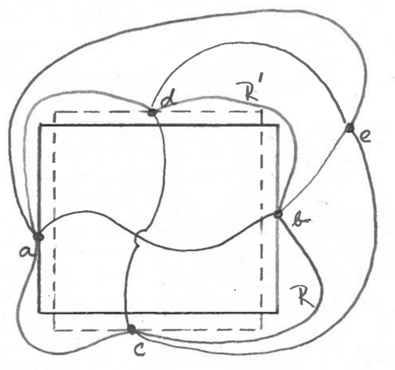


FIGURE 5.2. Planar drawing of $K_5$ coming from the simultaneous realisation of $H(R)$ and $V(R')$.

**Step 2. At least one of $H(R)$ and $V(R')$ occurs.** We use the argument of [**BR06b**]. Consider a partial tiling of the plane by squares and octagons as shown on Figure 5.3 In this tiling all vertices of $R$ and $R'$ are centres of regular octagons. The octagons centred at vertices of $R$ are coloured black (grey), the octagons centred at $R'$ are coloured white. The edges of both rectangles are represented by small squares occupying the space between the octagons. Each square represent one edge of $R$ together with its dual edge $e^\star$. It is coloured black, if the standard edge is open, it is white if the dual edge is open. The squares on the left and right side of the tiling are all black by definition, the squares on the top and

[2] This can be proved, e.g., by applying so called *Euler's formula*. By this formula, every planar graph should satisfy $\#vertices - \#edges + \#faces = 2$. The graph $K_5$ has 5 vertices $\binom{5}{2} = 10$ edges. Moreover, every three vertices of $K_5$ form a triangle, so a planar drawing of $K_5$ should have $\binom{5}{3} = 10$ faces. But, $5 - 10 + 10 \neq 2$, so $K_5$ cannot be drawn in the plane.

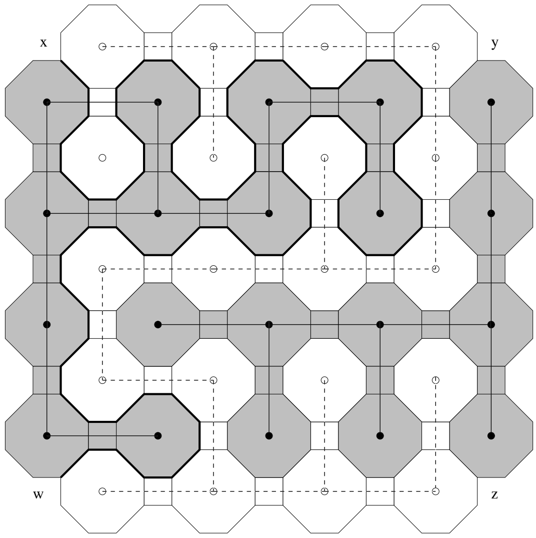


FIGURE 5.3. The joint representation of the percolation on $R$ and $R'$ in terms of a particular partial tiling of the plane (points of $R$ - filled circles/grey octagons, points of $R'$ - empty circles/white octagons, open edges - solid lines, open dual edges - dashed lines, exploration process of interface graph $I$ - thick broken line).

the bottom side white. The state of these edges/squares is irrelevant for events $H(R)$ and $V(R')$.

Observe that if a standard edge $e$ is open, the corresponding black square joins the black octagons representing the endpoints of $e$. Therefore, there is an open horizontal crossing of $R$ iff there is a path in $\mathbb{R}^2$ lying completely within black tiles joining a black octagon on the left side of $R$ with a black octagon on the right side of $R$. Analogously, $V(R')$ occurs iff top and bottom side of the tiling are joined by a path within the white region of the tiling.

We now define the interface graph $I$ as the graph consisting of all edges of the tiling separating the black and white region, together with their end-vertices. It is easy to see that every vertex of $I$ has degree two, except for four vertices $x$, $y$, $z$, $w$ which have degree one (see Figure 5.3). Hence, $I$ must be composed of several (possibly none) cycles and two paths joining the four special vertices.

We now explore the component of $I$ containing $x$ by a *local* algorithm. We start a walker in $x$ and let him walk around the edge from $x$ which separates black and white region. After his arrival at the 'crossing' the walker should decide if he turns left or right (since vertices of the tiling have all degree at most three, there are only these two possibilities, back-tracking is excluded). This decision can be made locally, the walker turns to the right if the object (octagon or square) that he sees before him is white, otherwise he turns left. After taking decision, the walker continues along the chosen edge; on reaching the next crossing he applies the same decission algorighm, etc. It is easy to see that this algorithm explores the component of $x$ in the interface graph $I$. Moreover, as the walker explores this component, it has always black object on his right-hand side and white object on its left-hand side. Eventually, the walker reaches one of the vertices $y$, $z$ or $w$. It is however impossible that it reaches $z$, since walking along the edge going to $z$ he should have black octagon on its left-hand side, which is impossible.

He thus exists at $y$ or at $w$. In the first case we see that the black region on the right-hand side of the walker contains an horizontal crossing of $R$. In the second case, the white region on his left-hand size contains an open vertical crossing of $R'$. Hence one of $V(R')$, $H(R)$ must occur, completing the proof of the lemma. □

**Remark 5.3.** The algorithm of the last proof give us more than just a horizontal of vertical crossing. If the walker exits at $y$, the open horizontal crossing contained in the black region neighbouring with his trajectory is the *top-most* horizontal crossing $P$ of $R$. Moreover, the locality of the decision making of the algorithm implies that this crossing can be found without examining the state of the edges that are 'below' $P$. Similarly, if the walker exits at $w$, the white region on his left contains *left-most* vertical crossing of $R'$, and this crossing can be found without examining the state of edges that to the right of it.

**Corollary 5.4.** *(i) Let $R$ be a rectangle in $\mathbb{Z}^2$ and $R'$ its dual. Then*

$$\mathbb{P}_p[H(R)] + \mathbb{P}_{1-p}[V(R')] = 1. \tag{5.2}$$

*(ii) If $R = R_{n+1,n}$ is $n+1 \times n$ rectangle ($n+1$ columns, $n$ rows), then*

$$\mathbb{P}_{1/2}(H(R)) = 1/2. \tag{5.3}$$

*(iii) If $S = S_n$ is a $n \times n$ square, then*

$$\mathbb{P}_{1/2}(H(S)) = \mathbb{P}_{1/2}(V(S)) \geq 1/2. \tag{5.4}$$

Proof. The claim (i) follows directly from the previous lemma, recalling that percolation with parameter $p$ on $R$ induces the percolation with parameter $1-p$ on $R'$. For (ii), observe that if $R$ is $n+1 \times n$ rectangle, then $R'$ is $n \times n+1$ rectangle, and thus $\mathbb{P}_{1/2}[V(R')] = \mathbb{P}_{1/2}[H(R)]$. The claim (iii) follows from the fact that every crossing of $R_{n+1,n}$ crosses $S_n$, implying $\mathbb{P}_{1/2}[H(S_n)] \geq \mathbb{P}_{1/2}[H(R_{n+1,n})]$. □

**1.1. Russo-Seymour-Welsh Theory.** In the last corollary we obtained a uniform (in $n$) lower bound on the crossing probability of squares. Our next goal is to extend this lower bound to rectangles of fixed aspect ratio. The hardest step is to pass from the squares to the rectangles $3n \times 2n$. We start with the following lemma.

**Lemma 5.5.** *Let $R = R_{m,2n}$, $m \geq n$ be a $m \times 2n$ rectangle. Denote by $X(R)$ the event (see Figure 5.4)*

$$X(R) = \begin{Bmatrix} \text{there are open paths } \gamma_1, \gamma_2, \text{ such that } \gamma_1 \text{ crosses } S_n \text{ vertically} \\ \text{and } \gamma_2 \text{ joins } \gamma_1 \text{ with the right edge of } R \end{Bmatrix}. \tag{5.5}$$

*Then*

$$\mathbb{P}_p[X(R)] \geq \mathbb{P}_p[H(R)]\mathbb{P}_p[V(S)]/2. \tag{5.6}$$

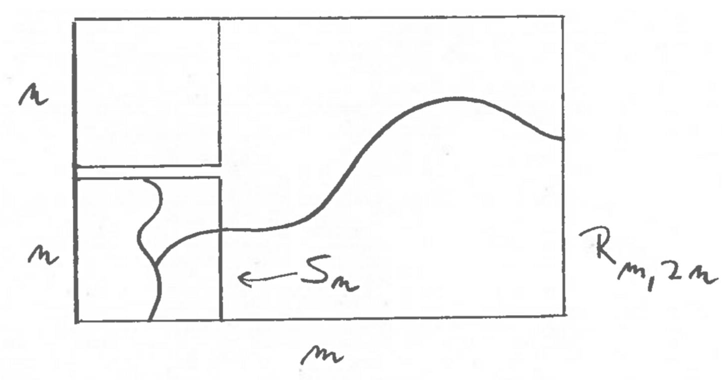


Figure 5.4. The event $X(R)$.

Proof. Let $\gamma$ be any vertical crossing (not necessarily open) of $S$ and let $LV(S,\omega)$ be the left-most vertical open crossing of $S$ in $\omega$ (if it does not exist, we set $LV(S,\omega) = \emptyset$.). By Remark 5.3, the event $\{LV(S,\omega) = \gamma\}$ is independent of the state of the edges that are to the right of $\gamma$. We use this property to prove the following claim.

**Claim 5.6.**

$$\mathbb{P}_p[X(R)|LV(S,\omega) = \gamma] \geq \mathbb{P}_p[H(R)]/2. \tag{5.7}$$

Proof. Let $\bar{\gamma}$ be the path formed by $\gamma$, its reflection $\gamma'$ around the horizontal symmetry axis of $R$, and the edge connecting $\gamma$ with $\gamma'$. (See Figure 5.5)

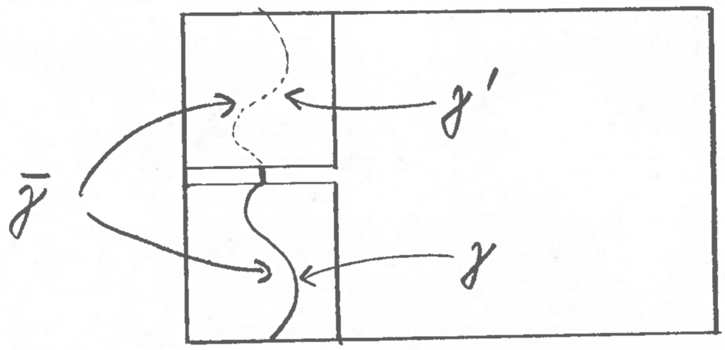


Figure 5.5. Construction of $\bar{\gamma}$.

With probability $\mathbb{P}_p[H(R)]$ there exists a open horizontal crossing $\gamma_3$ of $R$. The crossing $\gamma_3$ must meet the path $\bar{\gamma}$. By symmetry, it meets $\gamma$ before $\gamma'$ with probability $\mathbb{P}_p[H(R)]/2$. Therefore, for all vertical crossings $\gamma$ of $S$,

$$\mathbb{P}_p\begin{bmatrix}\text{there exists an open path connecing } \gamma \text{ with the right-hand} \\ \text{side of } R\text{, lying to the right of } \bar{\gamma}\end{bmatrix} \geq \mathbb{P}_p[H(R)]/2. \tag{5.8}$$

Let $Y(\gamma)$ denotes the event in the last display. Since $Y(\gamma)$ depends only on the edges that are to the right of $\gamma$, $Y(\gamma)$ is independent of $\{LV(S,\omega) = \gamma\}$. Hence,

$$\mathbb{P}_p[Y(\gamma)|LV(S,\omega) = \gamma] = \mathbb{P}_p[Y(\gamma)] \geq \mathbb{P}_p[H(R)]/2. \tag{5.9}$$

Since $\{LV(S,\omega) = \gamma\} \cap Y(\gamma) \subset X(R)$, we obtain

$$\mathbb{P}_p[X(R)|LV(S,\omega) = \gamma] \geq \mathbb{P}_p[H(R)]/2. \tag{5.10}$$

This proves the claim. □

The proof of the lemma is then trivial. Summing over all possible values $\gamma$ of the left-most vertical crossing of $S$ we obtain,

$$\begin{aligned}\mathbb{P}_p[X(R)] &= \sum_\gamma \mathbb{P}_p[X(R)|LV(S) = \gamma]\mathbb{P}_p[LV(S) = \gamma] \\ &\geq \tfrac{1}{2}\mathbb{P}_p[H(R)]\sum_\gamma \mathbb{P}_p[LV(S) = \gamma] \\ &= \tfrac{1}{2}\mathbb{P}_p[H(R)]\mathbb{P}_p[V(S)].\end{aligned} \tag{5.11}$$

□

We now study the crossing probability of $3n \times 2n$ rectangle along its longer direction.

**Lemma 5.7.** *For all $n \geq 1$,*

$$\mathbb{P}_{1/2}[H(R_{3n,2n})] \geq 2^{-7}. \tag{5.12}$$

PROOF. Let $R$, $R'$ and $S$ be as at Figure 5.6. $R$ and $R'$ are $2n \times 2n$ squares and $S$ is $n \times n$ square. Let $X(R)$ be as before and $X'(R')$ be the same event in $R'$, reflected horizontally. Then, by Lemma 5.5 and Corollary 5.4,

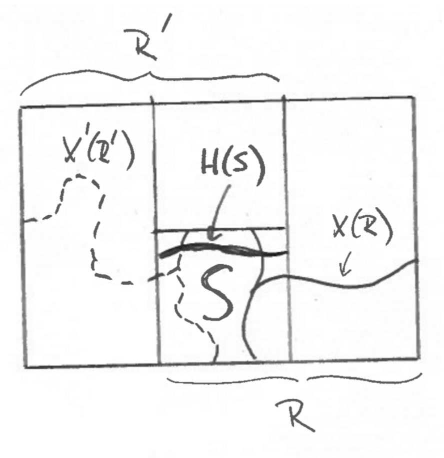


FIGURE 5.6. Construction of the horizontal crossing of $R_{3n,2n}$.

$$\mathbb{P}_{1/2}[X'(R')] = \mathbb{P}_{1/2}[X(R)] \geq \mathbb{P}_{1/2}[H(R)]\mathbb{P}_{1/2}[V(S)]/2 \geq 2^{-3}. \tag{5.13}$$

Moreover, by the FKG inequality (see Figure 5.6 again),

$$\begin{aligned}\mathbb{P}_{1/2}[H(R_{3n,2n})] &\geq \mathbb{P}_{1/2}[X(R) \cap X'(R') \cap H(S)] \\ &\geq \mathbb{P}_{1/2}[X(R)]\mathbb{P}_{1/2}[X'(R)']\mathbb{P}_{1/2}[H(S)] \geq 2^{-7}.\end{aligned} \tag{5.14}$$

This completes the proof. □

It is much easier to pass from $3 \times 2$ rectangles to rectangles of larger aspect ratio.

**Lemma 5.8.** *For all integers $k \geq 2$, $n \geq 1$,*

$$\mathbb{P}_{1/2}[H(R_{kn,2n})] \geq 2^{17-8k}. \tag{5.15}$$

PROOF. We cover the $kn \times 2n$ rectangle by $(k-2)$ rectangles $R_i$ of size $3n \times 2n$, so that the intersection of two neighbouring rectangles $R_i$, $R_{i+1}$ is a square denoted by $S_i$ (see Figure 5.7).

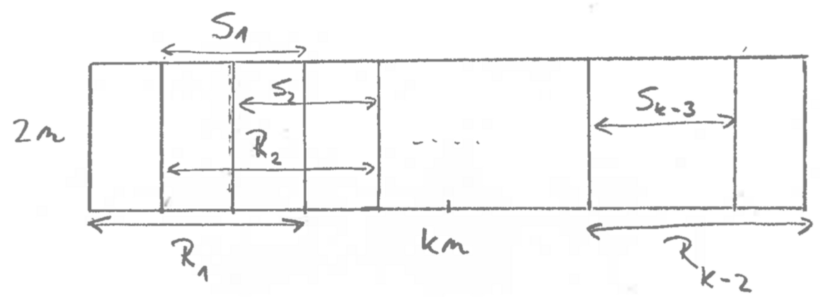


FIGURE 5.7. Tiling a long rectangle by $3 \times 2$ rectangles and squares.

Then, by the FKG inequality again,

$$\begin{aligned}\mathbb{P}_{1/2}[H(R_{kn,2n})] &\geq \mathbb{P}_{1/2}\Big[\bigcap_{i=1}^{k-2} H(R_i) \cap \bigcap_{i=1}^{k-3} V(S_i)\Big] \\ &\geq \mathbb{P}_{1/2}[H(R_{3n,2n})]^{k-2}\mathbb{P}_{1/2}[S_{2n}]^{k-3} \geq 2^{-7(k-2)}2^{-(k-3)} = 2^{17-8k},\end{aligned} \tag{5.16}$$

and the proof is completed. □

The last two lemmas are originally due to Russo [**Rus78**] and Seymour and Welsh [**SW78**]. The proof we presented is due to Bollobás and Riordan.

**1.2. Proof of Kesten's theorem.** We can now prove the Kesten's theorem (Theorem 5.1). We start with a lower bound on $p_c(\mathbb{Z}^2)$.

**Lemma 5.9.** *On* $\mathbb{Z}^2$, $\theta(\frac{1}{2}) = 0$. *This implies* $p_c(\mathbb{Z}^2) \geq \frac{1}{2}$.

PROOF. Consider the dual lattice $\mathbb{Z}^2_\star$ and the dual percolation as in the proof of Peierl's argument (recall that any dual edge is open iff the unique 'standard' edge that it is crossing is closed). Let $S = S_{6n} \subset \mathbb{Z}^2_\star$ by $6n \times 6n$ square centred at $(\frac{1}{2}, \frac{1}{2})$. Let $R^1_n, \ldots, R^4_n$ by $2n \times 6n$ rectangles as on Figure 5.8, and let $F_n$ be the event 'there is an open crossing in every $R^i_n$, $i = 1, \ldots, 4$ along the $6n$ direction' (see Figure 5.8 again).

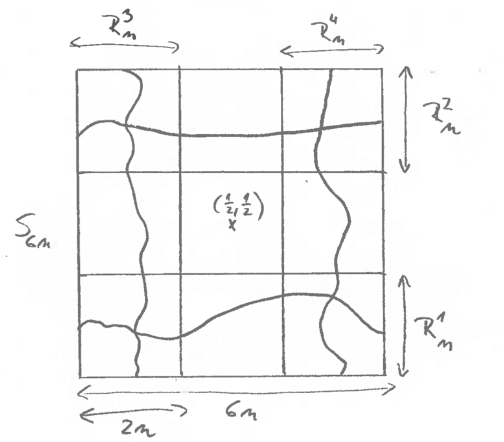

FIGURE 5.8. The event $F_n$.

By the FKG inequality and Lemma 5.8,

$$\mathbb{P}_{1/2}[F_n] \geq \mathbb{P}_{1/2}[H(R^i_n)]^4 \geq (2^{17-8\cdot 8})^4 = 2^{-100}. \tag{5.17}$$

If $F_n$ occurs, then the open cluster of the origin (in the standard percolation) is contained in $S_{6n}$. Therefore,

$$\theta(1/2) = \mathbb{P}_{1/2}[0 \leftrightarrow \infty] \leq \mathbb{P}_{1/2}[\cap_{i \in \mathbb{N}} F^c_{4^i}]. \tag{5.18}$$

Since, for any $i \neq j$ we have $(\cup_{k=1}^4 R^k_{4^i}) \cap (\cup_{k=1}^4 R^k_{4^j}) = \emptyset$, the events $F_{4^i}$, $i \in \mathbb{N}$ are independent. Therefore,

$$\theta(1/2) \leq \mathbb{P}_{1/2}[\cap_{i \in \mathbb{N}} F^c_{4^i}] = \prod_{i \in \mathbb{N}} \mathbb{P}_{1/2}[F^c_{4^i}] \leq (1 - 2^{-100})^\infty = 0 \tag{5.19}$$

and the lemma is proved. □

**Exercise 5.10.** Deduce from the presented proof that $\mathbb{P}_{1/2}[\sup\{\|x\|_\infty : x \in \mathcal{C}_0\} \geq k] \leq k^{-c}$, for some $c > 0$.

It remains to prove an upper bound on $p_c(\mathbb{Z}^2)$, that is $p_c(\mathbb{Z}^2) \leq \frac{1}{2}$. Suppose that this is not true, that is $p_c(\mathbb{Z}^2) > \frac{1}{2}$. Then $p = \frac{1}{2}$ is sub-critical and we can apply Menshikov's theorem (Theorem 3.1),

$$\mathbb{P}_{1/2}[H(S_n)] \leq \sum_{x \in \text{left edge of } S_n} \mathbb{P}_{1/2}[x \leftrightarrow \partial B(n, x)] \leq n e^{-cn} \xrightarrow{n \to \infty} 0. \tag{5.20}$$

(Here, $B(n, x)$ is the box centred at $x$ with side $2n$). However, this contradicts Corollary 5.4 which states $\mathbb{P}_{1/2}[H(S_n)] \geq \frac{1}{2}$ for all $n$. This completes the proof of Kesten's theorem.

The application of the Menshikov's theorem in the previous argument might appear unsatisfactory, given the complexity of the proof of this theorem. There are however simple proofs of $p_c(\mathbb{Z}^2) \leq 1/2$, the presented one is the shortest given our previous work. Different proofs can be found in [**Gri99**] and [**BR06a**].

## 2. Conformal invariance of critical percolation: Smirnov's Theorem

This section follows closely Chapter 7 of the book of Bollobás and Riordan [**BR06a**]. In these notes, we do not give detailed proofs of all statements, some of them we even state without proof. The goal is to explain the main ideas of the proof and to give details on those places where we found them important for understanding these ideas.

**2.1. Conformal invariance conjecture.** We summarise first some facts from the complex analysis needed to formulate the conformal invariance of the critical percolation. In this section we consider lattices in the plane $\mathbb{R}^2$ that we identify with the complex plane $\mathbb{C}$. A domain $D \subset \mathbb{C}$ is a non-empty connected open subset of $\mathbb{C}$. If $D$, $D'$ are domains, then a *conformal map* from $D$ to $D'$ is a bijection $f : D \to D'$ which is analytic with non-vanishing derivative at every point of $D$. Note that then the inversion $f^{-1}$ is analytic on $D'$. The analyticity of $f$ implies that conformal maps preserve angles (which explains the terminology 'conformal'): the images by $f$ of two crossing line segments in $D$ are two curves crossing at the same angle as the segments.

By *Riemann Mapping Theorem* if $D$, $D' \neq \mathbb{C}$ are two simply connected domains (domains without holes), then there exists a conformal map $f$ from $D$ to $D'$. Actually, there exist infinitely many of such maps.

We write $\bar{D}$ and $\partial D$ for the closure and the boundary of $D$. We say that $D$ is a *Jordan domain* if $\partial D$ is a *Jordan curve*, that is the image of a continuous injection $\gamma : \mathbb{S}^1 \to \mathbb{C}$, where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is the circle. By theorem of Carathéodory, if $D$, $D'$ are Jordan domains and $f$ a conformal map from $D$ to $D'$, then $f$ can be uniquely extended to a continuous bijection $\bar{f}$ from $\bar{D}$ to $\bar{D}'$.

By *k-marked domain* $(D; P_1, \dots, P_k)$ we understand a Jordan domain $D$ together with $k$ points $P_1, \dots, P_k$ on its boundary. We always assume that these points are ordered anti-clockwise on $\partial D$. Another consequence of the Riemann Mapping Theorem is the following claim.

**Theorem 5.11.** *If $(D; P_1, P_2, P_3)$, $(D'; P'_1, P'_2, P'_3)$ are two 3-marked domains, then there exists a unique conformal map $f : D \to D'$ such that its extension $\bar{f}$ satisfies $\bar{f}(P_i) = P'_i$, $i = 1, 2, 3$.*

We now consider 4-marked domains $D_4 = (D; P_1, \dots, P_4)$, $D'_4 = (D'; P'_1, \dots, P'_4)$. We call $D_4$ and $D'_4$ *conformally equivalent* if there exists a conformal mapping $f : D \to D'$ which fixes all four boundary points, that is $\bar{f}(P_i) = P'_i$ for all $i = 1, \dots, 4$. (This mapping is either unique or does not exists, as can be seen from the last theorem.) We denote by $A_1, \dots, A_4$ the arcs of $\partial D_4$; $A_i$ joins $P_i$ with $P_{(i+1)}$, $i = 1, \dots, 4$[3].

We now formulate the conformal invariance conjecture. Let $\Lambda$ be any 'suitable[4]' lattice in the plane, $\delta\Lambda$ its scaling by factor $\delta > 0$, $\mathbb{P}_p$ a percolation (bond or site) on $\Lambda$ (or its scaling), and $p_c(\Lambda)$ its critical parameter. For a 4-marked domain $D_4$ we denote by

[3]In these notes, we always identify $P_5$ with $P_1$ when considering 4-marked domains. When considering 3-marked domain we identify $P_4$ with $P_1$

[4]For what 'suitable' means see page 182 of [**BR06a**].

$P_\delta(D_4, \Lambda, p_c(\Lambda))$ to be the probability that in the percolation on $\delta\Lambda$ with parameter $p_c(\Lambda)$ there exists an open path joining arcs $A_1$ and $A_3$[5]

**Conjecture 5.12.** *The limit*

$$P(D_4) := \lim_{\delta \to 0} P_\delta(D_4, \Lambda, p_c(\Lambda)) \tag{5.21}$$

*exists, lies in* $(0, 1)$*, and is independent of the lattice* $\Lambda$*. Moreover, it is conformal invariant: If* $D_4$ *and* $D_4'$ *are two conformally invariant 4-marked domains, then*

$$P(D_4) = P(D_4'). \tag{5.22}$$

From our previous results it follows that $\lim_{\delta\to 0} P_\delta(D_4, \Lambda, p) = 0$ (resp. 1) when $p < p_c(\Lambda)$ (resp. $p > p_c(\Lambda)$). Moreover, the RSW theory (which can be easily extended to any reasonable lattice $\Lambda$) can be used to show that $0 < \liminf_{\delta\to 0} P_\delta(D_4, \Lambda, p_c(\Lambda)) \leq \limsup_{\delta\to 0} P_\delta(D_4, \Lambda, p_c(\Lambda)) < 1$. The conjecture thus predicts, in addition, the existence of the limit and its conformal invariance.

**2.2. Smirnov's Theorem.** Smirnov's Theorem [**Smi01**] states that the conformal invariance conjecture holds in one special case, namely for site percolation on the equilateral triangular lattice $\mathbb{T}$.

**Theorem 5.13.** *For the site percolation on the triangular lattice* $\mathbb{T}$*, the limit*

$$P(D_4) := \lim_{\delta \to 0} P_\delta(D_4, \mathbb{T}, p_c(\Lambda)) \tag{5.23}$$

*exists and is conformally invariant.*

Moreover, the value $P(D_4)$ can be 'computed' as follows. Consider 4-marked domain $D_4'$ which is equilateral triangle with vertices $P_1' = (1, 0)$, $P_2' = (1/2, \sqrt{3}/2)$, $P_3' = (0, 0)$, with an additional point $P_4' = (x, 0)$, $x \in (0, 1)$. For an arbitrary 4-marked domain $D_4$, let $f_D$ be the unique conformal map from $D$ to $D'$ which fixes first three vertices of $D_4$[6], that is $\bar{f}(P_i) = P_i'$, $i = 1, \ldots, 3$. This maps sends $P_4$ onto segment joining $P_3'$, $P_1'$: $f_D(P_4) =: (x_D, 0)$, $x_D \in (0, 1)$.

**Theorem 5.14** (Cardy's formula in Carleson formulation)**.** (i) *The domain* $D_4'$ *satisfies* $P(D_4') = x$.
(ii) *Let* $D_4$ *be an arbitrary 4-marked domain. Then* $P(D_4) = x_D$.

We now give (non-complete) the proofs of these two theorems. We start by several preliminary remarks. We use the fact that site percolation on triangular lattice can be represented as a tiling of the hexagonal lattice $\mathbb{H}$, see Figure 1.3. This representation can be then used to prove that $p_c(\mathbb{T}) = 1/2$.

**Exercise 5.15.** Adapt the proof of Kesten's theorem to show the last claim.

In the same way the RSW theory can be extended to the critical site percolation on $\mathbb{T}$. One of its consequences is the following lemma.

**Lemma 5.16** (Radial RSW lemma)**.** *Let* $A$ *be the annulus in the plain with inner radius* $r_-$ *and outer radius* $r_+$*,* $A = \{z \in \mathbb{C} : r_- < |z| < r_+\}$*. We say that* $A$ *has a radial open (closed) crossing in* $\delta\mathbb{T}$ *if there exists nearest-neighbour path of open (closed) vertices of* $\delta\mathbb{T}$ *intersecting the inner and outer boundary of* $A$*.*

[5]We do not give the exact definition of 'joining' here. Any reasonable definition does the job.

[6]This map is generally very hard to find explicitly

*Let $\delta < r_-/1000$. Then there exists an absolute constant $\alpha > 0$ such that*

$$\mathbb{P}[A \textit{ has a radial open (closed) crossing on } \delta\mathbb{T}] \le \left(\frac{r_-}{r_+}\right)^{\alpha}. \tag{5.24}$$

*Here and later, $\mathbb{P}$ denotes the critical site percolation on $\mathbb{T}$.*

SKETCH OF THE PROOF. Consider a sequence $A_i$, $i = 1, \dots, k$, of concentric annuli: inner radius of $A_i$ equals $2^{i-1}r_-$, the outer radius of $A_i$ is $2^i r_-$. If $k = \lfloor \log_2(r_+/r_-) \rfloor$, then all $A_i$'s are contained in $A$. In every $A_i$ draw six rectangles as on Figure 5.9. By standard

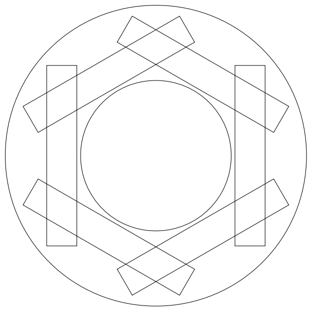

FIGURE 5.9. Illustration of the proof of the radial RSW lemma.

RSW lemma, the probability that any of these rectangles has a closed crossing along the longer direction is bounded below by a positive constant $c$. Let $E_i$ be the event 'there is a closed connection in all six rectangles drawn in $A_i$'. By the FKG inequality, $\mathbb{P}[E_i] \ge c^6$. If any of $E_i$ occurs, than there cannot be an open radial crossing. Therefore, the probability that $A$ has an open radial crossing is bounded from above by

$$\mathbb{P}[\cap_{i=1}^k E_i^c] \le (1 - c^6)^k \le (r_-/r_+)^{\alpha}. \tag{5.25}$$

(For the first inequality we used the fact that the annuli $A_i$ are 'disjoint', and thus $E_i$ are independent). ☐

**2.3. Discrete domains and colour switching.** We now present so called colour switching lemma that is at heart of the proof of Smirnov's theorem. To present this lemma we need however precise what we mean by open crossing of a domain.

We consider triangular lattice $\delta\mathbb{T}$ and the corresponding hexagonal lattice $\delta\mathbb{H}$. We call $G = G_\delta \subset \delta\mathbb{T}$ *discrete domain* if the union of corresponding closed hexagons is simply connected in $\mathbb{C}$. We write $\partial_- G = \{x \in G : \exists y \sim x, y \notin G\}$ for the inner boundary of $G$ and $\partial_+ G = \{x \notin G : \exists y \sim x, y \in G\}$ for the outer boundary of $G$. We always assume that discrete domain is 'nice' in the sense that both inner and outer boundary are simple loops (closed path without repetitions) in $\delta\mathbb{T}$.

$k$-marked discrete domain $G$ is a discrete domain together with $k$ points $v_1, \dots, v_k \in \partial_- G$ ordered in anti-clockwise order around the loop $\partial_- G$. For convenience, we always assume that $v_i$ has at least two neighbours in $\partial_+ G$ (This is quite mild, since any site in $\partial_- G$ that does not have two such neighbours is adjacent to a site in $\partial_- G$ which does).

We denote by $A_i$ the part of $\delta_- G$ between $v_i$, $v_{i+1}$[7], $v_i$ and $v_{i+1}$ included. We partition $\partial_+ G$ into sets $A_i^+$ in the way that $A_i^+$ is a path starting at a neighbour of $v_i$ and ending at a neighbour of $v_{i+1}$. This is possible since we assume that all $v_i$'s have two neighbours in $\partial_+ G$.

Let $G$ be a 4-marked domain. We say that $G$ has an open crossing from $A_1$ to $A_3$ if there is an open path in $G$ starting in $A_1$ and ending in $A_3$. Other crossings are defined analogously.

[7]with obvious identification $v_{k+1} = v_1$

**Lemma 5.17.** *$G$ has either open (black) crossing from $A_1$ to $A_3$ of white crossing from $A_2$ to $A_4$.*

**Exercise 5.18.** Show the lemma. *Hint.* Consider the tiling containing all hexagons corresponding to $G \cup \partial_+ G$. Colour all hexagons of $G$ black iff the corresponding site is open, all hexagons of $A_1^+$ and $A_3^+$ black and all hexagons of $A_2^+$, $A_4^+$ white. Use the argument as in the proof of Kesten's theorem to show the claim.

We now state the colour-switching lemma. We consider 3-marked domain $G \subset \mathbb{T}$ (Here $\delta$ is irrelevant so we set $\delta = 1$.) and three points $x_1, x_2, x_3 \in G \setminus \partial_- G$ which form a triangle (in the anti-clockwise order). We write $B_1B_2W_3$ for the event 'there exist three disjoint paths $\gamma_1$, $\gamma_2$ and $\gamma_3$ in $G$ such that $\gamma_i$ starts in $x_i$ and ends in $A_i$, $i = 1, 2, 3$; $\gamma_1$ and $\gamma_2$ are black (open), and $\gamma_3$ is white (closed)'. We define events $B_1W_2B_3$, $W_1W_2W_3$, etc. analogously. We are interested in probabilities of (in total eight) events we have just defined.

By black/white symmetry (remember $p_c = 1/2$) we have identities of the type

$$\mathbb{P}[W_1W_2B_3] = \mathbb{P}[B_1B_2W_3]. \tag{5.26}$$

So there are potentially only four distinct probabilities. The colour-switching lemma states that three of them are equal.

**Lemma 5.19** (Colour switching)**.**

$$\mathbb{P}[B_1B_2W_3] = \mathbb{P}[B_1W_2B_3] = \mathbb{P}[W_1B_2B_3]. \tag{5.27}$$

**Remark 5.20.** In general $\mathbb{P}[W_1W_2W_3] \neq \mathbb{P}[W_1W_2B_3]$.

Proof. We will show

$$\mathbb{P}[B_1W_2B_3] = \mathbb{P}[W_1B_2B_3] = \mathbb{P}[B_1W_2W_3]. \tag{5.28}$$

The remaining identities follows by relabelling.

We consider tiling of $G \cup \partial_+ G$ by black, white and grey tiles. The hexagons of $G$ are black iff the corresponding sites are open, the hexagons of $A_1^+$ black, $A_2^+$ white and $A_3^+$ grey. Let $I$ be the interface graph, that is the subgraph of $\mathbb{H}$ consisting of all edges separating black and white tiles together with their endpoints. Every vertex of $I$ has degree two, possible exceptions are the vertex $y$ on the boundary of $A_1^+$ and $A^+2$ and all vertices incident to grey region. This implies that the path $P$ in $I$ starting in $y$ should end at the boundary of the grey region.

Let $w \in \mathbb{H}$ be the barycentre of $x_1, x_2, x_3$ and let $z$ be the other vertex of the edge separating the hexagons corresponding to $x_1$ and $x_2$. Let $\vec{e}$ be the oriented edge $(z, w)$.

**Claim 5.21.** *On $B_1W_2$, the path $P$ starting in $y$ traverses $\vec{e}$ in the positive direction.*

Proof. Let $\gamma_1$ realises $B_1$ and $\gamma_2$ realises $W_2$ be two necessarily disjoint paths. We form a cycle $C \subset \mathbb{T}$ by following $\gamma_1$ from $x_1$ to $A_1$, then $A_1^+$ to its end, then $A_2^+$ to the neighbour of the endpoint of $\gamma_2$, and then $\gamma_2$ from $A_2$ to $x_2$. Walking along this cycle, the colour changes exactly twice, once from black to white when crossing the edge adjacent to $y$, and once from white to black, when crossing $\vec{e}$.

The path $P$ enters the interior of $C$ at its first step. It cannot end inside of $C$, since there are no grey hexagons there. So it must exit it and the only possibility how to exit the interior of $C$ is via edge $\vec{e}$. Moreover, it follows from the ordering of $x_i$'s and $A_i$'s that $z$ is in the interior of $C$ and $w$ not, so $\vec{e}$ should be traversed in the positive direction. □

**Claim 5.22.** *Let $P'$ be defined as follows: start at $y$, continue along $I$ and stop if $\vec{e}$ is traversed in the positive direction or if we reach the grey region. Let $N(P') \subset \mathbb{T}$ be the set of sites whose hexagons are adjacent to $P'$. Then if $P'$ ends in $\vec{e}$, then $N(P')$ contains two paths $\gamma_1$, $\gamma_2$ witnessing $B_1$ and $W_2$.*

Sketch of the proof. The path $P'$ has the property that it has black hexagons on its left-hand side (when walked starting from $y$) and white hexagons on its right-hand side. Moreover, white and black parts of $N(P')$ are connected. If $P'$ traverses $\vec{e}$ in the positive direction, then $x_1$ is white, $x_2$ is black, and it is possible to extract a path $\gamma_1$ ($\gamma_2$) from the white (black) part of $N(P')$ joining $x_1$ ($x_2$) to $A_1$ ($A_2$). The paths $\gamma_1$, $\gamma_2$ are necessarily disjoint since they have different colours. $\square$

From the previous two claims we obtain

**Claim 5.23.** *$B_1W_2$ holds iff $P'$ ends in $\vec{e}$.*

**Claim 5.24.** *$B_1W_2B_3$ (resp. $B_1W_2W_3$) holds iff $P'$ ends in $\vec{e}$ and there is a path $\gamma_3 \subset G$ of black (resp. white) vertices joining $x_3$ to $A_3$ using no vertex of $N(P')$.*

Sketch of the proof. Suppose that $B_1W_2B_3$ holds and $\gamma_1$, $\gamma_2$, $\gamma_3$ are the witnessing paths. Construct the cycle $C$ as in the proof of Claim 5.21. From this proof it follows that $P'$ lies entirely within $C$, apart from its initial and terminal edge. In particular, every site of $N(P')$ is inside or on $C$. But $\gamma_3$ cannot cross $C$, so $\gamma_3$ lies entirely outside $C$ and is disjoint from $N(P')$.

The reverse implication is immediate from the previous claim. The proof for the event $B_1W_2W_3$ is analogous. $\square$

We can now finish the proof of the colour-switching lemma. Let $\Lambda = 2^G$ be the space of all possible arangement of colours of hexagons in $G$, every arangement has the same probability. Let $\omega \in \Omega$ and let $P'(\omega)$ be the path $P'$ as above. Define the arangement $\omega'$ by

$$\omega'(x) = \begin{cases} \omega(x) & \text{if } x \in N(P'(\omega)), \\ 1 - \omega(x) & \text{if } x \notin N(P'(\omega)). \end{cases} \tag{5.29}$$

The path $P'$ depends only on the state of sites in $N(P')$, by the locality of the algorithm constructing it. Hence $P'(\omega) = P'(\omega')$ and thus $\omega'' = \omega$. Therefore, the map $\omega \mapsto \omega'$ is a bijection on $\Omega$. By colour symmetry, this map is also measure preserving. But if $\omega \in B_1W_2B_3$, then $\omega' \in B_1W_2W_3$. This implies

$$\mathbb{P}[B_1W_2B_3] = \mathbb{P}[B_1W_2W_3], \tag{5.30}$$

and completes the proof of the lemma. $\square$

**2.4. Separating events.** Consider now a discrete domain $G_\delta \in \delta\mathbb{T}$ a point $z \in \delta\mathbb{H}$ that lies in the 'interior' of $G$. We define events

$$E^3_\delta(z) = \{G_\delta \text{ contains a black path from } A_1 \text{ to } A_2 \text{ separating } z \text{ from } A_3^+\}. \tag{5.31}$$

and $E^2_\delta(z)$, $E^1_\delta(z)$ by relabelling cyclically. Observe that we require a separating path, that is the situation depicted on Figure 5.10(b) is not contained in $E^3_\delta(z)$.

**Claim 5.25.** *Let $x_1, x_2, x_3$ and $w, z$ be as in the previous claims. Then $E^3_\delta(z) \setminus E^3_\delta(w)$ occurs iff $B_1B_2W_3$ occurs.*

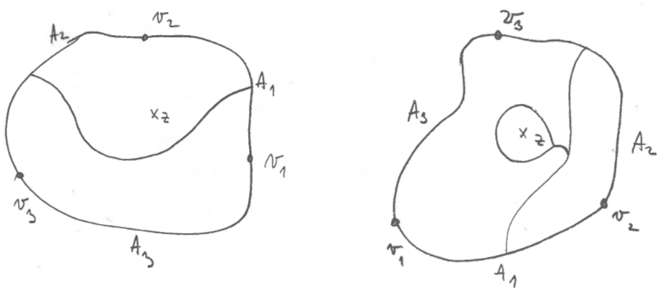

FIGURE 5.10. (a) Separating event $E_\delta^3(z)$. (b) a situation not in $E_\delta^3(z)$.

SKETCH OF THE PROOF. ($E_\delta^3(z) \setminus E_\delta^3(w) \implies B_1 B_2 W_3$). Since $z$ is separated from $A_3^+$ and $w$ not, there must be black paths $\gamma_1$, $\gamma_2$ witnessing $B_1$ and $B_2$ (recall that $x$'s and $A$'s are ordered anti-clockwise). We need thus to find a paths $\gamma_3$ witnessing $W_3$. Obviously, $x_3$ is white since otherwise $\gamma_1 x_3 \gamma_2$ would separate $w$ from $A_3^+$. By duality, it is also easy to see that if we cannot realise $W_3$ there is a black cycle around $x_3$. The hard part of the proof is to show that it is possible to extract a path from this cycle and from $\gamma_1$ and $\gamma_2$. For details see [**BR06a**], p. 205.

($B_1 B_2 W_3 \implies E_\delta^3(z) \setminus E_\delta^3(w)$). Trivial. □

LAST LECTURE WILL BE COMPLETED.

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